On the Hasse Principle for the Brauer group of a purely transcendental extension field in one variable over an arbitrary field

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Abstract

In this paper we show the Hasse principle for the Brauer group of a purely transcendental extension field in one variable over an arbitrary field.

1 Introduction

For a field k, let k_s be the separable closure of k and $\bar{k}$ the algebraic closure of k. Let K be a global field (i.e., an algebraic number field or an algebraic function field of transcendental degree one over a finite field), S the set of all primes of K and $\widehat{K}_{\mathfrak{p}}$ the completion of K at $\mathfrak{p} \in S$. For a ring A, let $\mathrm{Br}(A)$ be the Brauer group of A (see [6, p.141, IV, §2]). Then, the local-global map

$$\mathrm{Br}(K) \to \prod_{\mathfrak{p} \in S} \mathrm{Br}(\widehat{K}_{\mathfrak{p}})$$

is injective (see [5, Theorem 8.42 (2)]). We call a statement of this form the Hasse principle. It is also known that the Hasse principle holds if K is a purely transcendental extension field in one variable over a perfect field k (see [8]). We show that it also holds without any assumption on k. The following is our main theorem.

Theorem 3.5. Let k be an arbitrary field, $k(t)$ the purely transcendental extension field in one variable t over k and $\widehat{k(t)}_{\mathfrak{p}}$ the quotient field of the completion of $\mathcal{O}_{\mathbb{P}^1_k, \mathfrak{p}}$. Then, the local-global map

$$\mathrm{Br}(k(t)) \to \prod_{\substack{\mathfrak{p} \in \mathbb{P}^1_k \\ \mathrm{ht}(\mathfrak{p})=1}} \mathrm{Br}(\widehat{k(t)}_{\mathfrak{p}})$$

is injective.

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Moreover, if k is a separably closed field, the Hasse principle for the Brauer group of any algebraic function fields in one variable over k is shown by using [2, Corollaire (5.8)] as in the case of Theorem 3.5.

For the defference between the case of perfect fields and Theorem 3.5, see Remark 3.7.

2 Notations

For a field k and a Galois extension field k' of k, $G(k'/k)$ denotes the Galois group of k'/k and k_s denotes the separable closure of k. We denote $G(k_s/k)$ by G_k and the category of (discrete) G_k-modules (cf, [7, p.10, I]) by G_k-mod. For a discrete $G(k'/k)$-module A (but the action is continuous) and a positive integer q, $\mathrm{H}^q(k'/k, A)$ denotes the q-th cohomology group of $G(k'/k)$ with coefficients in A (see [7, p.10, I, §2]). We put $\mathrm{H}^q(k, A) = \mathrm{H}^q(k_s/k, A)$. $\mathrm{Res} : \mathrm{H}^p(k, A) \to \mathrm{H}^p(k', A)$ denotes the restriction homomorphism. For a group G, we put $G_q = \{g \in G \mid g^q = 1\}$ and $X(G)$ the group of characters of G.

For a scheme X, $X^{(i)}$ is the set of points of codimension i and $X_{(i)}$ is the set of points of dimension i. We denote the étale site (resp. finite étale site) on X by X_{et} (resp. X_{fet}) and the category of sheaves over X_{et} (resp. X_{fet}) by $\mathbb{S}_{X_{et}}$ (resp. $\mathbb{S}_{X_{fet}}$). For $\mathcal{F} \in \mathbb{S}_{X_{et}}$ (resp. $\mathbb{S}_{X_{fet}}$), we denote the q-th cohomology group of X_{et} (X_{fet}) with values in $\mathcal{F}$ by $\mathrm{H}^q_{et}(X, \mathcal{F})$ or even simply $\mathrm{H}^q(X, \mathcal{F})$ (resp. $\mathrm{H}^q_{fet}(X, \mathcal{F})$). If $Y \subset X$ is a closed subscheme, we denote the q-th local (étale) cohomology with support in Y by $\mathrm{H}^q_Y(X, \mathcal{F})$. For an integral scheme X and $\mathfrak{p} \in X^{(1)}$, let $R(X)$ be the function field of X, $\mathcal{O}_{X,\mathfrak{p}}$ the local ring at $\mathfrak{p}$ of X, $\widehat{\mathcal{O}}_{X,\mathfrak{p}}$ the completion of $\mathcal{O}_{X,\mathfrak{p}}$, $\widehat{R(X)}_{\mathfrak{p}}$ its quotient field , $\widetilde{\mathcal{O}}_{X,\mathfrak{p}}$ the Henselization of $\mathcal{O}_{X,\mathfrak{p}}$, $\widetilde{R(X)}_{\mathfrak{p}}$ its quotient field , $\mathcal{O}_{X,\bar{\mathfrak{p}}}$ the strictly Henselization of $\mathcal{O}_{X,\mathfrak{p}}$ and $R(X)_{\bar{\mathfrak{p}}}$ its quotient field.

3 Main theorem

Theorem 3.1. Let X be a 1-dimensional connected regular scheme, K its quotient field. Then

$$0 \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(K) \longrightarrow \prod_{\mathfrak{p} \in X^{(1)}} \mathrm{Br}(\widetilde{R(X)}_{\mathfrak{p}}) / \mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}) \tag{1}$$

is exact.

Proof. Suppose that B is a discrete valuation ring, L is its quotient field, $Y = \operatorname{Spec} B$ and $Z = Y \setminus \operatorname{Spec} L = \{\mathfrak{p}\}$. Then we have the exact sequence

$$\mathrm{H}^p(Y, \mathbb{G}_m) \to \mathrm{H}^p(\operatorname{Spec} L, \mathbb{G}_m) \to \mathrm{H}^{p+1}_Z(Y, \mathbb{G}_m) \tag{2}$$

by [6, p.92, III, Proposition 1.25] and $\mathrm{H}^2(Y, \mathbb{G}_m) \to \mathrm{H}^2(\operatorname{Spec} L, \mathbb{G}_m)$ is injective by [6, p.145, IV, §2]. Moreover we have

$$\mathrm{H}^p_Z(Y, \mathbb{G}_m) \simeq \mathrm{H}^p_{\{\mathfrak{p}\}}(\operatorname{Spec}(\widetilde{\mathcal{O}}_{Y,\mathfrak{p}}), \mathbb{G}_m) \tag{3}$$

by [6, p.93, III, Corollary 1.28]. Moreover, the diagram

$$\begin{tikzcd}
\mathrm{Br}(K)/\mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \arrow[r] \arrow[d] & \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\mathrm{Br}(\widetilde{\mathcal{O}}_{\mathfrak{p}}) \arrow[d] \\
\mathrm{H}^3_{\{\mathfrak{p}\}}(\mathrm{Spec}(\mathcal{O}_{X,\mathfrak{p}}), \mathbb{G}_m) \arrow[r, "\simeq", "{cf,(3)}"'] & \mathrm{H}^3_{\{\mathfrak{p}\}}(\mathrm{Spec}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}), \mathbb{G}_m)
\end{tikzcd}$$

is commutative. Therefore

$$\mathrm{Br}(K)/\mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \to \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}})$$

is injective. So the statement follows from [2, p.77, II, Proposition 2.3]. □

LEMMA 3.2. Let A be a Henselian discrete valuation ring, K its quotient field , k its residue field and K_{nr} its maximal unramified extension. Then

$$\mathrm{H}^p(\mathrm{Spec}(A), g_*(\mathbb{G}_m)) = \mathrm{H}^p(K_{nr}/K, (K_{nr})^*)$$

for any $p > 0$ and the sequence

$$0 \to \mathrm{H}^p(\mathrm{Spec}(A), \mathbb{G}_m) \to \mathrm{H}^p(K_{nr}/K, (K_{nr})^*) \to \mathrm{H}^p(k, \mathbb{Z}) \to 0 \tag{4}$$

is exact.

Proof. Let i: $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ be the natural map. Then, i_* is exact. Let (set) be the class of all separated etale morphisms and f: $X_{et} \to X_{set}$ the continuous morphism which is induced by identity map on X. Then f_* is exact by [6, p.112, (b) of Examples 3.4]. Let (fet) be the class of all finite etale morphisms and f': $X_{set} \to X_{fet}$ the continuous morphism which is induced by identity map on X.

Let $Y \to X$ be a separated etale morphism with Y connected, $R(Y)$ the ring of rational functions of Y, $A \to B$ the normalization of A in $R(Y)$ and $X' = \mathrm{Spec}(B)$. Then $R(Y)/K$ is a finite separable extension and Y is an open subscheme of X' by [6, p.29, I, Theorem 3.20]. Moreover $X' \to X$ is finite by [6, p.4, I, Proposition 1.1]. Then, since A is a Henselian discrete valuation ring, B is a Henselian discrete valuation ring by [6, p.33, I, (b) of Theorem 4.2] and [6, p.34, I, Corollary 4.3]. Also $R(X')/R(X)$ is an unramfied extension. Therefore f'_* is exact by [6, p.111, III, Proposition 3.3]. So $f'_* \circ f_*$ is exact and

$$\mathrm{H}^p_{fet}(X, (f' \circ f)_*(\mathcal{F})) \simeq \mathrm{H}^p_{et}(X, \mathcal{F})$$

for any $\mathcal{F} \in \mathbb{S}_{X_{et}}$.

We have the isomorphism G_K-**mod** $\simeq \mathbb{S}_{\mathrm{Spec}(K)_{et}}$ by [6, p.53, II.§1,Theorem1.9]. Let the functor N be defined as

$$(G_K\text{-}\mathbf{mod}) \ni M \longmapsto M^{\mathrm{Gal}(K_s/K_{nr})} \in (G_k\text{-}\mathbf{mod})$$

and $N' : \mathbb{S}_{\mathrm{Spec}(K)_{et}} \to \mathbb{S}_{\mathrm{Spec}(k)_{et}}$ the functor which corresponds to N. Let $Y'' \in X_{fet}$ be connected. Moreover, let $K'' = R(Y'')$ and k'' the finite extension field of k which corresponds to the closed point of Y''. Then

$$N'(F)(\mathrm{Spec}(k'')) = F(\mathrm{Spec}(K''))$$

for $F \in \mathbb{S}_{\mathrm{Spec}(K)_{et}}$ because

$$G(K_{nr}/K'') \simeq G_{k''}, \quad G(K_{nr}/K'') \simeq G_{K''}/G_{K_{nr}}.$$

Therefore the diagram

$$\begin{tikzcd}
G_K\text{-}\mathbf{mod} \arrow[r, phantom, "\simeq"] \arrow[d, "N"'] & \mathbb{S}_{\mathrm{Spec}(K)_{et}} \arrow[r, "f'_*\circ f_*\circ g_*"] \arrow[d, "N'"] & \mathbb{S}_{X_{fet}} \\
G_k\text{-}\mathbf{mod} \arrow[r, phantom, "\simeq"] & \mathbb{S}_{\mathrm{Spec}(k)_{et}} \arrow[ur, "f'_*\circ f_*\circ i_*"'] & .
\end{tikzcd}$$

is commutative. So

$$\begin{aligned}
\mathrm{H}^p_{et}(X, g_*(\mathbb{G}_m)) &= \mathrm{H}^p_{fet}(X, f^{'} \circ f \circ g_*(\mathbb{G}_m)) \\
&= \mathrm{H}^p_{fet}(X, f^{'} \circ f \circ i_*(N'(\mathbb{G}_m))) \\
&= \mathrm{H}^p_{et}(X, i_*(N'(\mathbb{G}_m))) \\
&= \mathrm{H}^p_{et}(\mathrm{Spec}(k), N'(\mathbb{G}_m)) \\
&= \mathrm{H}^p(k, (K_{nr})^*) = \mathrm{H}^p(K_{nr}/K, (K_{nr})^*).
\end{aligned}$$

If we want to show where we consider the sheaf $\mathbb{G}_m$, we use the notation such as $\mathbb{G}_{m,A}$. Then the exact sequence (4) follows from the exact sequence of sheaves

$$0 \to \mathbb{G}_{m,A} \to g_*(\mathbb{G}_{m,K}) \to i_*(\mathbb{Z}) \to 0$$

(cf, [6, p.106, III, Example 2.22]). So the proof is complete. □

Corollary 3.3. Consider the situation of Theorem 3.1 and

$$\mathrm{Br}_{un}(X) = \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{\mathfrak{p}\in X_{(0)}} \mathrm{Br}(\widetilde{R(X)_{\bar{\mathfrak{p}}}})\right).$$

Then the sequence

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}_{un}(X) \to \prod_{\mathfrak{p}\in X^{(1)}} \mathrm{X}(G_{\kappa(\mathfrak{p})}) \tag{5}$$

is exact.

Proof. It follow from [2, p.76, II, Corollaire 2.2] and [6, p.147, IV, Proposition 2.11 (b)] that $\mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \subset \mathrm{Br}_{un}(\mathrm{Spec}(\mathcal{O}_{X,\mathfrak{p}}))$. So the sequence

$$0 \to \mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \to \mathrm{Br}_{un}(\mathrm{Spec}(\mathcal{O}_{X,\mathfrak{p}})) \to \mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\,\mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}})$$

is exact by Theorem 3.1. Moreover, $\mathrm{Br}(\widetilde{R(X)_{\mathfrak{p}}})/\,\mathrm{Br}(\widetilde{\mathcal{O}}_{X,\mathfrak{p}}) \simeq X(G_{\kappa(\mathfrak{p})})$ by Lemma 3.2. Therefore the sequence

$$0 \to \mathrm{Br}(\mathcal{O}_{X,\mathfrak{p}}) \to \mathrm{Br}_{un}(\mathrm{Spec}(\mathcal{O}_{X,\mathfrak{p}})) \to X(G_{\kappa(\mathfrak{p})}) \tag{6}$$

is exact. So the statement follows from (6) and [2, p.77, II, Proposition 2.3]. □

Remark 3.4. 1. Suppose that X is a regular algebraic curve over a field k. If k is perfect, $\mathrm{Br}_{un}(X) = \mathrm{Br}(K)$ by [7, p.80, II, 3.3]. If $(n, \mathrm{ch}(k)) = 1$, $\mathrm{Br}_{un}(X)_n = \mathrm{Br}(K)_n$ by [7, p.111, Appendix, §2, (2.2)].

2. Corollary 3.3 is true even if $\dim X \neq 1$ because

$$\mathrm{H}^2(X, g_*(\mathbb{G}_{m,K})) = \mathrm{Ker}\left(\mathrm{Br}(K) \overset{\mathrm{Res}}{\to} \prod_{x \in X_{(0)}} \mathrm{Br}(K_{\bar{x}})\right)$$

where $g : \mathrm{Spec}\, K \to X$ is the generic point of X.

Theorem 3.5. Let k be an arbitrary field k and $k(x)$ the purely transcendental extension field in one variable x over k. Then, the local-global map

$$\mathrm{Br}(k(x)) \to \prod_{\mathfrak{p} \in \mathbb{P}_k^{1(1)}} \mathrm{Br}(\widehat{k(x)}_{\mathfrak{p}})$$

is injective.

Proof. By using the facts [4, proof of Theorem 1] and [3, p.674, §3.4, Lemma 16], we see that $\mathrm{Br}(\widetilde{k(x)}_{\mathfrak{p}}) \simeq \mathrm{Br}(\widehat{k(x)}_{\mathfrak{p}})$. So it is sufficient for the proof of the statement to prove that

$$\mathrm{Br}(k(x)) \to \prod_{\mathfrak{p} \in \mathbb{P}_k^{1(1)}} \mathrm{Br}(\widetilde{k(x)}_{\mathfrak{p}})$$

is injective. We denote the point which corresponds to $(\frac{1}{x}) \in \mathrm{Spec}(k[\frac{1}{x}]) \subset \mathbb{P}_k^1$ by ∞. Then, by Theorem 3.1,

$$\begin{aligned}
&\mathrm{Ker}\left(\mathrm{Br}(k(x)) \to \prod_{\mathfrak{p} \in \mathbb{P}_k^{1(1)}} \mathrm{Br}(\widetilde{k(x)}_{\mathfrak{p}})\right) \\
&\subset \mathrm{Ker}\left(\mathrm{Br}(k(x)) \to \prod_{\mathfrak{p} \in \left((\mathbb{P}_k^1)^{(1)} \setminus \infty\right)} \mathrm{Br}(\widetilde{R(\mathbb{P}_k^1)_{\mathfrak{p}}})/\,\mathrm{Br}(\widetilde{\mathcal{O}}_{\mathbb{P}_k^1,\mathfrak{p}})\right) \\
&= \mathrm{Br}(k[x]).
\end{aligned}$$

Moreover

$$\operatorname{Ker}\left(\operatorname{Br}(k(x)) \to \prod_{\mathfrak{p}\in\mathbb{P}_k^{1(1)}} \operatorname{Br}(\widetilde{k(x)}_{\mathfrak{p}})\right) \subset \operatorname{Ker}\Big(\operatorname{Br}(k[x]) \to \operatorname{Br}(k(x)) \to \operatorname{Br}(\widetilde{R(\mathbb{P}_k^1)_\infty})\Big)$$

and $\operatorname{Ker}\Big(\operatorname{Br}(k[x]) \to \operatorname{Br}(k(x)) \to \operatorname{Br}(\widetilde{R(\mathbb{P}_k^1)_\infty})\Big) = 0$ by [6, p.153, IV, Exercise 2.20 (d)] or [9]. Therefore

$$\operatorname{Ker}\left(\operatorname{Br}(k(x)) \to \prod_{\mathfrak{p}\in\mathbb{P}_k^{1(1)}} \operatorname{Br}(\widetilde{k(x)}_{\mathfrak{p}})\right) = 0.$$

So the statement follows. □

Corollary 3.6. Let X be an algebraic curve over a seperably closed field such that regular and proper. Then, the local-global map

$$\operatorname{Br}(R(X)) \to \prod_{\mathfrak{p}\in X^{(1)}} \operatorname{Br}(\widehat{R(X)}_{\mathfrak{p}})$$

is injective.

Proof. The statement follows from Theorem 3.1 and [2, III, Corollary 5.8]. □

Remark 3.7. If k is perfect, Theorem 3.5 is proved by using the exact sequence

$$0 \to \operatorname{Br}(\mathbb{P}_k^1) \to \operatorname{Br}(k(x)) \to \bigoplus_{\mathfrak{p}\in\mathbb{P}_k^{1(1)}} X(G_{\kappa(\mathfrak{p})}) \tag{7}$$

in [8]. But it is unknown fact whether (7) is exact or not in the case where k is not perfect and Theorem 3.5 has not been proved. The sequence (5) is exact in Corollary 3.3, but the sequence (7) is not exact in the case where k is not perfect as follows.

It is known that k is perfect if and only if $\operatorname{Br}(k) = \operatorname{Br}(k[x])$ (cf, [1, p.389, Theorem 7.5]). So $\operatorname{Br}(k[x]) \neq 0$ in the case where k is the separable closure of an imperfect field and $\operatorname{Br}(k(x)) \neq 0$ because $\operatorname{Br}(k[x]) \subset \operatorname{Br}(k(x))$. On the other hand, $X(G_{\kappa(\mathfrak{p})}) = \{1\}$ and $\operatorname{Br}(\mathbb{P}_k^1) = \operatorname{Br}(k) = \{0\}$. So the sequence (7) is not exact.

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