Transnormal functions on a Riemannian manifold

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Abstract

We extend theorems of É. Cartan, Nomizu, Münzner, Q.M. Wang, and Ge–Tang on isoparametric functions to transnormal functions on a general Riemannian manifold. We show that if a complete Riemannian manifold $M$ admits a transnormal function, then $M$ is diffeomorphic to either a vector bundle over a submanifold, or a union of two disk bundles over two submanifolds. Moreover, a singular level set $Q$ is austere and minimal, if exists, and generic level sets are tubes over $Q$. We give a criterion for a transnormal function to be an isoparametric function.

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1. Introduction

A cohomogeneity one action on a manifold $M$ is used to construct special metrics or submanifolds with a certain property [3,27]. More generally, one parameter family of hypersurfaces, not necessarily homogeneous, would play an important role to reduce some PDE to an ODE. Therefore, to investigate when and where such a family exists is important. In this paper, we consider a family of parallel hypersurfaces given by the level sets of a certain function, and investigate the geometric properties of the level sets as well as of $M$ itself. Throughout the paper, $M$ denotes a complete connected smooth Riemannian manifold without boundary, and $\nabla$ and $\Delta$ denote the Levi-Civita connection and the Laplacian of $M$, respectively.

Definition 1. (See [32].) A globally defined non-constant $C^2$ function $f$ on $M$ satisfying

(I) $|\nabla f|^2 = b(f)$

for a $C^2$ function $b$ on the range of $f$ in $\mathbb{R}$, is called a transnormal function. If $f$ satisfies, in addition to (I),

(II) $\Delta f = a(f)$

for a continuous function $a$ on the range of $f$ in $\mathbb{R}$, $f$ is called an isoparametric function.
The condition (I) implies that the level sets are parallel to each other. The condition (II) implies that a level hypersurface has constant mean curvature (see Section 6). When \( a \equiv 0 \), (II) is the Laplace equation, and when \( b \equiv 1 \), (I) is the Eikonal equation which relates with the geometric optics.

**Definition 2.** When \( f \) is a transnormal (resp., isoparametric) function, a component of a level set is called a foil (resp., isoparametric hypersurface) if it has codimension one, and a singular foil (resp., focal submanifold) if it has codimension bigger than one.

Obviously, an isoparametric hypersurface is a foil, and a focal submanifold is a singular foil. We do not call level sets of a transnormal function transnormal hypersurfaces, as there is another notion of transnormal hypersurfaces [22].

Isoparametric hypersurfaces in the space forms are well-investigated. Those in \( E^n \), \( H^n \) are classified [4], and those in \( S^n \) are almost classified [5,6,13,9,15,17,8]. Summaries and related topics are in [29,16]. Y. Nagatomo [21] constructs isoparametric functions on compact symmetric spaces \( SU(n)/SO(n), Sp(n)/U(n), Gr_4(\mathbb{R}^8) \). There exist isoparametric and/or transnormal functions on various manifolds, see Section 2.

The role of condition (I) is rather essential, in the sense that some properties satisfied by isoparametric functions have already been satisfied by transnormal functions. In fact, we show:

**Theorem 1.1.** Let \( M \) be a complete connected Riemannian manifold which admits a transnormal function \( f \). Then either one of the following holds:

1. \( M \) is diffeomorphic to a vector bundle over a submanifold \( Q \) of \( M \).
2. \( M \) is diffeomorphic to a union of two disk bundles over two submanifolds \( Q \) and \( Q' \) of \( M \), where \( Q \) and/or \( Q' \) may be hypersurface (s).

(2) is a generalization of the well-known fact of Münzner [20] in the case \( M = S^n \) and \( f \) is an isoparametric function. The proof is obtained by combining the fundamental results of Q.M. Wang [32] and J. Bolton [2] on transnormal functions and systems, respectively. Now, we introduce transnormal systems:

**Definition 3.** (See [21].) A transnormal system \( F \) on a complete connected Riemannian manifold \( M \) is a partition of \( M \) into non-empty connected submanifolds called “foils”, so that any geodesic segment of \( M \) cuts the “foils” orthogonally at none or all of its points. \( F \) is non-singular if all “foils” have equal dimension. Otherwise \( F \) is singular.

Here, we restrict our argument to transnormal systems having codimension one “foils” with some exceptional lower dimensional “foils”. Trivial transnormal systems and functions are given by \( M = N \times \mathbb{R} \) or \( M = N \times S^1 \) with the product metric where \( N \) is a Riemannian manifold, and \( f(x, t) = t \) for \( (x, t) \in M \). Another case is when there is a cohomogeneity one group action on \( M \). However, these are not so interesting in our context, and we treat more general situations.

By Wang's regularity theorem (Fact 2 in Section 3), a transnormal function \( f \) induces a transnormal system, which we denote by \( F_f \). In fact, all the components of the level sets of \( f \) generate \( F_f \), and a geodesic normal to \( N \in F_f \) is in the direction \( \nabla f \), which is orthogonal to every foil including singular foils. In this case, we may regard foils as “foils”, where the former is a component of a level set of \( f \), and the latter is an element of a transnormal system.

The converse way, namely, to construct \( f \) from \( F \) by no means trivial. As a necessary condition for a hypersurface to be defined by a level set of a function is \( t \)-regular, that is, regular in the sense of topology, by which we mean that the topology of the hypersurface coincides with the topology induced from the ambient space. The irrational flow on a flat torus is a transnormal system, but “foils” are not \( t \)-regular, and there are no transnormal functions generating this system. We prove a correct version of Wang’s Theorem C in [32]:

**Theorem 1.2.** For a transnormal system \( F \) with \( t \)-regular foils, there exists a transnormal function \( f \) such that \( F = F_f \).

Thus we can discuss transnormal functions instead of transnormal systems with \( t \)-regular foils. Transnormal systems with “foils” of higher codimension are investigated under the name of singular Riemannian foliations (see [1] and its references), which we do not treat here, but some of our results may hold in that case.

Now, we extend Nomizu’s theorem [23] on isoparametric functions in the space forms to a surprisingly stronger version:

**Theorem 1.3.** Let \( f \) be a transnormal function on a connected complete Riemannian manifold \( M \). Then a singular foil \( Q \) of \( f \), if exists, is austere and minimal.

Here by austere, we mean a submanifold of which shape operators have eigenvalues in pairs \( \pm \mu \) or 0. This is a generalization of the following recent result proved by Ge and Tang in the isoparametric case (see also Theorem D in [32]).
**Fact 1.** (See [11,12].) Let $f$ be an isoparametric function on a complete Riemannian manifold $M$. Then the following hold:

1. A focal submanifold is austere and minimal.
2. When $M$ is closed, there exists at least one minimal hypersurface as a level set.

The codimension condition in (1) is necessary, as we have counter-examples in Section 2(ii). We also give a new proof of (2) using the mean curvature flow in Section 6. In this paper, “closed” means compact without boundary.

Now, when does a transnormal function become an isoparametric function? An immediate consequence of the condition (II) is that isoparametric hypersurfaces have constant mean curvature (CMC for short), see Section 6. Is a transnormal function $f$ isoparametric if the level sets have CMC? This is not true in general as is shown in (i) of Section 2.

When $f$ is a transnormal function, let $S(f)$ be the set of singular foils, and put $V_+ = \{ x \in M \mid f(x) = \max f \}$, and $V_- = \{ x \in M \mid f(x) = \min f \}$, which are called the focal varieties (possibly disconnected, or empty). Q.M. Wang shows that $S(f) \subset V_+ \cup V_-$. When $F_f$ is non-singular and $M$ is closed, $S(f) = \emptyset \neq V_+ \cup V_-$ follows. Thus the equality does not necessarily hold.

**Theorem 1.4.** Let $f$ be a transnormal function on a complete connected Riemannian manifold $M$, which satisfies $S(f) = V_+ \cup V_-$. Then $f$ is an isoparametric function if and only if every foil has constant mean curvature.

Next, we ask when foils of a transnormal function have CMC. This has been discussed in the case of symmetric spaces of compact type ([28], p. 675 in [30]). Here, we consider the problem in the space forms, and prove Q.M. Wang’s Theorems B and C [32] in a correct statement:

**Theorem 1.5.**

1. Foils of a transnormal function $f$ on $E^n$ and $S^n$ have CMC and are isoparametric hypersurfaces.
2. Foils of a transnormal function $f$ on $H^n$ have CMC and are isoparametric hypersurfaces, if all the principal curvatures of some foil have absolute value not less than 1.
3. There exists a transnormal function on $H^n$ with foils which are not isoparametric hypersurfaces.

(1) and (2) do not mean that $f$ itself is an isoparametric function. (3) implies that in the hyperbolic space, there is an essential difference between transnormal functions and isoparametric functions. We give a reason for this fact in Remark 7.1.

The paper is organized as follows: In Section 2, examples of transnormal and isoparametric functions are given in order to overview our argument. Then we introduce Q.M. Wang’s and J. Bolton’s results in Section 3 which are essential to prove Theorem 1.1. In Section 4, we prove Theorem 1.2, and use such standard transnormal function thereafter. For the proof of Theorem 1.3, we need Jacobi fields, which will be discussed in Section 5. The proof of Theorem 1.4 is given in Section 6, and that of Theorem 1.5 is in Section 7.

2. **Examples of transnormal functions**

Examples of transnormal and isoparametric functions on $M = E^n$, $S^n$, $T^2$, $K^2$ and $\mathbb{R}P^n$, will give typical models of our argument.

(i) First, note that the level sets do not determine $f$, since for instance, both $f(x) = |x|^2$ and $g(x) = \cos |x|$, $x \in E^n$, have the same level sets; the round spheres. These two functions play an important role in Section 4. Obviously, $\nabla f = 2x$, $|\nabla f|^2 = 4f^2$, and $\Delta f = 2n$. Thus $f$ is an isoparametric function on $E^n$. Although $|x|$ is not differentiable at $x = 0$, $g(x) = \cos |x|$ is of class $C^0$ because of its Taylor expansion:

$$
\cos |x| = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} |x|^{2j}.
$$

(1)

Moreover from

$$
\nabla g(x) = \frac{\sin |x|}{|x|} x,
$$

(2)

we obtain $|\nabla g|^2 = 1 - g^2$, and $g$ is transnormal. However, $g$ is not an isoparametric function, since the second term of $\Delta g = -g + (1 - n\sin |x|/|x|)$ is not a function of $g$. Note that each level set of $f$ is connected, but that of $g$ has infinitely many connected components. For $g$, the points satisfying $|x| = n\pi$ belong to $V_- \cup V_+$, but the level set $g^{-1}(\pm 1)$ consists of hyperspheres and so are not singular foils except for the origin. Thus we have $S(g) \neq V_- \cup V_+$. Obviously, $S(f) = V_+ \cup V_-$ holds, where $V_+ = \emptyset$. To describe the transnormal system $F_f = F_{\overline{F}}$, a use of $f$ seems more natural.
(ii) Next we consider as $M$ a rotational torus $T^2$ given by 
\[ T^2 = \{(x, y, z) = \left((R + r \cos \theta) \cos \varphi, (R + r \cos \theta) \sin \varphi, r \sin \theta\right) \} \subset \mathbb{E}^3, \]
where $0 < r < R$. Consider a function $f : T^2 \to \mathbb{R}$ by restricting $F(x, y, z) = z$ to $T^2$. With respect to the induced metric on $T^2$, we have 
\[ |\nabla f|^2 = \frac{r^2 - f^2}{r^2}, \quad \Delta f = -\frac{f}{r}, \]
and $f$ is an isoparametric function. In this case, there are no focal submanifolds, i.e., $\mathcal{S}(f) = \emptyset$ but $V_\pm(f) \neq \emptyset$. Note that this $T^2$ is not a symmetric nor homogeneous manifold.

Concerning (2) of Theorem 11, $T^2$ is decomposed into two 1-disk bundles over top and bottom circles, which are critical set of $f$ but are not minimal. This reflects the fact that a focal variety of codimension one is not necessarily minimal, and the assumption in Theorem 13 is necessary. However, two geodesics, the outer and inner circles are minimal hypersurfaces, which is implied by (2) of Fact 1, although the uniqueness, which holds when Ricci $> 0$ [12], does not hold since $T^2$ has not positive Ricci curvature.

(iii) Consider a complete flat Möbius strip $M = [0, 1] \times \mathbb{R}/\sim$, where $(0, y)$ and $(1, -y)$ are identified. Let $f(x, y) = y^2$ be a function on $M$. It satisfies $\nabla f = (0, 2y)$ and $\Delta f = 2$, and hence is an isoparametric function on the flat $M$. The core circle $C = f^{-1}(0)$ is non-orientable in $M$, and for positive $t$, $f^{-1}(t)$ is a double cover of $C$, and is connected and orientable. Certainly, $M$ is a non-trivial $S^1$-bundle over $C$.

Similarly, on the flat Klein bottle $K^2 = [0, 1] \times [-1, 1]/\sim$ obtained by identifying $(x, -1)$ with $(x, 1)$ and $(-1, y)$ with $(1, -y)$, the above $f$ is an isoparametric function. Here, $K^2$ is a non-trivial $S^1$-bundle over $C$.

(iv) Let $\theta$ be the angle between a point $p \in S^2$ and the north pole of $S^2$. Then $f(p) = \cos \theta : S^2 \to [-1, 1]$ is the simplest isoparametric function which is the restriction of the linear function $F(x) = x_{e+1}$ to $S^2$. It satisfies $|\nabla f|^2 = \sin^2 \theta = 1 - f^2$, $\Delta f = -f$ with respect to the connection of $S^2$. The focal submanifolds are south and north poles $f^{-1}(\pm 1)$, which are minimal. Now, consider $h(p) = f^2(p) : S^2 \to [0, 1]$. Then $h$ satisfies
\[ |\nabla h|^2 = 4f^2|\nabla f|^2 = 4h(1 - h), \]
\[ \Delta h = 2(f \Delta f + |\nabla f|^2) = 2(1 - 2h), \]
and hence $h$ is also an isoparametric function. The focal varieties associated to $h$ are, in addition to points $h^{-1}(1)$, the equator $h^{-1}(0)$, which is also minimal (totally geodesic). In this way, isoparametric functions are not uniquely determined by isoparametric hypersurfaces.

(v) The function $h$ given above is an isoparametric function on $\mathbb{R}P^n$, the real projective space obtained by identifying the antipodal points. The projection $\pi : S^n \to \mathbb{R}P^n = S^n/\sim$ is a local isometry. Note that $h^{-1}(1)$ is a point, and $h^{-1}(0) = \mathbb{R}P^{n-1}$. Therefore, $h^{-1}(1)$ is the unique singular foil. Note further that $h^{-1}(t) = S^{n-1}(\sqrt{1-t}), t \in (0, 1)$, is an $S^0$ bundle (i.e., a double cover) of $h^{-1}(0) = \mathbb{R}P^{n-1}$. In more general, we have:

**Proposition 2.1.** Every isoparametric function on $\mathbb{R}P^n$ corresponds to an isoparametric function on $S^n$, and vice versa. In particular, every isoparametric hypersurface in $\mathbb{R}P^n$ has constant principal curvatures.

**Proof.** An isoparametric function $f$ on $S^n$ is essentially given by restricting a Cartan–Münzner polynomial $F$ to $S^n$, where $F$ is a homogeneous polynomial on $\mathbb{R}^{n+1}$ with degree $g$ satisfying two PDEs [20]. Here, $g$ is the number of distinct principal curvatures taking values in $\{1, 2, 3, 4, 6\}$. When $g$ is even, $f = F|_{S^2}$ descends to a function on $\mathbb{R}P^n$, and $f$ is isoparametric since $\pi$ is a local isometry. When $g = 1$, we have done. When $g = 3$, $h = f^2$ is well-defined on $\mathbb{R}P^n$ by $f(p) = -f(-p)$. This is isoparametric because $|\nabla f(\pm p)|^2 = |\nabla f(p)|^2$ implies that $|\nabla h|^2$ is a function of $h$, and $\Delta h$ is also a function of $h$ since $\Delta f = 0$ holds when $g = 3$ [4]. Because a lift of an isoparametric function on $\mathbb{R}P^n$ to $S^n$ is isoparametric as $\pi$ is a local isometry, all the isoparametric functions of $\mathbb{R}P^n$ come from those in $S^n$. For the same reason, the principal curvatures are constant as in the case of $S^n$ (see Section 7). \(\square\)

**Remark 2.2.** In general, an isoparametric hypersurface does not have constant principal curvatures [31], although they have constant mean curvature (see Section 6).

**Remark 2.3.** Among isoparametric hypersurfaces in $S^n$, there exist the so-called OT–FKM type hypersurfaces with $g = 4$ [24,10,19], obtained from each representation of Clifford algebras. They include infinitely many homogeneous and non-homogeneous isoparametric hypersurfaces. Hence the same is true for $\mathbb{R}P^n$.

### 3. Results of Q.M. Wang and J. Bolton

The following is fundamental:
Fact 3. (See [27].) Let $F$ be a transnormal system on $M$. For a “foil” $F$ of $F$, we denote by $e_F$ the normal exponential map $T^*F \to M$. Since a normal geodesic of $F$ cuts other “foils” orthogonally, we see that the distance between two “foils” of $F$ is constant by the Gauss lemma. We put $N_1F = \{ e_F(\xi) \mid \xi \in T^*F, \| \xi \| = 1 \}$, $N_{\leq s}F = \{ e_F(\xi) \mid \xi \in T^*F, \| \xi \| < 1, s < t \}$, and $N_{\leq s}F = N_{\leq s}F \cup N_F$. The conjugate locus of $F$ is the set of critical points of $e_F$ in $T^*F$. Note that when $F$ has a dense foil, the distance between two foils is not well-defined. However, including this case, J. Bolton proved:

Proof of Theorem 1.1. When $f$ is a transnormal function on $M$, each foil is $t$-regular. Then applying Fact 3 to $F_f$, we can see:

(1) If $F_f$ is non-singular, $F$ foliates $M$ and $M$ is a manifold with a bundle-like metric (see [25] for a bundle-like metric).

(II) When $F$ contains a singular “foil” $Q$, let $C(Q)$ be the image of the first conjugate locus of $Q$ under $e_Q$.

(a) When $C(Q)$ is empty, $Q$ is the only singular “foil” of $M$, and $M$ has the structure of a vector bundle over $Q$ (Theorem 2, [2]).

(b) When $C(Q)$ is non-empty, and

(i) if $C(Q) = Q$, holds, $F$ has only one singular “foil”, and normal geodesics of $Q$ return to $Q$ after travelling a distance $2a$. In this case, $M$ is diffeomorphic to $N_{\leq a}Q \cup Q' = N_{\leq a}Q$, where $Q' = N_{a}Q$ is a hypersurface;

(ii) if $C(Q) = Q' \neq Q$ and $\text{dist}(Q, Q') = 2a$, $Q' = N_{2a}Q$ is another singular “foil”, and $M$ is diffeomorphic to $N_{\leq a}Q \cup N_{\leq a}Q'$, or more generally, to $N_{\leq a}Q \cup N_{\leq a}Q' \cup N_{\leq a}Q$ for each $u \in (-a, a)$.

In all the cases, $N_{\leq a}Q$ (a $\to \infty$ in (II)(a)) and $N_{\leq a}Q'$ have bundle-like metrics. An irrational flow on a flat torus generates a non-singular transnormal system, which belongs to (I). In this case, $e_F: T^*F \to M$ is an $\mathcal{S}$ isomorphism in the sense of Bolton, but is not an open map.

Proof of Theorem 1.1. When $f$ is a transnormal function on $M$, each foil is $t$-regular. Then applying Fact 3 to $F_f$, we can see:

(1) If $F_f$ is non-singular, $M$ is an $\mathbb{R}$-bundle or an $S^1$ bundle over a foil $F$. For the latter, we can apply an argument similar to the proof of Lemma 3.2 below.

(II) If $F_f$ contains a singular foil $Q$, one of the following occurs:

(i) $M$ is diffeomorphic to a vector bundle over the unique singular foil $Q$.

(ii) $M$ is diffeomorphic to either

(a) a union of a disk bundle over the unique singular foil $Q$ and a hypersurface $Q'$ given by the boundary tube of the disk bundle, or

(b) a union of two disk bundles over two singular foils $Q$ and $Q'$.

Whenever $M$ is not diffeomorphic to a vector bundle, $M$ can be considered as a union of two disk bundles over $Q$ and $Q'$, where $Q$ and/or $Q'$ may possibly be of codimension one. \qed

Remark 3.1. An example having unique singular foil $Q$ with $C(Q) = Q$ is $\mathbb{R}^p$ in (v) of Section 2, where $Q$ is a point and $Q' = \mathbb{R}^{p-1}$. By this theorem, the number of focal submanifolds is at most two, but the number of principal curvatures can be more than two, as is the case $M = S^3$.

Note that the existence of a transnormal function on $M$ is related with not only the topology but also the differentiable structure of $M$ (see [11]).

For later use, we show:

Lemma 3.2. Let $f$ be a transnormal function on a closed manifold $M$. Then a geodesic $\gamma$ normal to the foils is a closed curve.

Proof. In this case, each level set of $f$ is $t$-regular. We may consider the singular case. Because $M$ is closed, $M$ is a union of two disk bundles over $Q$ and $Q'$ by Theorem 1.1. Put $\text{dist}(Q, Q') = b$ and let $\gamma$ be the normal geodesic of $Q$ at $p \in Q$. Then $\gamma$ cuts $Q$ so that the distance between adjacent points of $Q \cap \gamma$ is $2b$. If $\gamma$ is not closed, $Q \cap \gamma$ consists of infinitely many points, and there is a subsequence in $Q \cap \gamma$ which converges to some $q \in Q \cap \gamma$. However, then, an $\epsilon$-neighborhood of $q$ in $M$ has infinitely many connected components of $Q \cap \gamma$, which contradicts that $Q$ is a $t$-regular submanifold. \qed
4. Proof of Theorem 1.2

The topological structure of $M$ with transnormal system $F$ is clear by Fact 3, and now, when $F$ has $t$-regular foils, we construct a transnormal function $f$ such that $F = F_f$. Even if we start from a transnormal system associated with a transnormal function $g$, the new function $f$ is not necessarily equal to $g$. We call $f$ standard if each level set is connected.

Let $F$ be a transnormal system with $t$-regular “foils”. When there exists a singular “foil” $Q$ of $F$, we can identify $U = N_{\varepsilon} Q$ with $e_Q^{-1}(N_{\varepsilon} Q)$ in $T^2 Q$ for sufficiently small $\varepsilon > 0$. The following lemma is essential in the proof of Theorem 1.2.

Lemma 4.1. Let $U$ be as above, and define $g(x) = t(x)^2$, where $t(x)$ is given by $t(x)\xi_p = e_Q^{-1}(x), |\xi_p| = 1$. Then $g(x)$ is a $C^2$-function on $U$ satisfying $|\nabla g|^2 = 4g$. Also the function given by $f(x) = \cos(mt(x))$ ($m \in \mathbb{R}$) is of class $C^2$, and satisfies $|\nabla f(x)|^2 = 4m^2(1 - f(x)^2)$ on $U$.

Proof. We may show that $g(x)$ is of class $C^2$ along $Q$. It is well-known that for $p \in M$, $h(x) = d(x, p)^2$ is of class $C^\infty$ satisfying $|\nabla h(x)| = 2h(x)$ on $M \setminus C_p$, where $C_p$ is the cut locus of $p$ [26]. Since $t(x) = \text{dist}(x, Q)$, $g(x) = h(x)$ for $x = e_Q t(x)\xi_p$, and $g(x) = 0$ along $Q$. These imply that $g(x)$ is of class $C^\infty$, satisfying $|\nabla g|^2 = 4g$ on $U$. Also, since cost is a series of $t^2$ and the Taylor expansion converges uniformly, we obtain the lemma from $\nabla f(x) = -m\sin(mt(x))\nabla t(x)$. $\square$

Proof of Theorem 1.2. Note that when $F$ has $t$-regular “foils”, the conclusion of Theorem 1.1 holds, since $e_Q : T^2 Q \to M$ restricted to a small tubular neighborhood of $Q$ is a diffeomorphism onto its image.

(I) When $F$ is non-singular, $M$ is diffeomorphic to an $\mathbb{R}$ or $S^1$ bundle over some “foil” $F$. In the trivial $\mathbb{R}$-bundle case, taking a unit normal vector field $\xi$ along $F$, we can define $f : M \to \mathbb{R}$ by

$$f(x) = t(x), \quad e_F^{-1}(x) = t(x)\xi_p \in T^2 F.$$  \hspace{1cm} (4)

Then $f$ is a smooth transnormal function with $|\nabla f|^2 = 1$. When the $\mathbb{R}$-bundle is non-trivial (see [25, Corollary 4]), taking $f(x) = x^2$ where $t(x)$ is locally defined by (4), we obtain a global function $f(x)$, which satisfies $|\nabla f|^2 = 4f$, and $f(x)$ is a transnormal function on $M$. In the $S^1$-bundle case, let $l$ be the length of a geodesic normal to the foils. Then $e_F$ covers the same point of $M$ for $t \in \mathbb{R}$ modulo $l$. Thus

$$f(x) = \cos\left(\frac{2\pi t(x)}{l}\right), \quad e_N t(x)\xi_p = x,$$  \hspace{1cm} (5)

is well-defined on $M$, and $f$ is a transnormal function on $M$ by Lemma 4.1. The non-trivial $S^1$ bundle case is also settled by (5), as $\cos$ is an even function.

(II) Singular case: (i) When $F$ contains a singular foil $Q$ and $M$ is a vector bundle over $Q$, we can apply Lemma 4.1 to $M$, and

$$f(x) = |t(x)|^2, \quad t(x) = \text{dist}(x, Q), \quad x \in M,$$

is a transnormal function with $|\nabla f|^2 = 4f$.

(ii) In this case, the normal geodesic $\gamma$ of $Q$ starting from a point $q \in Q$, comes back to (possibly a different point of) $Q$ in a constant distance $t = 2a > 0$, and we may put $\varepsilon = 2a$ in Lemma 4.1. Thus putting $m = \pi/a$, we define $f$ by

$$f(x) = \cos(mt(x)) : M \to [-1, 1], \quad x = \exp_p t(x)\xi_p,$$

which is a transnormal function by Lemma 4.1.

(b) In this case, if we put $\text{dist}(Q, Q') = 2a$, the distance between adjacent points on $Q \cap \gamma$ is $4a$. Obviously, $t(x) = \text{dist}(x, Q) = 2a - t'(x)$ holds where $t'(x) = \text{dist}(x, Q')$. Now, for $m = 2\pi/4a = \pi/2a$, define $f : M \to \mathbb{R}$ by

$$f(x) = \cos(mt(x)), \quad t(x) = \text{dist}(x, Q) \in [0, 2a].$$

Then $f(x)$ is of class $C^2$ on $M \setminus Q$. On the other hand, since

$$f(x) = \cos(m(2a - t'(x))) = \cos(\pi - mt'(x)) = -\cos(mt'(x)),$$

the right hand side is of class $C^2$ on $M \setminus Q$. Thus it is easy to see that $f(x)$ is a transnormal function on $M$. $\square$

Remark 4.2. By our construction, $f$ is standard in the sense that each level set is connected. Note that we use $t$-regularity as we use Lemmas 3.2 and 4.1.
5. Jacobi fields and shape operators

Let \( M \) be a complete connected Riemannian manifold. Let \( \gamma(t) \) be a geodesic of \( M \) with \( \gamma(0) = p \), and denote \( \xi = \dot{\gamma}(t) \) and \( \xi = \dot{\gamma}(0) \). A Jacobi field \( J(t) \) along \( \gamma(t) \) is a solution to
\[
\nabla^2 J(t) + R(J, \dot{\gamma}) \dot{\gamma}(t) = 0
\]
where \( \nabla = \nabla_p \) and \( R \) is the curvature operator of \( M \). This is the second order linear differential equation of \( J(t) \), and the solution is determined by two initial data.

When \( N \) is a submanifold of \( M \) and \( \gamma(t) \) is a normal geodesic of \( N \) at \( p = \gamma(0) \in N \), a Jacobi field \( J(t) \) along \( \gamma(t) \) is called an \( N \)-Jacobi field if it satisfies
\[
\nabla J(t) - A_t J(t) \in T_p^\perp N, \quad J(0) = \gamma(0) \in T_p N,
\]
where \( A_t = -\nabla J(t) \xi \) is the shape operator of \( N \). The initial data (6) determines the \( N \)-Jacobi field \( J(t) \) uniquely. The following is well-known:

**Fact 4.** (See Lemma 4.6 in II of [26].) Let \( J(t) \) be a vector field along a geodesic \( \gamma : [0, b] \rightarrow M \) with \( \gamma(0) = p \in N \) and \( \dot{\gamma}(0) = \dot{s} / \nabla, l e t \gamma(t) = \gamma(0) + \int_0^t \dot{\gamma}(s) ds \). Let \( J(t) \) be an \( N \)-Jacobi field along \( \gamma(t) \). Then \( J(t) \) is an \( N \)-Jacobi field along \( \gamma(t) \) if and only if there exists a \( C^\infty \) variation of \( \gamma \) given by
\[
\alpha : (-\epsilon, \epsilon) \times [0, b] \rightarrow M, \quad \alpha(t, s) = \gamma(t),
\]
where for each \( s \in (-\epsilon, \epsilon) \), \( \alpha_1(t, s) = \alpha(t, s) \) is a geodesic orthogonal to \( N \) at \( t = 0 \), and \( J(t) \) is given by \( \frac{\partial \alpha}{\partial t}(t, s) \bigg|_{s=0} \).

For a foil \( N = N_0 \) of a transnormal function \( f \), let \( \gamma = f \) be the normal geodesic of \( N_0 \) through \( \gamma(0) = p \in N_0 \) with the arclength parameter. In this section, we denote by \( N \) the level set of \( f \) through \( \gamma(t) \), and we denote \( \alpha \) as \( \alpha f \). Then \( J(t) \) is an \( N \)-Jacobi field along \( \gamma(t) \). Let \( J(t) \) be an \( N \)-Jacobi field along \( \gamma(t) \). Then \( J(t) \) is an \( N \)-Jacobi field for every \( t \) such that \( J(t) \neq 0 \).

**Proof.** Since \( F_F \) is a transnormal system, a geodesic of \( M \) is orthogonal to each \( N_t \), if it is orthogonal to \( N \). Then \( J(t) \in T_{\gamma(t)} N_t \) follows. Now, let \( Y(t) \) be a parallel vector field along \( \gamma(t) \) such that \( Y = Y(0) \in T_p N \). Then \( Y(t) \) is tangent to each \( N_t \), and moreover from (6), it follows that
\[
\left\langle \nabla J(0) - A_t J(0), Y(t) \right\rangle = 0.
\]

On the other hand, we have
\[
\frac{d}{dt} \left\langle \nabla J(t) - A_t J(t), Y(t) \right\rangle = \nabla^2 J(t) + \nabla^2 J(t) \dot{\gamma}(t), Y(t) \bigg|_{s=0} = \left\langle \nabla^2 J(t) + \nabla^2 J(t) \dot{\gamma}(t), Y(t) \bigg|_{s=0} = 0,
\]

since \( J(t) \) is a Jacobi field and \( \alpha_1(t, s) \) is a geodesic. Therefore, (7) implies
\[
\left\langle \nabla J(t) - A_t J(t), Y(t) \right\rangle = 0
\]
for all \( Y(t) \) tangent to \( N_t \), and hence \( J(t) \) is an \( N \)-Jacobi field for \( t \), \( J(t) \neq 0 \).

**Definition 4.** A point \( c = \gamma(t_0), t_0 \neq 0 \), is called a focal point of \( N \) when there exists a non-trivial \( N \)-Jacobi field along \( \gamma(t) \) such that \( J(t_0) = 0 \).

A focal point of \( N \) is a critical value of \( e_N \) since \( d(e_N) J(0) = 0 \).

**Proof of Theorem 1.3.** Shifting a parameter, we may consider that \( c = \gamma(0) \) belongs to a singular foil \( Q \). Let \( t \in (0, \varepsilon) \) be so that the first focal points of \( N_t = N_{\varepsilon-t} \) lie in \( Q \). \( N_t \) is the foil through \( \gamma(t) \). For some fixed \( t \in (0, \varepsilon) \), take a normal geodesic \( \gamma \) of \( N_t \) at \( p \in N_t \). Since \( c \) is the focal point of \( N_t \), there exist independent \( N_t \)-Jacobi fields \( J_1(t), \ldots, J_m(t) \) which are tangent to \( N_t \) and vanish at \( c \), where \( m \) is the multiplicity of the focal point. Let \( Y_1, \ldots, Y_k \in T_p N_t \), \( k = n - 1 - m \), be orthogonal to \( J_1(t), \ldots, J_m(t) \), which generating a basis of \( T_p N_t \) with \( J_1(t), \ldots, J_m(t) \). Let \( J_{m+1}(t), \ldots, J_{n-1}(t) \) be the \( N_t \)-Jacobi fields such that \( J_{m+1}(t) = Y_1 \). Then \( Proposition 5.1 \) implies
\[

abla J_{m+1}(t) \equiv A_t J_{m+1}(t), \mod T^\perp N_t,
\]

for each \( t \neq 0 \), where \( A_t = A_0 \). Here, \( f_{m+1}(t), \ldots, f_{n-1}(t) \) never vanish and are independent for \( |t| \in [0, \varepsilon) \), because the rank of the normal exponential map is constant for \( |t| \in (0, \varepsilon) \). If we put
\[
a_{ij}(t) = \langle \nabla f J(t), J(t) \rangle, \quad 1 \leq i, j \leq n - 1,
\]
then these determine the shape operator \( A_t \) completely at all \( |t| \in (0, \varepsilon) \). Note that as \( t \to 0 \) tends to 0, \( a_{ij}(t) \) \((m + 1 \leq i, j \leq n - 1)\) determine the shape operator \( B_t \) of \( Q \), operating on \( T_t Q = \text{span} \{ f_{m+1}(t), \ldots, f_{n-1}(t) \} \). On the other hand, as \( t \to 0 \) tends to 0, \( a_{ij}(t) \) \((m + 1 \leq i, j \leq n - 1)\) determine \(-B_t\), because the unit normal vector field of \( N_t \) continuously defined along the fiber sphere \( S_t \) of the tube \( N_t \) is given by \(-\xi_t \) at \( \gamma(-t) \), \( t > 0 \), namely, outward to the tube \( N_t \). Since \( a_{ij}(t) \) is continuous at \( t = 0 \), this means that the eigenvalues of \( B_t \) and \(-B_t\) coincide in total, namely, the eigenvalues of \( B_t \) consist in pairs \( \pm \mu_t \), or 0. Therefore, \( Q \) is austere and is minimal. \( \square \)

**Remark 5.2.** We do not need an explicit eigenvalues nor symmetry of \( M \), which is used in the case of \( M = S^n \) [14,17]. The importance is the connectedness of the fiber sphere \( S_t \). We cannot apply the same argument to a codimension one foil in \( V_L \).

**Remark 5.3.** The tubes of \( Q \) do not necessarily have constant mean curvature nor constant principal curvatures.

### 6. Proof of Theorem 1.4

In a general Riemannian manifold \( M \), isoparametric hypersurfaces have not necessarily constant principal curvatures [31]. However, by the following well-known fact, they have constant mean curvature.

**Fact 5.** (See [18.]) When \( f \) is a \( C^2 \) function on a Riemannian manifold \( M \), a level set \( N_t = f^{-1}(t) \) of a regular value \( t \) has the mean curvature \( H(t) \)
\[
(n - 1)H(t) = \frac{\nabla f(\nabla f) - |\nabla f|^2}{|\nabla f|^2}.
\]

When \( f \) is an isoparametric function, the condition \( (\|) \) implies that \( \Delta f \) is constant on \( N_t \), and so is
\[
\nabla f(\nabla f) = \frac{b'(f)}{2\sqrt{b(f)}} \nabla f(f) = \frac{b'(f)\sqrt{b(f)}}{2}.
\]

where \( b'(f) \) means the differential w.r.t. the variable of \( b \). Thus we obtain
\[
(n - 1)H(t) = \frac{b'(f) - 2\alpha(f)}{2\sqrt{b(f)}},
\]
and \( N_t \) has constant mean curvature.

**Proof of Theorem 1.4.** When a transnormal function \( f \) satisfies the assumption of the theorem, each level set is connected since it is a sphere bundles over a connected submanifold with connected fiber sphere [11, Proposition 2.1]. Here we denote \( F_t = f^{-1}(t) \). If each \( F_t \) has constant mean curvature, then the mean curvature is a continuous function of \( t \). On the other hand, it follows from Lemma 6 in [32], that the eigenvalues of the Hess \( f \) of \( f \) on \( V_- (V_+, \text{resp.}) \) are zeros or \( \frac{1}{2}b'(\alpha) \left( \frac{1}{2}b'(\beta) \right. \), resp.) with multiplicities being the dimension and codimension of \( V_- (V_+, \text{resp.}) \), respectively. Thus we obtain
\[
\lim_{t \to 0} \Delta f = \frac{1}{2}b'(\alpha) \cdot \text{codim} V_- \quad \left( \lim_{t \to 0} \Delta f = \frac{1}{2}b'(\alpha) \cdot \text{codim} V_+ \text{ resp.} \right),
\]
which is a finite number. Since \( f \) is of class \( C^2 \) on \( M \), \( \Delta f \) is continuous, and the above value coincides with the value of \( \Delta f \) on \( V_- (V_+, \text{resp.}) \). Therefore, \( \Delta f \) is a continuous function of \( f \), namely, \( f \) is an isoparametric function. \( \square \)

**Remark 6.1.** The condition \( S(f) = V_+ \cup V_- \) is necessary, since the transnormal function \( g \) in Section 2(ii) with CMC foils is not isoparametric.

By the way, we give a proof of (2) of Fact 1 by using the mean curvature flow, namely, the flow with a variation vector field given by the mean curvature vector field \( H \). Since \( H \) is constant in the isoparametric case, the family of isoparametric hypersurfaces gives the solution of the mean curvature flow, with a suitable change of \( t \), if necessary. The first variation formula of the volume is given by
\[
\frac{d}{dt} \operatorname{vol}(F_t) = - \int_{F_t} |H(t)|^2 \, dv_t.
\]
Note that this holds for each $F_i$. When $M$ is closed, we have $f(M) = [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$, and $V_+ \neq \emptyset$. If $S(f) \neq V_+ \cup V_-$, $S(f) = \emptyset$, or one of $Q$ and $Q'$ is of codimension one. In the former case, the volume of a level set attains its minimum and/or maximum for some $t \in [\alpha, \beta]$, which implies $H(t) = 0$. In the latter case, for instance, when $Q = f^{-1}(\alpha)$ is singular, any level sets other than $Q$ are hypersurfaces of which volume $V(t)$ is a continuous function of $t$ and vanishes at $t = \alpha$. Thus there exists a critical point at which we have again $H \equiv 0$. If $S(f) = V_+ \cup V_-$ holds, we have two level sets $Q = f^{-1}(\alpha)$ and $Q' = f^{-1}(\beta)$ of which $(n - 1)$ volumes are 0. Thus $\text{vol}(F_i)$ tends to 0 as $t$ goes to $\alpha$ and $\beta$, and so $\text{vol}(F_i)$ cannot be monotone on $[\alpha, \beta]$, and takes a critical value at some $t$, where $H(t) \equiv 0$. \qed

7. Proof of Theorem 1.5

In this section, we treat the space forms.

Fact 6. (See É. Cartan [4].) Let $M$ be a space form $(E^n, S^n$ or $H^n$), and consider a family of parallel hypersurfaces $\{N_i\}$. Then the following are equivalent:

(i) $\{N_i\}$ is a family of isoparametric hypersurfaces.

(ii) All $N_i$ have constant mean curvatures.

(iii) One of $N_i$ has constant principal curvatures.

Although (i) is a global notion, (iii) is a local notion. Hence the implication from (iii) to (i) is non-trivial, which is, in our context, suggested by Theorem 1.2.

The following is well-known [7]: Let $N$ be a hypersurface of $M = E^n, S^n$, $H^n$, and for a (local) unit normal vector $\xi$, consider a map $\phi_i(p) = e_N(t\xi_p)$ for $t \in \mathbb{R}$. If $x = e_N(t\xi_p) \in M$ is a focal point of $N$ at $p$, where

$$
\lambda(t) = \begin{cases} 
\frac{1}{t}, & M = E^n, \\
\cot t, & M = S^n, \\
\coth t, & M = H^n.
\end{cases}
$$

In fact, since $\phi_i$ is given, respectively, by

$$
\phi_i(p) = \begin{cases} 
p + t\xi_p, & E^n, \\
cost p + \sin t\xi_p, & S^n, \\
\cosh t p + \sinh t\xi_p, & H^n,
\end{cases}
$$

when $X$ is a principal vector of $N$ such that $A_\xi X = \lambda X$, the Jacobi operator is given by

$$
J\phi_i(X) = \begin{cases} 
(1 - t\lambda)X, & E^n, \\
(\sin t)(\cot t - \lambda)X, & S^n, \\
(\sinh t)(\coth t - \lambda)X, & H^n,
\end{cases}
$$

which becomes zero when $\lambda = \lambda(t)$. The converse is easily shown.

Remark 7.1. The range of $1/t$ or $\cot t$ is the whole $\mathbb{R}$, where we allow $t = \pm\infty$. On the other hand, the range of $\coth t$ is $(-\infty, -1) \cup (1, \infty)$. Thus when $M = H^n$, a principal curvature $\lambda$ with $|\lambda| \leq 1$ does not correspond to any focal point of $N$. This causes a difference between the case $H^n$ and the cases $E^n, S^n$.

Proof of Theorem 1.5. (1) We may put $f(M) = [\alpha, \beta]$ allowing $\alpha = -\infty$ and/or $\beta = \infty$, which means $V_- = \emptyset$ and/or $V_+ = \emptyset$, respectively.

Consider a component $F$ of a level $f^{-1}(c)$, $c \in (\alpha, \beta)$. For $p \in F$, let $\gamma$ be a normal geodesic of $F$ through $p$. Then we can take $q_1, \ldots, q_g \in y \cap V_{\pm}$ so that $q_i$ is the focal point corresponding to the principal curvature $\lambda_i$, where we allow $q_i = \infty$ for $\lambda_i = 0$ in the Euclidean case. Then $1 \leq g \leq n - 1$ holds where $g$ is the number of distinct principal curvatures at $p$. Let $Q_i \in S(f) \subset V_{\pm}$ be the component on which $q_i$ lies. Fact 3 implies that $F$ is a tube over $Q_i$ with constant radius equal to the distance from $p$ to $q_i$ along $\gamma$ (not necessarily equal to $\text{dist}(F, Q_i)$). Since the rank of the focal map is constant, the multiplicity of $\lambda_i$ is constant. Even when one of $\lambda_1, \ldots, \lambda_g$, say $\lambda_g = 0$ in the Euclidean case, $\lambda_g$ should be identically zero on $F$ since otherwise a new focal point appears, which is impossible. Thus all the principal curvatures are constant with constant multiplicities, and $F$ is an isoparametric hypersurface by Fact 6(iii).

(2) If $F \in \mathcal{F}_f$ has principal curvatures $\lambda_i$ with $|\lambda_i| > 1$, each $\lambda_i$ corresponds to a focal point on the normal geodesic, and so we can apply the argument in (1) to conclude that $F$ is isoparametric. Even if we weaken the condition to $|\lambda_i| \geq 1$, a principal curvature $\lambda_i = \pm 1$ should be constant on $F$ since otherwise a new focal point appears in $H^n$, which is impossible. Thus $F$ is also isoparametric in this case.
Take a totally geodesic $H^{n-1}$, then all the principal curvatures are 0. Let $F$ be a hypersurface deformed slightly from $H^{n-1}$ so that the principal curvatures $\lambda_i$, $1 \leq i \leq n-1$ depend on points keeping $|\lambda_i| < 1$ (see Fig. 1, by S. Fujimori). Then $\lambda_i$ is written as $\tanh \theta_i(p)$, $p \in F$, and a hypersurface $F_t = \phi_t(F)$, where $\phi_t(p) = t_\theta \xi(p)$ and $\xi$ is a unit normal, is defined for all $t \in \mathbb{R}$, as $F$ has no focal points. Then $\mathcal{F} = [F_t]$ is a non-singular transnormal system with $t$-regular “foils”, and by Theorem 1.2, the function $f(x) = t(x)$, $x = \phi_t(x) \in H^n$ is a transnormal function on $H^n$. However, the level sets $F_t$ are not isoparametric hypersurfaces because $\lambda_i$ and so the mean curvature depends on points. □

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References