解析学 A （石田，2017/6/26）

4 章 5 節， 広義積分，問 1 (p.110)

(1) $x=t^2$ と置いて $dx=2tdt$ より

$$\begin{aligned}&\int_0^1 \frac{(1-x)^2}{\sqrt{x}}dx\\ &= 2\int_0^1 (1-t^2)^2 dt\\ &= 2\int_0^1 (t^4-2t^2+1)dt\\ &= 2\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right]_0^1\\ &= 2\left(\frac{1}{5}-\frac{2}{3}+1\right)\\ &= \frac{16}{15}\end{aligned}$$

(2) $x=\sin t$ と置いて $dx=\cos t\,dt$ より

$$\begin{aligned}&\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}\\ &= \int_{-\pi/2}^{\pi/2} dt\\ &= \pi\end{aligned}$$

(3) $x=\sin^2 t$ と置いて $dx=2\cos t\sin t$ より

$$\begin{aligned}&\int_0^1 \sqrt{\frac{x}{1-x}}dx\\ &= 2\int_0^{\pi/2}\frac{\sin t}{\cos t}\cos t\sin t dt\\ &= 2\int_0^{\pi/2}\sin^2 t dt\\ &= \int_0^{\pi/2}(1-\cos 2t)dt\\ &= \left[t-\frac{\sin 2t}{2}\right]_0^{\pi/2}\\ &= \frac{\pi}{2}\end{aligned}$$

(4) $x=t^6$ と置いて $dx=6t^5\,dt$ より

$$\begin{aligned}&\int_0^1 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}\\ &= \int_0^1 \frac{6t^3}{t+1}dt\\ &= \int_0^1\left(6t^2-6t+6-\frac{6}{t+1}\right)dt\\ &= [2t^3-3t^2+6t-6\log(t+1)]_0^1\\ &= 5-6\log 2\end{aligned}$$

(5) 部分積分により

$$\int \log x\,dx = x\log x - \int dx = x\log x - x$$

であるから，$\lim_{x\to 0} x\log x = 0$ より

$$\begin{aligned}&\int_0^1 \log x\,dx\\ &= [x\log x - x]_0^1\\ &= -1\end{aligned}$$

(6)

$$\frac{1}{x(1+x^2)} = \frac{1}{x}-\frac{x}{1+x^2}$$

より

$$\int \frac{dx}{x(1+x^2)} = \log x - \frac{1}{2}\log(1+x^2)$$

となる．したがって

$$\begin{aligned}&\int_1^\infty \frac{dx}{x(1+x^2)}\\ &= \left[\log x - \frac{1}{2}\log(1+x^2)\right]_1^\infty\\ &= \left[\log\left(\frac{x}{\sqrt{1+x^2}}\right)\right]_1^\infty\\ &= \left[\log\left(\frac{1}{\sqrt{(1/x^2)+1}}\right)\right]_1^\infty\\ &= \log\sqrt{2}\end{aligned}$$

(7)

$$\frac{x+1}{x(1+x^2)} = \frac{1}{x(1+x^2)}+\frac{1}{1+x^2}$$

より (6) を使って

$$\int_1^\infty \frac{dx}{x(1+x^2)} = \log\sqrt{2}+\int_1^\infty \frac{dx}{1+x^2}$$

となる．$x=\tan t$ と置いて

$$\begin{aligned}
&\log\sqrt{2}+\int_1^\infty \frac{dx}{1+x^2}\\
&= \log\sqrt{2}+\int_{\pi/4}^{\pi/2} dt\\
&= \log\sqrt{2}+\frac{\pi}{4}
\end{aligned}$$

(8) p.88 より

$$\int \frac{dx}{(1+x^2)^2}=\frac{x}{1+x^2}+\frac{1}{2}\int\frac{dx}{1+x^2}$$

となる．よって $x=\tan t$ として

$$\begin{aligned}
&\int_0^\infty \frac{dx}{(1+x^2)^2}\\
&= \left[\frac{x}{1+x^2}\right]_0^\infty+\frac{1}{2}\left[\int dt\right]_0^{\pi/2}\\
&= \frac{\pi}{4}
\end{aligned}$$

(9)

$$\begin{aligned}
&\int_0^\infty xe^{-x^2}\\
&= \left[-\frac{1}{2}e^{-x^2}\right]_0^\infty\\
&= \frac{1}{2}
\end{aligned}$$

(10) $x=\dfrac{t}{\sqrt{2}}$ と置けば，途中 $s=-t$ と $t=u+1$, $u=\tan y$ の変換により

$$\begin{aligned}
&\int_0^\infty \frac{x^2}{1+x^4}dx\\
&= \sqrt{2}\int_0^\infty \frac{t^2}{4+t^4}dt\\
&= \sqrt{2}\int_0^\infty \frac{t^2}{(2-2t+t^2)(2+2t+t^2)}dt\\
&= \frac{\sqrt{2}}{4}\int_0^\infty\left(\frac{t}{2-2t+t^2}-\frac{t}{2+2t+t^2}\right)dt\\
&= \frac{\sqrt{2}}{8}\int_0^\infty\left(\frac{2t-2}{2-2t+t^2}-\frac{2t+2}{2+2t+t^2}\right)dt\\
&+ \frac{\sqrt{2}}{4}\int_0^\infty\left(\frac{1}{2-2t+t^2}+\frac{1}{2+2t+t^2}\right)dt\\
&= \frac{\sqrt{2}}{8}\left[\log(2-2t+t^2)-\log(2+2t+t^2)\right]_0^\infty\\
&+ \frac{\sqrt{2}}{4}\left(\int_0^\infty\frac{dt}{2-2t+t^2}+\int_{-\infty}^0\frac{ds}{2-2s+s^2}\right)\\
&= \frac{\sqrt{2}}{8}\left[\log\frac{2-2t+t^2}{2+2t+t^2}\right]_0^\infty\\
&+ \frac{\sqrt{2}}{4}\int_{-\infty}^\infty\frac{dt}{1+(t-1)^2}\\
&= \frac{\sqrt{2}}{4}\int_{-\infty}^\infty\frac{du}{1+u^2}\\
&= \frac{\sqrt{2}}{4}\int_{-\pi/2}^{\pi/2}dy\\
&= \frac{\sqrt{2}\,\pi}{4}
\end{aligned}$$