Remarks on the Dorfmeister-Neher theorem on isoparametric hypersurfaces

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Acknowledgment. The author cordially appreciate U. Abresch and A. Sifert’s questions and indication.

1 Introduction

§7 and §8 of [M] are the heart of the paper, but a lack of clear argument causes some questions, although the statement is true. The purpose of the present paper is to make it clear. We follow the notation and the argument in [M].

2 Dim $E = 2$ (§7 [M])

We call a vector field $v(t)$ along $L_6$ parametrized by $p(t)$, even when $v(t + \pi) = v(t)$, and odd when $v(t + \pi) = -v(t)$. Note that $E$ consists of $\nabla^k e_6(t)$, $k = 0, 1, \ldots$ which are odd or even at the same time, and $W$ consists of $\nabla^k e_3 e_6(t)$ of which evenness and oddness is the opossite of $E$, since $L(t + \pi) = -L(t)$.

Proposition 7.1 [M] dim $E = 2$ does not occur at any point of $M_+$. 

Proof: dim $E = 2$ implies dim $W = 1$, and so $W$ consists of even vector ($\nabla e_6 e_6$ never vanish by Remark 5.3 of [M]). Thus $E$ consists of odd vectors. For $X_1, Z_1, X_2, Z_2$ in p.709, $X_1$ is parallel to $\nabla e_6 e_3$ at $p_0 = p(0)$ and $p(\pi)$, and so has opposite sign at $p(0)$ and $p(\pi)$. Note that $Z_1 \in W$ is a constant unit vector parallel to $\nabla e_3 e_6(t)$. Also, span$\{X_2, Z_2\}$ is parallel since this is the orthogonal complement of $E \oplus W$. Because $D_1(\pi) = D_3(0)$ and
$D_2(\pi) = D_4(0)$ etc. hold, four cases occur:

\[
(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0), \\
(e_1 + e_5)(\pi) = (e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0), \\
(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = (e_2 + e_4)(0), \\
(e_1 + e_5)(\pi) = -(e_1 + e_5)(0) \quad \text{and} \quad (e_2 + e_4)(\pi) = -(e_2 + e_4)(0).
\]

In the first case, $\alpha(\pi) = -\alpha(0)$ and $\beta(\pi) = -\beta(0)$ follow. Then $X_2$ becomes even and $Z_2$ becomes odd, contradicts that $\text{span}\{X_2, Z_2\}$, is parallel. In the second case, $\alpha(\pi) = -\alpha(0)$ and $\beta(\pi) = \beta(0)$ hold, and so $X_2$ is odd, and $Z_2$ is even, again a contradiction. Other cases are similar.

\section{Dim $E = 3$ (§8 [M])}

When $\dim E = 3$, $e_3(t)$ is an even vector, since $E$ is parallel along $L_6$. Using Proposition 8.1 [M], we extend $e_1, e_2, e_4, e_5$ as follows: Taking the double cover $\tilde{c}(t)$ of $c(t)$, i.e., $t \in [0, 4\pi)$, if necessary, we choose a differentiable frame $e_i(t)$ as follows: First take $e_1(t), e_2(t)$ continuously for $t \in [0, 4\pi)$. Then we define $e_5(t) = e_1(t + \pi)$ and $e_4(t) = e_2(t + \pi)$ for $t \in [0, 3\pi)$. Thus we have a differentiable frame $e_i(t)$ for $t \in [0, 3\pi)$, though we only need $t \in [0, 2\pi)$.

With respect to this frame, we can take a differentiable orthonormal frame of $E$ and $E^\perp$ by

\[
e_3(t), \quad X_1 = \alpha(t)(e_1 + e_5)(t) + \beta(t)(e_2 + e_4)(t) \\
X_2(t) = \frac{1}{\sigma(t)} \left( \frac{\beta(t)}{\sqrt{3}} (e_1 - e_5)(t) - \sqrt{3}\alpha(t)(e_2 - e_4)(t) \right)
\]

and

\[
Z_1(t) = \frac{1}{\sigma(t)} \left( \sqrt{3}\alpha(t)(e_1 - e_5)(t) + \frac{\beta(t)}{\sqrt{3}} (e_2 - e_4)(t) \right) \\
Z_2(t) = \beta(t)(e_1 + e_5) - \alpha(t)(e_2 + e_4)(t),
\]

where $\alpha(t), \beta(t), \sigma(t)$ are differentiable for $t \in [0, 3\pi]$, satisfying

\[
\alpha^2(t) + \beta^2(t) = 1/2, \quad \sigma(t) = 2(3\alpha^2(t) + \beta^2(t)/3).
\]

Note that $\sigma(t) = \sigma(t + \pi)$ holds, since $\sigma(t)$ is an eigenvalue of $T(t) = tRR(t)$ (see (45) [M] and the statement after it).

Proposition 8.2 [M] $\sigma(t)$ is constant and takes values 1/3 or 3.

Remark. We need not specialize the case $\sigma = 1$ in the proof.
Proof: From (3), the conclusion follows if we show \( \alpha(t)\beta(t) \equiv 0 \). Suppose \( \alpha(t)\beta(t) \neq 0 \). By definition, we have
\[
eq 1 = e_5(0), \quad e_2(\pi) = e_4(0). \tag{4}
\]
We must be careful for
\[
eq 1(2\pi) = \epsilon_1 e_1(0), \quad e_4(\pi) = e_2(2\pi) = e_2 e_2(0),
\]
where \( \epsilon_i = \pm 1 \). However, since \( e_3 \) is even and by (4), we obtain
\[
eq := \epsilon_1 = \epsilon_2.
\]

**[Case 1]** \( \epsilon = 1 \). In this case, we have
\[
X_1(\pi) = \frac{1}{\sqrt{\sigma(\pi)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_1(\pi) - e_5(\pi)) - \sqrt{3} \alpha(\pi) (e_2(\pi) - e_4(\pi)) \right)
\]
which belongs to \( E \), and is orthogonal to \( e_3(0) \) and \( X_2(0) \). Thus we obtain
\[
X_1(\pi) = \epsilon X_1(0), \quad \text{namely, } \alpha(\pi) = \bar{\epsilon} \alpha(0), \beta(\pi) = \bar{\epsilon} \beta(0), \tag{6}
\]
\( \bar{\epsilon} = \pm 1 \). On the other hand, we have
\[
X_2(\pi) = -\bar{\epsilon} X_2(0).
\]
However, because \( E \) is parallel, \( X_1 \) and \( X_2 \) should be even or odd at the same time, a contradiction.

**[Case 2]** \( \epsilon = -1 \). In this case, we have
\[
X_1(\pi) = \frac{1}{\sqrt{\sigma(\pi)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_1(\pi) - e_5(\pi)) - \sqrt{3} \alpha(\pi) (e_2(\pi) - e_4(\pi)) \right)
\]
which belongs to \( E \), and is orthogonal to \( e_3(0) \) and \( X_1(0) \). Thus we obtain
\[
X_1(\pi) = \bar{\epsilon} X_2(0), \quad \text{namely, } \alpha(\pi) = -\bar{\epsilon} \frac{\beta(0)}{\sqrt{3} \sigma(0)}, \quad \text{and } \beta(\pi) = \bar{\epsilon} \frac{\sqrt{3} \alpha(0)}{\sqrt{\sigma(0)}}, \tag{9}
\]

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for $\bar{c} = \pm 1$. On the other hand, we see that

$$X_2(\pi) = \frac{1}{\sqrt{\sigma(\pi)}} \left( \frac{\beta(\pi)}{\sqrt{3}} (e_5(\pi) - e_3(\pi)) - \sqrt{3} \alpha(\pi)(e_2(\pi) - e_4(\pi)) \right)$$

where we use $\sigma(\pi) = \sigma(0)$. Because it belongs to $E$ and is orthogonal to $e_3(0)$ and $X_2(0)$, and further because $(X_1(0), X_2(0)) \rightarrow (X_1(\pi), X_2(\pi))$ should be orientation preserving, we obtain,

$$X_2(\pi) = -\bar{c} X_1(0), \text{ namely, } \frac{\beta(\pi)}{\sqrt{3}\sigma(0)} = -\bar{c} \alpha(0) \text{ and } -\sqrt{3} \alpha(\pi) \sqrt{\sigma(0)} = -\bar{c} \beta(0).$$

(11)

However, then (9) and (11) have no solution.

These contradiction is caused by the assumption $\alpha(t) \beta(t) \neq 0$. Thus $\alpha(t) \beta(t) \equiv 0$ follows. Now, by the argument in §9 [M], we obtain

**Theorem 3.1** [DN], [M] Isoparametric hypersurfaces with $(g, m) = (6, 1)$ are homogeneous.

**References**


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