ON THE DENSITY OF UNNORMALIZED TAMAGAWA NUMBERS OF ORTHOGONAL GROUPS III

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1. Introduction

This is part III of a series of three papers. In this series of papers, we determine the density of unnormalized Tamagawa numbers of projective special orthogonal groups defined over a fixed number field.

Let k be a number field and $\mathbb{A}$ its ring of adèles. Throughout this paper,

$$G = \mathrm{GL}(1) \times \mathrm{GL}(n),\ V = \mathrm{Sym}^2 \mathrm{Aff}(n). \tag{1.1}$$

We regard V as the space of quadratic forms in $n \geq 1$ variables. Let $V_k^{\mathrm{ss}} = \{x \in V_k \mid \det x \neq 0\}$. If $x \in V_k^{\mathrm{ss}}$ then we define the special orthogonal group $\mathrm{SO}(x)$ in the well-known manner. We define $\mathrm{PSO}(x)$ to be $\mathrm{SO}(x)$ modulo its center. Then

$$\mathrm{PSO}(x) = \begin{cases} \mathrm{SO}(x) & n \text{ odd}, \\ \mathrm{SO}(x)/\{\pm I_n\} & n \text{ even}. \end{cases}$$

We denote the set of k-isomorphism classes of algebraic groups over k of the form $\mathrm{SO}(x)$ by S_n. Then S_n is in bijective correspondence with the set of k-isomorphism classes of algebraic groups over k of the form $\mathrm{PSO}(x)$.

In part I, we proved that the correspondence $G_k \backslash V_k^{\mathrm{ss}} \ni x \to \mathrm{SO}(x) \in S_n$ is a bijective map. We also defined the notion of the discriminant $\Delta_x \in \mathbb{Z}_{>0}$ for $x \in V_k^{\mathrm{ss}}$ and an invariant measure $d\widetilde{g}''_x$ on the adèlization $\mathrm{PSO}(x)_{\mathbb{A}}$. Roughly speaking, it is the measure defined by the Iwasawa decomposition. The volume $\mathrm{vol}(\mathrm{PSO}(x)_{\mathbb{A}}/\mathrm{PSO}(x)_k)$ with respect to $d\widetilde{g}''_x$ is finite, and we call it the *unnormalized Tamagawa number* $\mathrm{PSO}(x)$.

Our main theorems are Theorem 6.12 in part II [9] and Theorem 5.9 in this part. Our results are over an arbitrary number field k, but we state them here assuming that $k = \mathbb{Q}$ for simplicity.

For convenience, we put $r = \left[\frac{n}{2}\right]$, i.e., $r = \frac{n-1}{2}$ (n odd) and $r = \frac{n}{2}$ (n even). For a prime number p, we put

$$E_p = 1 + \frac{5}{4}p^{-2} + \frac{1}{4}p^{-3} - p^{-r-1} - \frac{3}{2}p^{-r-2} - \frac{1}{2}p^{-r-3} + \frac{1}{4}p^{-2r-2} + \frac{1}{4}p^{-2r-3},$$
$$E'_p = 1 - p^{-2} - p^{-2r-1} + p^{-2r-2} + \frac{1}{4}p^{-3}\frac{(1-p^{-1})^2(1-p^{-(r-1)})(1-p^{-2r})}{1-p^{-2}}.$$

Let $\Gamma(s)$ be the classical gamma function. We put

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2}),\ \Gamma_{\mathbb{C}}(s) = (2\pi)^{1-s}\Gamma(s). \tag{1.2}$$

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For $0 \leq i \leq r$, let $S_{n,\mathbb{i}}$ be the subset of S_n consisting of groups of the form $\mathrm{PSO}(x)$ where x is a quadratic form with signature $(\mathbb{i}, n-\mathbb{i})$.

Theorem 1.3. *Suppose that* $n = 2r+1 \geq 3$ *is odd. Then*

$$\lim_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\substack{x,y\in S_{n,\mathbb{i}} \\ \Delta_x\Delta_y<X}} \mathrm{vol}(\mathrm{SO}(x)_{\mathbb{A}}/\mathrm{SO}(x)_{\mathbb{Q}})\mathrm{vol}(\mathrm{SO}(y)_{\mathbb{A}}/\mathrm{SO}(y)_{\mathbb{Q}})$$

$$= \frac{2^{-n+\mathbb{i}(n-\mathbb{i}+1)+2}}{n+1} \left(\prod_{1\leq j\leq \mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1\leq j\leq n-\mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1\leq j\leq r} \zeta(2j) \right)^2 \prod_p E_p.$$

Note that $\mathrm{PSO}(x) \cong \mathrm{SO}(x)$ if n is odd.

Theorem 1.4. *Suppose that* $n = 2r \geq 4$ *is even. Then*

$$\lim_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\substack{x\in S_{n,\mathbb{i}} \\ \Delta_x<X}} \mathrm{vol}(\mathrm{PSO}(x)_{\mathbb{A}}/\,\mathrm{PSO}(x)_{\mathbb{Q}})$$

$$= \frac{2^{-n+\frac{\mathbb{i}(n-\mathbb{i}+1)}{2}+2}}{n+1} \prod_{1\leq j\leq \mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1\leq j\leq n-\mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1\leq j\leq r} \zeta(2j) \prod_p E'_p.$$

Since our work is a generalization of Datskovsky's work [1], our method works for $n = 2$ also, and can prove the following known result of Goldfeld–Hoffstein [6].

Theorem 1.5. (Goldfeld–Hoffstein)

$$\lim_{X\to\infty} X^{-3/2} \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<\Delta_F\leq X}} h_F R_F = \frac{\pi^2}{36} \prod_p (1-p^{-2}-p^{-3}+p^{-4}),$$

$$\lim_{X\to\infty} X^{-3/2} \sum_{\substack{[F:\mathbb{Q}]=2 \\ 0<-\Delta_F\leq X}} h_F = \frac{\pi}{18} \prod_p (1-p^{-2}-p^{-3}+p^{-4})$$

where h_F, R_F *are the class number and the regulator of of the quadratic field* F *respectively.*

In this part we prove Theorem 1.4. For a general introduction, the reader should see the introduction of part I. Here we discuss issues specific to this part.

We use the zeta function theory for the case (1.1) to prove Theorem 1.4. There is a possibility that one can use the explicit method as in part II (which uses a technique in [10]). However, we do not restrict ourselves to the ground field $\mathbb{Q}$ and the pole structure of the zeta function for the space of quadratic forms has been determined by the author in Chapter 4 of [24]. So at this point, using the zeta function theory is probably the easiest way to prove Theorem 1.4.

The notion of prehomogeneous vector spaces was introduced by M. Sato in the early 1960's. The principal parts of the global zeta functions for some prehomogeneous vector spaces were determined by Shintani [19], [20], Sato-Shintani [18] and the author [24], [25], [26]. As for the present case, Siegel determined the density of $\mathrm{vol}(\mathrm{SO}(x)_{\mathbb{R}}/\mathrm{SO}(x)_{\mathbb{Z}})$ for integral equivalence classes of quadratic forms in [21]. Siegel's approach was quite classical and does not use the zeta function theory. Shintani [20] tried to interpret Siegel's result from the viewpoint of zeta function theory and obtained a partial result. He succeeded in determining the principal parts of the zeta function for the set of positive definite quadratic forms. The author obtained the principal parts of the zeta function without any condition on the signature in [24]. In the meantime, Ibukiyama–Saito [10] proved an “explicit formula” for the zeta function for this case when the ground field is $\mathbb{Q}$.

Shintani’s method in [19] and the second author’s approach in [24] were similar and use the Eisenstein series in an essential manner. Ibukiyama–Saito’s method was quite different from that in [19], [24]. Roughly speaking, their approach uses a relatively simple orbit structure for this case and computes the global zeta function using the local zeta functions. In the case where n is odd, they expressed the zeta function as a sum of two functions which are products of Riemann zeta functions. In the case where n is even, which is the main concern of this part, they expressed the zeta function using Riemann zeta functions and the Eisenstein series of half integral weight. It is possible to figure out the pole structure of the Eisenstein series of half integral weight, but is of about the same difficulty as to figure out the pole structure of the zeta function for the case $n = 2$.

Theorems 1.3, 1.4 are density theorems for rational equivalence classes in comparison with Siegel’s result. Generally speaking, it is more difficult to count objects which are sparse. Since infinitely many integral equivalence classes are rationally equivalent, one has to filter out the ambiguity to deduce the density for rational equivalence classes. The method which achieves this task is called the filtering process (see [2], [3], [13], etc.). In order to apply this filtering process, one has to know the residue at the rightmost pole of the zeta function and find a uniform estimate of local zeta functions. As we stated above, we use the author’s result for the global zeta function.

In [10], the Igusa zeta function was computed explicitly. However, local orbital zeta functions were not computed and we do need their information. If we extremely simplify the situation, when p is an odd prime, the Igusa zeta function for the present case is in the form

$$\begin{aligned}F(s) &= A_1(p)f_1(s) + A_2(p)f_2(s)\\ &\quad + p^{-(s-\frac{n-1}{2})}A_3(p)f_3(s) + p^{-(s-\frac{n-1}{2})}A_4(p)f_4(s) + p^{-(2s-n+2)}A_5(p)f_5(s),\end{aligned}$$

where $A_1(p), \ldots, A_5(p)$ are constants which are close to 1 and $f_1(s), \ldots, f_5(s)$ are the local orbital zeta functions. We have to estimate $f_1(s), \ldots, f_5(s)$ for $\mathrm{Re}(s) > \frac{n}{2}$, and the coefficients of $f_3(s), f_4(s), f_5(s)$ at $s = \frac{n}{2}$ are close to $p^{-\frac{1}{2}}, p^{-\frac{1}{2}}, p^{-2}$.

So for example, we can bound $f_5(s)$ by something like $p^2F(s)$ around $s = \frac{n}{2}$, but this does not accomplish what we need. We speculate that further analysis along the line of [10] computes the local orbital zeta functions explicitly. However, even though we use the formulation of [10], we only estimate the local orbital zeta functions without finding their explicit forms, which is fairly simple.

Even though we do not have to determine the explicit form of the local orbital zeta functions, we do have to prove that the constant term of the q-expansion of the local orbital zeta function is 1 except for a finite number of places. In [12] Kable and the author generalized Datskovsky’s approach and introduced the notion of the “omega set” for that purpose. However, if the stabilizer is not a torus then it is not easy to construct omega sets. We prove in Section 7 a proposition (Proposition 7.3) which makes it possible to prove that the constant term of the q-expansion of the local orbital zeta function is 1 for general prehomogeneous vector spaces. For the present case, it has already been proved in part I that the assumption of Proposition 7.3 is satisfied.

There have been many works on Siegel's "local density". We compute the local factor of the constant in Theorem 1.4 at finite places in Section 6 and the computation is essentially the same as that for the local density. If the ground field is $\mathbb{Q}_p$, the local density is known for all cases. However, the local density over an arbitrary dyadic field seems unknown and difficult. To deal with ramified cases over dyadic fields, we group together those orbits, which makes the computation simple and uniform for all finite places.

The computation of the local factor of the constant in Theorem 1.4 at infinite places is essentially done in part II. It can basically be reduced to the comparison between the measure defined by the Iwasawa decomposition and the measure defined by the Tamagawa measure. We briefly discuss how to relate the computation to that in part II in Section 9.

For the rest of the introduction, we discuss the organization of this part. Throughout this paper except for Section 6, $n \geq 2$ is an even integer. In Section 2 we discuss notations used throughout this part. In Section 3 we review results from parts I,II which are needed in this part. In Sections 4, 5, we deduce Theorem 1.4 assuming results in later sections. In Section 6 we determine the local factor at finite places of the constant in Theorem 1.4. We need this result to prove a uniform estimate of the local zeta functions in Section 8 and so Section 6 had to come earlier. We have discussed the content of Section 7 above. In Section 8 we prove a uniform estimate of the local zeta functions using results in [10]. In Section 9 we determine the local factor at infinite places of the constant in Theorem 1.4. In Section 10 we specialize to the case $k = \mathbb{Q}$, $n = 2$, deduce the theorem of Goldfeld–hoffstein and explain that the values of $\widetilde{\mathcal{C}}'_{v,x}$ we obtained are compatible with the constants in [1].

2. NOTATION

In this section we define basic notations. More specialized notation will be introduced in the section where it is required.

If X is a finite set then $\sharp X$ will denote its cardinality. The standard symbols $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}$ will denote respectively the set of rational, real and complex numbers and the rational integers. The set of positive real numbers is denoted by $\mathbb{R}_+$. If R is any ring then $R^\times$ is the set of invertible elements of R, and if V is a variety defined over R then V_R denotes the set of R-valued points. If G is an algebraic group then G° denotes its identity component.

Throughout this paper, k is a fixed number field except in Section 3 where k is an arbitrary field. Let $\mathfrak{M}$, $\mathfrak{M}_\infty$, $\mathfrak{M}_{\mathrm{f}}$, $\mathfrak{M}_{\mathrm{dy}}$, $\mathfrak{M}_{\mathbb{R}}$ and $\mathfrak{M}_{\mathbb{C}}$ denote respectively the set of all places of k, all infinite places, all finite places, all dyadic places (those dividing the place of $\mathbb{Q}$ at 2), all real places and all imaginary places.

Let $\mathcal{O}$ be the ring of integers of k. If $v \in \mathfrak{M}$ then k_v denotes the completion of k at v and $|\ |_v$ denotes the normalized absolute value on k_v. If $v \in \mathfrak{M}_{\mathrm{f}}$ then $\mathcal{O}_v$ denotes the ring of integers of k_v, π_v a uniformizer in $\mathcal{O}_v$, $\mathfrak{p}_v$ the maximal ideal of $\mathcal{O}_v$ and q_v the cardinality of $\mathcal{O}_v/\mathfrak{p}_v$. If $a \in k_v$ and $(a) = \mathfrak{p}_v^i$ then we write $\mathrm{ord}_v(a) = i$ (or simply $\mathrm{ord}(a) = i$). If $\mathfrak{i}$ is a fractional ideal in k_v and $a - b \in \mathfrak{i}$ then we write $a \equiv b \mod \mathfrak{i}$.

We denote the absolute discriminant of k by Δ_k. Let r_1, r_2, h_k, R_k and e_k be, respectively, the number of real places, the number of complex places, the class number, the regulator and the number of roots of unity contained in k. It will be convenient to set

$$\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} h_k R_k e_k^{-1}. \tag{2.1}$$

We assume that the reader is familiar with basic definitions and facts concerning adèles and idèles. These may be found in [22]. The ring of adèles, the group of idèles and the adèlic absolute value of k are denoted by $\mathbb{A}$, $\mathbb{A}^\times$ and $|\ |$ respectively. Let $\mathbb{A}^1 = \{t \in \mathbb{A}^\times \mid |t| = 1\}$. Suppose that $[k : \mathbb{Q}] = n$. For $\lambda \in \mathbb{R}_+$, $\underline{\lambda} \in \mathbb{A}^\times$ is the idèle whose component at any infinite place is $\lambda^{1/n}$ and whose component at any finite place is 1. Clearly, $|\underline{\lambda}| = \lambda$.

If V is a vector space over k then we let $V_\mathbb{A}$ be its adèlization, and V_∞ and V_f its infinite and finite parts. Let $\mathcal{S}(V_\mathbb{A})$, $\mathcal{S}(V_\infty)$, $\mathcal{S}(V_\mathrm{f})$ and $\mathcal{S}(V_{k_v})$ be the spaces of Schwartz–Bruhat functions on each of the indicated domains.

We choose a Haar measure dx on $\mathbb{A}$ so that $\int_{\mathbb{A}/k} dx = 1$. For any $v \in \mathfrak{M}_\mathrm{f}$, we choose a Haar measure dx_v on k_v so that $\int_{\mathcal{O}_v} dx_v = 1$. Let dx_v be the Lebesgue measure if $v \in \mathfrak{M}_\mathbb{R}$, and two times the Lebesgue measure if $v \in \mathfrak{M}_\mathbb{C}$. It is known that $dx = |\Delta_k|^{-1/2} \prod_v dx_v$ (see [22, p. 91]).

We define a Haar measure $d^\times t^1$ on $\mathbb{A}^1$ so that $\int_{\mathbb{A}^1/k^\times} d^\times t^1 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that

$$\int_{\mathbb{A}^\times} f(t) d^\times t = \int_0^\infty \int_{\mathbb{A}^1} f(\underline{\lambda} t^1) d^\times \lambda d^\times t^1,$$

where $d^\times \lambda = \lambda^{-1} d\lambda$. For any $v \in \mathfrak{M}_\mathrm{f}$, we choose a Haar measure $d^\times t_v$ on $k_v^\times$ so that $\int_{\mathcal{O}_v^\times} d^\times t_v = 1$. Let $d^\times t_v = |t_v|_v^{-1} dt_v$ if $v \in \mathfrak{M}_\infty$.

We later have to compare the global measure and the product of local measures, and for that purpose it is convenient to denote the product of local measures on $\mathbb{A}, \mathbb{A}^\times$ as follows

$$d_\mathrm{pr} x = \prod_v dx_v\,,\ d_\mathrm{pr}^\times t = \prod_v d^\times t_v\,. \tag{2.2}$$

It is well-known (see [22, pp. 91,95]) that

$$dx = |\Delta_k|^{-\frac{1}{2}} d_\mathrm{pr} x,\ d^\times t = \mathfrak{C}_k^{-1} d_\mathrm{pr}^\times t. \tag{2.3}$$

Let $\zeta_k(s)$ be the Dedekind zeta function of k. We define

$$Z_k(s) = |\Delta_k|^{\frac{s}{2}} (\pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}))^{r_1} ((2\pi)^{1-s} \Gamma(s))^{r_2} \zeta_k(s)\,. \tag{2.4}$$

This definition differs from that in [22, p. 129] by the inclusion of the $|\Delta_k|^{s/2}$ factor and from that in [24] by a factor of $(2\pi)^{r_2}$. It is known ([22, p. 129]) that

$$\operatorname{Res}_{s=1} \zeta_k(s) = |\Delta_k|^{-1/2} \mathfrak{C}_k, \quad \text{and so} \quad \operatorname{Res}_{s=1} Z_k(s) = \mathfrak{C}_k\,. \tag{2.5}$$

3. REVIEW OF FACTS FROM PARTS I, II

In this section, we briefly review results on quadratic forms from parts I and II. Let k be an arbitrary field in this section. For the rest of this paper except for Section 6, n is an even integer.

Throughout this paper,

$$G = \mathrm{GL}(1) \times \mathrm{GL}(n),\ V = \mathrm{Sym}^2\mathrm{Aff}(n) \tag{3.1}$$

for $n \geq 1$. We regard V as the space of quadratic forms.

We denote the n-dimensional unit matrix by I_n. Let $\widetilde{T} = \mathrm{Ker}(G \to \mathrm{GL}(V))$. It is easy to see that

$$\widetilde{T} = \{(\widetilde{t_0}^{-2}, \widetilde{t_0} I_n) \mid \widetilde{t_0} \in \mathrm{GL}(1)\}. \tag{3.2}$$

We express elements of V as

$$x[v] = \sum_{i \leq j} x_{ij} v_i v_j \tag{3.3}$$

where $v = (v_1, \ldots, v_n)$ (v is an n-dimensional row vector) and $v_1, \ldots, v_n$ are variables. If $\mathrm{ch}\, k \neq 2$ then we may identify x with the following symmetric matrix

$$M_x = \begin{pmatrix} 2x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & 2x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & 2x_{nn} \end{pmatrix} \tag{3.4}$$

We choose (x_{ij}) as the coordinate system for V. It would be convenient to be able to use matrix computation even if $\mathrm{ch}\, k = 2$. If $\mathrm{ch}\, k = 2$ then by slight abuse of notation, we regard 2 in diagonal entries as a variable, and 2 in other entries as 0. This may be a little confusing but it will be more convenient than the definition (3.3) of $x[v]$ itself. For example, if $\mathrm{ch}\, k = 2$ and $y = (y_{ij})$ corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1+2u \\ 1+2u & 2(1+u+u^2) \end{pmatrix}$$

then $y_{11} = 1, y_{12} = 1, y_{22} = 1 + u + u^2$.

The group G acts on V by $(g_1, g_2)x[v] = g_1 x[vg_2]$. Let $P(x) = \det M_x$ and $\chi(g_1, g_2) = g_1^n (\det g_2)^2$. Then χ is a character of G. Let $V^{\mathrm{ss}} = \{x \in V \mid P(x) \neq 0\}$. If $x \in V^{\mathrm{ss}}$ then it is called a *semi-stable point*. It is easy to see that $P(gx) = \chi(g)P(x)$ for $g \in G$ and $x \in V$.

We put

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let w be the element of V such that

$$M_w = \begin{pmatrix} H & & \\ & \ddots & \\ & & H \end{pmatrix}. \tag{3.5}$$

Then $w \in V_k^{\mathrm{ss}}$. So this element is universally generic (this notion was defined in Section 2 [15]).

If G_1, G_2 are algebraic groups over k then G_2 is called a k-form of G_1 if there exists a finite separable extension F/k such that $G_1 \times F \cong G_2 \times F$. For $x \in V_k^{\mathrm{ss}}$ we define

$$\begin{aligned} \mathrm{GO}(x) &= \{g \in \mathrm{GL}(n) \mid \exists \alpha(g)(g) \in \mathrm{GL}(1)\ gx = \alpha(g)x\}, \\ \mathrm{O}(x) &= \{g \in \mathrm{GL}(n) \mid gx = \alpha(g)x\}, \\ \mathrm{SO}(x) &= \mathrm{O}(x) \cap \mathrm{SL}(n). \end{aligned} \tag{3.6}$$

These groups are smooth algebraic groups over k if $\mathrm{ch}\, k \neq 2$. $\mathrm{GO}(x)$ and $\mathrm{SO}(x)$ are smooth over k even if $\mathrm{ch}\, k = 2$, but $\mathrm{O}(x)$ is not. If $\mathrm{ch}\, k = 2$ then we consider these groups only set-theoretically.

Suppose that $\mathrm{ch}\, k \neq 2$. Let $\mathrm{GO}(x)^\circ$ be the identity component and Z its subgroup of scalar matrices. We define

$$\mathrm{PGO}(x)^\circ = \mathrm{GO}(x)^\circ / Z, \quad \mathrm{PSO}(x) = \mathrm{SO}(x)/\{\pm I_n\}. \tag{3.7}$$

As algebraic groups over k,

$$\mathrm{PGO}(x)^\circ \cong \mathrm{PSO}(x).$$

Since we assumed that n is even, $[\mathrm{GO}(x) : \mathrm{GO}(x)^\circ] = 2$.

Remark 3.8. Note that if $F \supset k$ is a field then $(\mathrm{GO}(x)^\circ/\widetilde{T})_F = \mathrm{GO}(x)^\circ_F/\widetilde{T}_F$, but $(\mathrm{SO}(x)/\{\pm I_n\})_F$ may not coincide with $\mathrm{SO}(x)_F/\{\pm I_n\}$. Therefore, the set-theoretic quotients $\mathrm{GO}(x)^\circ_F/\widetilde{T}_F$, $\mathrm{SO}(x)_F/\{\pm I_n\}$ may not coincide.

The following proposition is proved in Proposition 3.12 [8].

Proposition 3.9. *Suppose that* $\mathrm{ch}\, k \neq 2$ *and that* $n \geq 4$. *Then the set* $G_k \backslash V_k^{\mathrm{ss}}$ *is in bijective correspondence with the set of* k*-forms of* $\mathrm{PGO}(w)^\circ$ *of the form* $\mathrm{PGO}(x)^\circ$. *If* $x \in V_k^{\mathrm{ss}}$ *then the corresponding* k*-form is* $\mathrm{PGO}(x)^\circ$.

If $n = 2$ then it is known that the set $G_k \backslash V_k^{\mathrm{ss}}$ is in bijective correspondence with the set of isomorphism classes of separable extensions of k of degree at most two. Moreover, if F corresponds to $x \in G_k \backslash V_k^{\mathrm{ss}}$ then $G_x^\circ \cong \mathrm{GL}(1)_F$ as algebraic groups over k (see [23, pp. 285, 309–310]). For even $n \geq 4$, let S_n be the set of k-isomorphism classes of algebraic groups over k of the form $\mathrm{PGO}(x)^\circ$.

We assume that k is a number field for the rest of this section. We review the choice of a set of representatives for $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ for $v \in \mathfrak{M}_{\mathrm{f}}$ in part I and for $v \in \mathfrak{M}_\infty$ in part II in the following.

We first consider the case $v \in \mathfrak{M}_{\mathrm{f}}$. We put $\lambda_v = \sharp(k_v^\times/(k_v^\times)^2) - 2$. Note that $\lambda_v = 2$ if v is not dyadic. Suppose that $a_0, b_0 \in \mathcal{O}_v^\times$ and a root of $z^2 + a_0 z + b_0 = 0$ generates the unramified quadratic extension of k_v. We choose Eisenstein polynomials $p_j(z) = z^2 + a_j z + b_j = 0$ $(j = 1, \ldots, \lambda_v)$ so that if F_j is the splitting field of $p_j(z)$ over k_v then $\{F_j \mid j = 1, \ldots, \lambda_v\}$ is a complete set of representatives of isomorphism classes of ramified quadratic extensions of k_v.

We put

$$A_{v,\mathrm{in}} = \begin{pmatrix} 2 & a_0 \\ a_0 & b_0 \end{pmatrix}, \ A_{v,(\mathrm{rm},j)} = \begin{pmatrix} 2 & a_j \\ a_j & b_j \end{pmatrix} \ j = 1, \ldots, \lambda_v, \ A_{v,\mathrm{dq}} = \begin{pmatrix} A_{v,\mathrm{in}} & \\ & \pi_v A_{v,\mathrm{in}} \end{pmatrix}.$$

Note that “in”and “dq” stand for “inert” and “division quarternion” respectively. Let $w_{v,\mathrm{sp}}$ be the element w in (3.5). If $n \geq 2$ is even then we define

$$w_{v,\mathrm{in}} = \begin{pmatrix} H & & & \\ & \ddots & & \\ & & H & \\ & & & A_{v,\mathrm{in}} \end{pmatrix}, \; w_{v,(\mathrm{rm},j)} = \begin{pmatrix} H & & & \\ & \ddots & & \\ & & H & \\ & & & A_{v,(\mathrm{rm},j)} \end{pmatrix} \tag{3.10}$$

where $j = 1, \ldots, \lambda_v$. If $n \geq 4$ is even then we define

$$w_{v,\mathrm{dq}} = \begin{pmatrix} H & & & \\ & \ddots & & \\ & & H & \\ & & & A_{v,\mathrm{dq}} \end{pmatrix}. \tag{3.11}$$

If we have to show the dimension n then we may write $w_{n,v,\mathrm{sp}}$, etc.

If $n = 2$ then

$$\{w_{v,\mathrm{sp}}, w_{v,\mathrm{in}} (= A_{v,\mathrm{in}}), w_{v,(\mathrm{rm},1)}, \ldots, w_{v,(\mathrm{rm},\lambda_v)}\}$$

is a complete set of representatives for $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ ([23, pp. 285, 309–310]). The following proposition is Proposition 4.16 in [8].

Proposition 3.12. *If $n \geq 4$ is even and $v \in \mathfrak{M}_f$ then*

$$\{w_{v,\mathrm{sp}}, w_{v,\mathrm{in}}, w_{v,(\mathrm{rm},1)}, \ldots, w_{v,(\mathrm{rm},\lambda_v)}, w_{v,\mathrm{dq}}\}$$

is a complete set of representatives for $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$.

If $v \in \mathfrak{M}_{\mathrm{f}}$ and $x \in G_{k_v} w_{v,\mathrm{sp}}, G_{k_v} w_{v,\mathrm{in}}, G_{k_v} w_{v,(\mathrm{rm},j)}, G_{k_v} w_{v,\mathrm{dq}}$ then we say that the *type* of x is (sp), (in), (rm), (dq) respectively.

Suppose that $v \in \mathfrak{M}_{\mathbb{R}}$. Then $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ is represented by quadratic forms with signatures

$$(0, n), \ldots, (\tfrac{n}{2}, \tfrac{n}{2}).$$

Therefore, we regard $\mathbb{i} = 0, \ldots, \frac{n}{2}$ as the index which parametrizes the orbit space $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$. For $\mathbb{i} = 0, \ldots, \frac{n}{2}$, we put

$$w_{v,\mathbb{i}} = \begin{pmatrix} -I_{\mathbb{i}} & \\ & I_{n-\mathbb{i}} \end{pmatrix}. \tag{3.13}$$

It is well-known that $\{w_{v,0}, \ldots, w_{v,\frac{n}{2}}\}$ is a complete set of representatives of $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$. Clearly, $|P(w_{v,\mathbb{i}})|_v = 1$ for all $\mathbb{i}$. If $v \in \mathfrak{M}_{\mathbb{C}}$ then $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ consists of a single point and is represented by $w_{v,\mathrm{sp}} = I_n$. The condition $|P(w_{v,\mathrm{sp}})|_v = 1$ is satisfied also.

For all $v \in \mathfrak{M}$, we call $w_{v,\mathrm{sp}}$, etc., *standard orbital representatives* for $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$. We denote by $\mathbb{i}$ the index of the representatives, i.e.,

$$\mathbb{i} = \mathrm{sp},\ \mathrm{in},\ (\mathrm{rm}, j),\ \mathrm{dq}$$

if $v \in \mathfrak{M}_{\mathrm{f}}$, $\mathbb{i} = 0, \ldots, \frac{n}{2}$ if $v \in \mathfrak{M}_{\mathbb{R}}$ and $\mathbb{i} = \mathrm{sp}$ if $v \in \mathfrak{M}_{\mathbb{C}}$. The following proposition is Proposition 4.22 [8].

Proposition 3.14. *Let $n = 2r \geq 2$ be even. If $v \in \mathfrak{M}$ and $x \in V_{k_v}^{\mathrm{ss}}$ is a standard representative then there exists an element τ_x of $G_{x\,k_v}$ not in $G_{x\,k_v}^{\circ}$ whose order is two. If $v \in \mathfrak{M}_f$ then one can choose τ_x in $G_{x\,k_v} \cap K_v$.*

This completes a general review of parts I and II. More specific theorems and propositions will be quoted where they are needed.

4. THE DENSITY THEOREM AND THE FORMULATION OF THE PROOF

For the rest of this paper, k is a fixed number field. In this section, we discuss the formulation of the filtering process and deduce the density theorem assuming results in later sections.

We first recall the definition of the global zeta function for the space (G, V), and review results in [24] on the poles of the zeta function. The zeta function is approximately the Dirichlet generating series for the sequence $\mathrm{vol}(G^{\circ}_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G^{\circ}_{x\,k})$. However, each term has a factor which is basically the local orbital integral. In order to control the contribution from such factors, we have to use the filtering process used in works such as [2], [3], [1], [12], [13], [14]

We choose an invariant measure on $\mathrm{GL}(n)_{\mathbb{A}}$ now. Let $T \subseteq \mathrm{GL}(n)$ be the subgroup consisting of diagonal matrices, and $N \subseteq \mathrm{GL}(n)$ the subgroup consisting of lower triangular matrices with diagonal entries 1. Then $B = TN$ is a Borel subgroup of G. We use the notation

$$\begin{aligned} t = a(t_1, \dots, t_n) &= \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix},\ t_1, \dots, t_n \in \mathbb{A}^{\times}, \\ n(u) &= \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ u & & 1 \end{pmatrix},\ u = (u_{ij})_{i>j} \in \mathbb{A}^{\frac{n(n-1)}{2}} \end{aligned} \tag{4.1}$$

for elements of $T_{\mathbb{A}}, N_{\mathbb{A}}$ respectively. Let $K = \prod_{v\in\mathfrak{M}} K_v$, where $K_v = \mathrm{O}(n)$ (the classical orthogonal group) if $v \in \mathfrak{M}_{\mathbb{R}}$, $K_v = \mathrm{U}(n)$ (the classical unitary group) if $v \in \mathfrak{M}_{\mathbb{C}}$, and $K_v = \mathrm{GL}(n)_{\mathcal{O}_v}$ if $v \in \mathfrak{M}_{\mathrm{f}}$. The group $G_{\mathbb{A}}$ has an Iwasawa decomposition $G_{\mathbb{A}} = KT_{\mathbb{A}}N_{\mathbb{A}}$. So any element $g \in G_{\mathbb{A}}$ can be written as $g = \kappa(g)t(g)n(u(g))$, where $\kappa(g) \in K$, $t(g) = a(t_1(g), \dots, t_n(g)) \in (\mathbb{A}^{\times})^n$, and $u(g) \in \mathbb{A}^{\frac{n(n-1)}{2}}$. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $t = a(t_1, \dots, t_n) \in T_{\mathbb{A}}$, we define $t^z = |t_1|^{z_1} \cdots |t_n|^{z_n}$. Also if $v \in \mathfrak{M}$ then we define $t_v^z = |t_1|_v^{z_1} \cdots |t_n|_v^{z_n}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $t = a(t_1, \dots, t_n) \in T_{k_v}$. Let ρ be half the sum of the weights of N with respect to conjugations by elements of T. Let $du = \prod_{i>j} du_{ij}$. This is an invariant measure on $N_{\mathbb{A}}$. We choose an invariant measure $d\kappa$ on K so that $\int_K d\kappa = 1$. Let $d^{\times}t = d^{\times}t_1 \cdots d^{\times}t_n$ $(t = a(t_1, \dots, t_n))$ and $db = t^{-2\rho}d^{\times}t du$ $(t^{-2\rho} = \prod_{i<j} |t_i^{-1}t_j|)$. We choose $dg = t^{-2\rho}d\kappa d^{\times}t du$ as an invariant measure on $G_{\mathbb{A}}$.

Let $G^0_{\mathbb{A}} = \{g \in G_{\mathbb{A}} \mid |\det g| = 1\}$, and

$$d_n(\lambda) = a(\underline{\lambda}, \dots, \underline{\lambda}) \tag{4.2}$$

for $\lambda \in \mathbb{R}_+$. Let dg^0 be the invariant measure on $G^0_{\mathbb{A}}$ such that for any measurable function $f(g)$ on $G_{\mathbb{A}}$,

$$\int_{G_{\mathbb{A}}} f(g)dg = n\int_0^{\infty} \int_{G^0_{\mathbb{A}}} f(d_n(\lambda)g^0)d^{\times}\lambda dg^0,$$

where $d^\times\lambda = \lambda^{-1}d\lambda$. We define invariant measures on $G_{k_v}, K_v, B_{k_v}, N_{k_v}, T_{k_v}$ similarly, and denote them by $dg_v, dk_v, db_v, du_v, d^\times t_v$ respectively. Then

$$du = |\Delta_k|^{-\frac{n(n-1)}{4}} \prod_v du_v, \quad d^\times t = \mathfrak{C}_k^{-n} \prod_v d^\times t_v. \tag{4.3}$$

Let

$$\mathfrak{V}_n = \frac{Z_k(2)\cdots Z_k(n)}{\mathfrak{C}_k^{n-1}} \tag{4.4}$$

(see (2.1)). The constant $\mathfrak{V}_n$ is the volume of $G^0_{\mathbb{A}}/G_k$ with respect to the measure dg^0. We define $\mathfrak{V}_1 = 1$ for convenience.

Let $d^\times\widetilde{t}$ (resp. $d^\times_{\rm pr}\widetilde{t}$) be the measure on $\widetilde{T}_{\mathbb{A}}$ compatible under the isomorphism $\widetilde{T}_{\mathbb{A}} \cong \mathbb{A}^\times$ with the measure defined on $\mathbb{A}^\times$ in Section 2 (resp. the product measure (2.2)). We define a measure $d^\times\widetilde{t}_v$ on $k_v^\times$ similarly. Then $d^\times\widetilde{t} = \mathfrak{C}_k^{-1}d^\times_{\rm pr}\widetilde{t}$.

Let G be as in (1.1). Then $G_{\mathbb{A}} = \mathbb{A}^\times \times \mathrm{GL}(n)_{\mathbb{A}}$. Let $K_{n,v}$ be the maximal compact subgroup of $\mathrm{GL}(n)_{k_v}$ defined above. We put $K_v = K_{1,v} \times K_{n,v}$ and $K = \prod_v K_v$. Then K_v, K are maximal compact subgroups of $G_{k_v}, G_{\mathbb{A}}$, and $G_{k_v}, G_{\mathbb{A}}$ have Iwasawa decompositions with K_v, K as the maximal compact subgroups respectively.

We express elements of $G_{\mathbb{A}}$ or G_{k_v} as $g = (t_0, g_1)$. If we have to indicate the place v, we may use the index v. Using the measures on $\mathbb{A}^\times$ and $\mathrm{GL}(n)_{\mathbb{A}}$ defined above, we define $dg = d^\times t_0 dg_1$. Write $\widetilde{G} = G/\widetilde{T}$ (see (3.2)), so that V is a faithful representation of $\widetilde{G}$. Since $\widetilde{T} \cong \mathrm{GL}(1)$ as groups over k, the first Galois cohomology group of $\widetilde{T}$ is trivial, and it follows that $\widetilde{G}_F \cong G_F/\widetilde{T}_F$ for any field $F \supseteq k$. Thus $\widetilde{G}_{\mathbb{A}} \cong G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}$ and $\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k \cong G_{\mathbb{A}}/\widetilde{T}_{\mathbb{A}}G_k$. For $x \in V_k^{\rm ss}$ or $V_{k_v}^{\rm ss}$, the identity component of the stabilizer of x in $\widetilde{G}_{\mathbb{A}}$ or $\widetilde{G}_{k_v}$ can be regarded as

$$G^\circ_{x\,\mathbb{A}}/\widetilde{T}_{\mathbb{A}} \cong \mathrm{PGO}(x)^\circ_{\mathbb{A}} \cong \mathrm{PSO}(x)_{\mathbb{A}}, \quad \text{or} \quad G^\circ_{x\,k_v}/\widetilde{T}_{k_v} \cong \mathrm{PGO}(x)^\circ_{k_v} \cong \mathrm{PSO}(x)_{k_v}$$

respectively. Since n is even, $[\mathrm{PGO}(x) : \mathrm{PGO}(x)^\circ] = 2$. Let $\widetilde{K}_v = K_v\widetilde{T}_{k_v} \subseteq \widetilde{G}_{k_v}, \widetilde{K} = K\widetilde{T}_{\mathbb{A}} \subseteq \widetilde{G}_{\mathbb{A}}$. It is an exercise problem to show that these are maximal compact subgroups using the Cartan decomposition in G_{k_v} or $G_{\mathbb{A}}$.

Let $\widetilde{t}_0$ be the parameter of $\widetilde{T}$ as in (3.2). Let $d\widetilde{g}$ (resp. $d\widetilde{g}_v$) be the measure on $\widetilde{G}_{\mathbb{A}}$ (resp. $\widetilde{G}_{k_v}$) which satisfies $dg = d\widetilde{g}\, d^\times\widetilde{t}_0$ (resp. $dg_v = d\widetilde{g}_v\, d^\times\widetilde{t}_{0,v}$). Let $d_{\rm pr}\widetilde{g} = \prod_v d\widetilde{g}_v$. By (2.3) and (4.3), we obtain the following relation

$$d\widetilde{g} = |\Delta_k|^{-\frac{n(n-1)}{4}}\mathfrak{C}_k^{-n}d_{\rm pr}\widetilde{g}. \tag{4.5}$$

If $v \in \mathfrak{M}_{\rm f}$ then measures on $G^\circ_{x\,k_v}, \widetilde{G}^\circ_{x\,k_v}$ were chosen in Sections 7,8 of [8]. If x is a standard orbital representative then $dg''_{x,v}$ (resp. $d\widetilde{g}''_{x,v}$) is the measure on $G^\circ_{x\,k_v}$ (resp. $\widetilde{G}^\circ_{x\,k_v}$) such that the volume of $G^\circ_{x\,k_v} \cap K_v$ (resp. $\widetilde{G}^\circ_{x\,k_v} \cap \widetilde{K}_v$) is 1. If $x \in V_{k_v}^{\rm ss}$ is arbitrary then we express $x = gw_{v,\mathbb{i}}$ where $w_{v,\mathbb{i}}$ is a standard orbital representative, and define measures on $G^\circ_{x\,k_v}$ and $\widetilde{G}^\circ_{x\,k_v}$ by the pull-back of the measures on $G^\circ_{w_{v,\mathbb{i}}\,k_v}$ and $\widetilde{G}^\circ_{w_{v,\mathbb{i}}\,k_v}$ by the conjugation by g. Since the volume of $\widetilde{T}_{k_v} \cap K_v \cong \mathcal{O}_v^\times$ is 1, $dg''_{x,v} = d\widetilde{g}''_{x,v}d^\times\widetilde{t}_v$.

If $v \in \mathfrak{M}_\infty$ then the choice of the measures $dg''_{x,v}, d\widetilde{g}''_{x,v}$ on $G^\circ_{x\,k_v}, \widetilde{G}^\circ_{x\,k_v}$ was discussed in Section 4 of part II. Since the computation at infinite places is essentially over in part II, we simply point out that $dg''_{x,v}$ is defined using an Iwasawa decomposition and $d\widetilde{g}''_{x,v}$ is chosen so that it is compatible with $dg''_{x,v}$ and the standard measure on $\widetilde{T}_{k_v}$.

For $v \in \mathfrak{M}$ and $x \in V^{\mathrm{ss}}_{k_v}$, let $dg'_{x,v}$ be the measure on $G_{k_v}/G^{\circ}_{x\,k_v} \cong \widetilde{G}_{k_v}/\widetilde{G}^{\circ}_{x\,k_v}$ such that if f is a measurable function on $G_{k_v}x$ then

$$\int_{G_{k_v}/G^{\circ}_{x\,k_v}} f(g'_{x,v}x)dg'_{x,v} = \int_{G_{k_v}x} f(y)|P(y)|_v^{-\frac{n+1}{2}}dy.$$

There exists a constant $b_{x,v} > 0$ such that

$$dg_v = b_{x,v}dg'_{x,v}dg''_{x,v}. \tag{4.6}$$

Since the measures are compatible with the measure $d^\times \widetilde{t}_v$,

$$d\widetilde{g}_v = b_{x,v}dg'_{x,v}d\widetilde{g}''_{x,v}$$

also.

If $x \in V^{\mathrm{ss}}_k$ then we put

$$d_{\mathrm{pr}}g'_x = \prod_{v\in\mathfrak{M}} dg'_{x,v}, \quad d_{\mathrm{pr}}g''_x = \prod_{v\in\mathfrak{M}} dg''_{x,v}.$$

We define $d_{\mathrm{pr}}\widetilde{g}''_x$ similarly.

Definition 4.7. For $\Phi \in \mathcal{S}(V_{\mathbb{A}})$ and $s \in \mathbb{C}$, we define

$$Z(\Phi, s) = \int_{\widetilde{G}_{\mathbb{A}}/\widetilde{G}_k} |\chi(\widetilde{g})|^s \sum_{x\in V^{\mathrm{ss}}_k} \Phi(\widetilde{g}x)d\widetilde{g}\,.$$

The integral $Z(\Phi, s)$ is called the *global zeta function* for (G, V). It has been proved in Proposition (4.1.5) [24, p. 110] that the integral converges absolutely and locally uniformly if $\mathrm{Re}(s) > \frac{n+1}{2}$. However, a slightly different formulation was used in [24] and we have to say a few words about the difference of the formulations.

We recall the definition of the zeta function used in [24] specializing to the case of the trivial character. Let $G^0_{\mathbb{A}} = \mathbb{A}^1 \times \mathrm{GL}(n)^0_{\mathbb{A}}$, and $dg^0 = d^\times \widetilde{t}^1_0 dg^0_1$ for $g^0 = (\widetilde{t}^1_0, g^0_1) \in G^0_{\mathbb{A}}$ where $d^\times \widetilde{t}^1_0$ is the measure on $\mathbb{A}^1$ in Section 2 and dg^0_1 is defined above.

In the situation of Definition 4.7, the zeta function for the present case in [24] is the following integral

$$Z^*(\Phi, s) = \int_{\mathbb{R}_+\times G^0_{\mathbb{A}}/G_k} \lambda^s \sum_{x\in V^{\mathrm{ss}}_k} \Phi(\underline{\lambda} g^0x)d^\times\lambda dg^0,$$

where $\underline{\lambda} g^0 x$ is the scalar multiplication of $\underline{\lambda}$ to g^0x.

Let $\widetilde{T}^1_{\mathbb{A}} \subseteq \widetilde{T}_{\mathbb{A}}$ be the subgroup which corresponds to $\mathbb{A}^1$ by the isomorphism $\widetilde{T}_{\mathbb{A}} \cong \mathbb{A}^\times$. We have $(\mathbb{R}_+ \times G^0_{\mathbb{A}})/\widetilde{T}^1_{\mathbb{A}} \cong \widetilde{G}_{\mathbb{A}}$ by the map which sends the class of (λ, g^0) to the class of $(\underline{\lambda}, I_n)g^0$. We have $\widetilde{G}_{\mathbb{A}} \cong (\mathbb{R}^2_+ \times G^0_{\mathbb{A}})/(\mathbb{R}_+ \times \widetilde{T}^1_{\mathbb{A}})$ where $\mathbb{R}_+ \times \widetilde{T}^1_{\mathbb{A}}$ is included in $\mathbb{R}^2_+ \times G^0_{\mathbb{A}}$ by $(\lambda_{\widetilde{T}}, \widetilde{t}^1) \to (\lambda^{-2}_{\widetilde{T}}, \lambda_{\widetilde{T}}, \widetilde{t}^1)$ and $\mathbb{R}^2_+ \times G^0_{\mathbb{A}}$ maps onto $\widetilde{G}_{\mathbb{A}}$ by $(\lambda_1, \lambda_2, g^0) \to (\underline{\lambda}_1, d_n(\lambda_2))g^0\widetilde{T}_{\mathbb{A}}$ (see (4.2) for the definition of $d_n(*)$). In this quotient we have chosen the measure $d\widetilde{g}$ to be compatible with the measures $nd^\times\lambda_1\, d^\times\lambda_2\, dg^0$ on $\mathbb{R}^2_+ \times G^0_{\mathbb{A}}$ and $d^\times\lambda_{\widetilde{T}}\, d^\times\widetilde{t}^1$ on $\mathbb{R}_+ \times \widetilde{T}^1_{\mathbb{A}}$. Since $d^\times(\lambda\lambda^{-2}_{\widetilde{T}})d^\times\lambda_{\widetilde{T}} = d^\times\lambda d^\times\lambda_{\widetilde{T}}$, it follows that the measures $nd^\times\lambda\, dg^0$ and $d^\times\widetilde{t}^1$ are compatible with the measure $d\widetilde{g}$ in the quotient $(\mathbb{R}_+ \times G^0_{\mathbb{A}})/\widetilde{T}^1_{\mathbb{A}} \cong \widetilde{G}_{\mathbb{A}}$.

Furthermore, $|\chi(\lambda,1)| = \lambda^n$ and the volume of $\widetilde{T}^1_{\mathbb{A}}/\widetilde{T}_k$ is 1, which imply that $Z(\Phi,s) = nZ^*(\Phi,ns)$. Let $\widehat{\Phi}(0)$ be the Fourier transform of Φ evaluated at the origin, and so is simply the integral of Φ over the $V_{\mathbb{A}}$. In Theorem (4.0.1) [24, p. 106], it is shown that $Z^*(\Phi,s)$ has a meromorphic continuation to the region $\mathrm{Re}(s) > \frac{n^2}{2}$ with a simple pole at $s = \frac{n(n+1)}{2}$ with residue $\mathfrak{V}_n\widehat{\Phi}(0)$. Thus, we arrive at the following theorem.

Theorem 4.8. *The zeta function $Z(\Phi,s)$ has an analytic continuation to the region $\mathrm{Re}(s) > \frac{n}{2}$ except for a possible simple pole at $s = \frac{n+1}{2}$ with residue $\mathfrak{V}_n\widehat{\Phi}(0)$.*

We define $\Sigma(\Phi) = \widehat{\Phi}(0)$ for $\Phi \in \mathcal{S}(V_{\mathbb{A}})$. For $v \in \mathfrak{M}$ and $\Phi_v \in \mathcal{S}(V_{k_v})$, we define the local version of the distribution $\Sigma(\Phi)$ by

$$\Sigma_v(\Phi_v) = \int_{V_{k_v}} \Phi_v(y)dy. \tag{4.9}$$

Since $\dim V = \frac{n(n+1)}{2}$, if $\Phi = \otimes_v \Phi_v$ then

$$\Sigma(\Phi) = |\Delta_k|^{-\frac{n(n+1)}{4}} \prod_v \Sigma_v(\Phi_v). \tag{4.10}$$

This completes our review of the analytic properties of the global zeta function. We now return briefly to the local situation.

For $x \in V^{\mathrm{ss}}_{k_v}$ and $\Phi_v \in \mathcal{S}(V_{k_v})$, we define the *local orbital zeta function* of x by the following integral

$$\begin{aligned} Z_{x,v}(\Phi_v,s) &= b_{x,v}\int_{G_{k_v}/G^\circ_{x\,k_v}} |\chi(g'_{x,v})|^s_v \Phi_v(g'_{x,v}x)dg'_{x,v} \\ &= b_{x,v}|P(x)|^{-s}_v \int_{G_{k_v}/G^\circ_{x\,k_v}} |P(x)|^s_v|\chi(g'_{x,v})|^s_v \Phi_v(g'_{x,v}x)dg'_{x,v} \\ &= b_{x,v}|P(x)|^{-s}_v \int_{G_{k_v}x} \Phi_v(y)|P(y)|_v^{s-\frac{n+1}{2}}dy. \end{aligned} \tag{4.11}$$

If $x \in V^{\mathrm{ss}}_k$ and $\Phi = \otimes\Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ then we define

$$Z_x(\Phi,s) = \prod_{v\in\mathfrak{M}} Z_{x,v}(\Phi_v,s) \tag{4.12}$$

and call it the *global orbital zeta function* of x. If $v \in \mathfrak{M}$ then there exists a unique index $\mathbb{i}$ such that $x \in G_{k_v}w_{v,\mathbb{i}}$ ($w_{v,\mathbb{i}}$ is a standard representative). We shall write $\Xi_{x,v}(\Phi_v,s) = Z_{w_{v,i},v}(\Phi_v,s)$ and $\Xi_x(\Phi,s) = \prod_{v\in\mathfrak{M}}\Xi_{x,v}(\Phi_v,s)$. We call $\Xi_{x,v}(\Phi_v,s)$ the *standard local orbital zeta function* and $\Xi_x(\Phi_v,s)$ the *standard global orbital zeta function.*

The following proposition can be proved by the same argument as that in Proposition 5.23 [12, p. 528] and (4.11), and we leave the proof to the reader.

Proposition 4.13. *If $x \in V^{\mathrm{ss}}_{k_v}$ and $y \in G_{k_v}x$ then*

(1) $b_{x,v} = b_{y,v}$,

(2) $Z_{x,v}(\Phi_v,s) = |P(x)|^{-s}_v|P(y)|^s_v Z_{y,v}(\Phi_v,s)$.

In particular, $Z_{x,v}(\Phi_v,s) = Z_{y,v}(\Phi_v,s)$ if $|P(x)|_v = |P(y)|_v$.

For $x \in V_k^{\mathrm{ss}}$, let Δ_x be the discriminant defined in Definitions 5.4, 5.6 in [8].

Proposition 4.14. *For $x \in V_k^{\mathrm{ss}}$ and $\Phi = \otimes \Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ we have*

$$Z_x(\Phi, s) = \Delta_x^{-s} \Xi_x(\Phi, s)\,.$$

Proof. For each $v \in \mathfrak{M}$ let $\mathbb{i}_v(x)$ be the index such that $x \in G_{k_v} w_{v,\mathbb{i}_v(x)}$. Then by Proposition 4.13,

$$\begin{aligned} Z_{x,v}(\Phi_v, s) &= \frac{|P(w_{v,\mathbb{i}_v(x)})|_v^s}{|P(x)|_v^s} \cdot Z_{w_{v,i_v(x)},v}(\Phi_v, s) \\ &= \frac{|P(w_{v,\mathbb{i}_v(x)})|_v^s}{|P(x)|_v^s} \cdot \Xi_{x,v}(\Phi_v, s). \end{aligned} \tag{4.15}$$

Since n is even, by the definition of Δ_x in [8], $\prod_{v \in \mathfrak{M}} |P(w_{v,\mathbb{i}_v(x)})|_v^s = \Delta_x^{-s}$. Note that $|P(w_{v,\mathbb{i}})|_v = 1$ for standard orbital representatives if $v \in \mathfrak{M}_\infty$. Since $x \in V_k^{\mathrm{ss}}$, $P(x) \in k^\times$, and so the Hasse principle implies that $\prod_{v \in \mathfrak{M}} |P(x)|_v = 1$. Now taking the product over all $v \in \mathfrak{M}$ on both sides of (4.15) proves the identity. □

For convenience we introduce the abbreviation

$$\mathfrak{R}_1 = \frac{1}{2} |\Delta_k|^{-\frac{n(n-1)}{4}} \mathfrak{C}_k^{-n}. \tag{4.16}$$

For $x \in V_k^{\mathrm{ss}}$, we put

$$V(x) = \mathrm{vol}(G^\circ_{x\,\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G^\circ_{x\,k}) = \mathrm{vol}(\widetilde{G}^\circ_{x\,\mathbb{A}} / \widetilde{G}^\circ_{x\,k}). \tag{4.17}$$

Proposition 4.18. *If $\Phi = \otimes \Phi_v \in \mathcal{S}(V_{\mathbb{A}})$ then we have*

$$Z(\Phi, s) = \mathfrak{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \Delta_x^{-s} V(x) \Xi_x(\Phi, s).$$

Proof. Since $G_k x \cong \{\gamma x \mid \gamma \in G_k / G_{x\,k}\}$, we have

$$\begin{aligned} Z(\Phi, s) &= \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \int_{G_{\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G_k} |\chi(\widetilde{g})|^s \sum_{\gamma \in G_k / G_{x\,k}} \Phi(\widetilde{g} \gamma x)\, d\widetilde{g} \\ &= \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \int_{G_{\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G_{x\,k}} |\chi(\widetilde{g})|^s \Phi(\widetilde{g} x)\, d\widetilde{g} \\ &= \frac{1}{2} \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \int_{G_{\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G^\circ_{x\,k}} |\chi(\widetilde{g})|^s \Phi(\widetilde{g} x)\, d\widetilde{g} \quad \text{since } [G_{x\,k} : G^\circ_{x\,k}] = 2 \\ &= \mathfrak{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \int_{G_{\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G^\circ_{x\,k}} |\chi(\widetilde{g})|^s \Phi(\widetilde{g} x)\, d_{\mathrm{pr}} \widetilde{g} \quad \text{by (4.5)} \\ &= \mathfrak{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \Big(\prod_v b_{x,v} \Big) \int_{G_{\mathbb{A}} / G^\circ_{x\,\mathbb{A}}} |\chi(\widetilde{g}')|^s \Phi(\widetilde{g}' x)\, d_{\mathrm{pr}} g'_x \\ &\qquad \cdot \int_{G^\circ_{x\,\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G^\circ_{x\,k}} d_{\mathrm{pr}} \widetilde{g}''_x \quad \text{by (4.6)} \\ &= \mathfrak{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \Big(\prod_v Z_{x,v}(\Phi_v, s) \Big) \cdot \mathrm{vol}(G^\circ_{x\,\mathbb{A}} / \widetilde{T}_{\mathbb{A}} G^\circ_{x\,k}) \quad \text{by (4.11)} \end{aligned}$$

$$= \mathcal{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} Z_x(\Phi, s) V(x) \quad \text{by (4.12)}$$
$$= \mathcal{R}_1 \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}} \Delta_x^{-s} V(x) \Xi_x(\Phi, s) \quad \text{by Proposition 4.14.}$$

□

The filtering process was originally used in [3] (or in [4], [5] implicitly). We describe the filtering process specializing to the present situation.

We set $S_0 = \mathfrak{M}_\infty \cup \mathfrak{M}_{\mathrm{dy}}$ and fix a finite set $S \supseteq S_0$ of places of k. For each finite subset $T \supseteq S$ of $\mathfrak{M}$ we consider T-tuples $\omega_T = (\omega_v)_{v \in T}$ where each ω_v is one of the standard orbital representatives. If $v \in \mathfrak{M}_{\mathrm{f}}$, $x, y \in V_{k_v}^{\mathrm{ss}}$ and $x \in G_{k_v} y$ (resp. x, y are of the same type) then we write $x \approx y$ (resp. $x \asymp y$). If $v \in \mathfrak{M}_\infty$ then we write $x \approx y$ or $x \asymp y$ if x, y are in the same G_{k_v}-orbit. So the notions $x \approx y$, $x \asymp y$ differ only if $v \in \mathfrak{M}_{\mathrm{f}}$ and they are of the type (rm)

If $x \approx \omega_v$ (resp. $x \asymp \omega_v$) for all $v \in T$ then we write $x \approx \omega_T$ (resp. $x \asymp \omega_T$).

Definition 4.19. For any $v \in \mathfrak{M}_{\mathrm{f}}$, $\Phi_{v,0}$ is the characteristic function of $V_{\mathcal{O}_v}$.

Let $\Xi_{x,v}(s) = \Xi_{x,v}(\Phi_{v,0}, s)$ and $\Xi_{x,T}(s) = \prod_{v \notin T} \Xi_{x,v}(s)$. From the integral defining $\Xi_{x,v}(s)$ it follows that for $v \notin S_0$ this function may be expressed as $\Xi_{x,v}(s) = \sum_{i=-\infty}^{\infty} a_{x,v,i} q_v^{-is}$ for certain real non-negative coefficients $a_{x,v,i} \geq 0$.

In Section 7 we shall prove that the following condition is satisfied.

Condition 4.20. For all $v \notin S_0$ and all $x \in V_{k_v}^{\mathrm{ss}}$, we have $a_{x,v,i} = 0$ for $i < 0$ and $a_{x,v,0} = 1$.

Suppose that we have Dirichlet series $L_i(s) = \sum_{m=1}^{\infty} l_{i,m} m^{-s}$ for $i = 1, 2$ where $\mathbb{R} \ni l_{i,m} \geq 0$. If $l_{1,m} \leq l_{2,m}$ for all $m \geq 1$ then we shall write $L_1(s) \preccurlyeq L_2(s)$. In Section 8 we shall establish that for every $v \notin S_0$ and a small number $\delta > 0$, there exists a Dirichlet series $L_{v,\delta}(s) = \sum_{i=0}^{\infty} l_{v,i} q_v^{-is}$ with real non-negative coefficients which satisfies the following condition (since the dependency on δ is not very serious, we do not include the index δ in $l_{v,i}$ to easy the notation).

Condition 4.21. (1) For all $v \notin S_0$ and $x \in V_{k_v}^{\mathrm{ss}}$, $\Xi_{x,v}(s) \preccurlyeq L_{v,\delta}(s)$.

(2) The series defining $L_{\delta,v}(s)$ converges to a holomorphic function and the product $\prod_{v \notin S_0} L_{\delta,v}(s)$ converges absolutely and uniformly if $\mathrm{Re}(s) \geq \frac{n}{2} + \delta$.

(3) For all $v \notin S_0$, $l_{v,0} = 1$ and $l_{v,i} \geq 0$ for all i.

For any $T \supseteq S$ we define $L_{\delta,T}(s) = \prod_{v \notin T} L_{v,\delta}(s)$. Both $\Xi_{x,T}(s)$ and $L_{\delta,T}(s)$ are Dirichlet series and if we let

$$\Xi_{x,T}(s) = \sum_{m=1}^{\infty} a_{x,T,m}^* m^{-s}, \quad L_{\delta,T}(s) = \sum_{m=1}^{\infty} l_{T,m}^* m^{-s}, \tag{4.22}$$

then $a_{x,T,m}^*$ (resp. $l_{T,m}^*$) is the sum of the terms $\prod_{v \notin T} a_{x,v,i_v}$ (resp. $\prod_{v \notin T} l_{v,i_v}$) over all possible factorizations $m = \prod_{v \notin T} q_v^{i_v}$. Since the number of such factorizations is finite, this sum is well-defined. It follows from Conditions 4.20 and 4.21 that $0 \leq a_{x,T,m}^* \leq l_{T,m}^*$ and that $a_{x,T,1}^* = 1$ for all $x \in V_k^{\mathrm{ss}}$, all $T \supseteq S$ and all $m \geq 1$. We shall use these observations in the proof of Theorem 4.29 below.

We define

$$\xi_{\omega_T}(s) = \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}, x \approx \omega_T} \Delta_x^{-s} V(x) \Xi_{x,T}(s) \tag{4.23}$$

and

$$\xi_{\omega_S,T}(s) = \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}, x \approx \omega_S} \Delta_x^{-s} V(x) \Xi_{x,T}(s), \tag{4.24}$$

which is the sum of $\xi_{\omega_T}(s)$ over all $\omega_T = (\omega_v)_{v \in T}$ which extends the fixed S-tuple ω_S. In order to determine the analytic properties of these Dirichlet series we require the following lemma. The proof is similar to that in Lemma 6.17 [12, p. 534], and so we leave it to the reader.

Lemma 4.25. *Let $v \in \mathfrak{M}$, $x \in V_{k_v}^{\mathrm{ss}}$ and $r \in \mathbb{C}$. Then there exists $\Phi_v \in \mathcal{S}(V_{k_v})$ such that the support of Φ_v is contained in $G_{k_v} x$, $Z_{x,v}(\Phi_v, s)$ is an entire function and $Z_{x,v}(\Phi_v, r) \neq 0$.*

Proposition 4.26. *Let $T \supseteq S$ be a finite set of places of k, and ω_T a T-tuple as above. The Dirichlet series $\xi_{\omega_T}(s)$ has a meromorphic continuation to the region $\mathrm{Re}(s) > \frac{n}{2}$. Its only possible singularity in this region is a simple pole at $s = \frac{n+1}{2}$ with residue*

$$\mathfrak{R}_2 \prod_{v \in T} b_{\omega_v,v}^{-1} |P(\omega_v)|_v^{\frac{n+1}{2}},$$

where

$$\mathfrak{R}_2 = \mathfrak{R}_1^{-1} |\Delta_k|^{-\frac{n(n+1)}{4}} \mathfrak{V}_n = 2|\Delta_k|^{-\frac{n}{2}} \mathfrak{C}_k^n \mathfrak{V}_n.$$

Proof. For each $v \in T$ we choose $\Phi_v \in \mathcal{S}(V_{k_v})$ so that $\mathrm{supp}(\Phi_v) \subseteq G_{k_v} \omega_v$. Let

$$\Phi = \bigotimes_{v \in T} \Phi_v \otimes \bigotimes_{v \notin T} \Phi_{v,0} \in \mathcal{S}(V_{\mathbb{A}}).$$

For $v \in T$ we have $\Xi_{x,v}(\Phi_v, s) = 0$ unless $x \approx \omega_v$ and hence

$$\begin{aligned} Z(\Phi, s) &= \mathfrak{R}_1 \big(\prod_{v \in T} \Xi_{\omega_v,v}(\Phi_v, s) \big) \sum_{x \in G_k \backslash V_k^{\mathrm{ss}}, x \approx \omega_T} \Delta_x^{-s} V(x) \Xi_{x,T}(s) \\ &= \mathfrak{R}_1 \big(\prod_{v \in T} \Xi_{\omega_v,v}(\Phi_v, s) \big) \xi_{\omega_T}(s) \end{aligned}$$

by Proposition 4.18. By Lemma 4.25 and Theorem 4.8, this formula implies the first statement.

Now choose Φ_v for $v \in T$ so that $\Xi_{\omega_v,v}(\Phi_v, \frac{n+1}{2}) \neq 0$. It follows directly from the definition (4.11) that $\Xi_{\omega_v,v}(\Phi_v, \frac{n+1}{2}) = b_{\omega_v,v} |P(\omega_v)|_v^{-\frac{n+1}{2}} \Sigma_v(\Phi_v)$ for all $v \in T$, and so the residue of $\xi_{\omega_T}(s)$ at $s = \frac{n+1}{2}$ is

$$\mathfrak{R}_1^{-1} \big(\prod_{v \in T} b_{\omega_v,v}^{-1} |P(\omega_v)|_v^{\frac{n+1}{2}} \big) \big(\prod_{v \in T} \Sigma_v(\Phi_v) \big)^{-1} \mathrm{Res}_{s=\frac{n+1}{2}} Z(\Phi, s).$$

We have $\Sigma_v(\Phi_{v,0}) = 1$ for $v \notin T$ and hence Theorem 4.8 and (4.10) imply that

$$\mathrm{Res}_{s=\frac{n+1}{2}} Z(\Phi, s) = |\Delta_k|^{-\frac{n(n+1)}{4}} \mathfrak{V}_n \prod_{v \in T} \Sigma_v(\Phi_v).$$

The last two equations imply that the residue of $\xi_{\omega_T}(s)$ at $s = \frac{n+1}{2}$ is

$$\mathfrak{R}_1^{-1}|\Delta_k|^{-\frac{n(n+1)}{4}}\mathfrak{V}_n\big(\prod_{v\in T} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^{\frac{n+1}{2}}\big),$$

which gives the second claim. ☐

Corollary 4.27. *The Dirichlet series $\xi_{\omega_S,T}(s)$ has a meromorphic continuation to the region* $\mathrm{Re}(s) > \frac{n}{2}$*. Its only possible singularity in this region is a simple pole at $s = \frac{n+1}{2}$ with residue*

$$\mathfrak{R}_2\big(\prod_{v\in S} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^{\frac{n+1}{2}}\big)\cdot\prod_{v\in T\setminus S}\sum_x\big(b_{x,v}^{-1}|P(x)|_v^{\frac{n+1}{2}}\big),$$

where the sum is over all standard orbital representatives for $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$.

Proof. We have $\xi_{\omega_S,T}(s) = \sum_{\omega_T}\xi_{\omega_T}(s)$ where the sum is over all T-tuples ω_T which extend the S-tuple ω_S. The claim follows immediately. ☐

We let $E_v = \sum_x b_{x,v}^{-1}|P(x)|_v^{\frac{n+1}{2}}$ for $v \notin S_0$, where the sum is over all standard orbital representatives, x, for orbits in $G_{k_v}\backslash V_{k_v}^{\mathrm{ss}}$. In Section 6 we shall prove that the following condition holds.

Condition 4.28. The product $\prod_{v\notin S_0} E_v$ converges to a positive number.

We are now ready to state and prove, subject to Conditions 4.20, 4.21 and 4.28, the theorem which is the goal of this section.

Theorem 4.29. *Let $S \supseteq S_0$ be a finite set of places of k and ω_S an S-tuple of standard orbital representatives. Then*

$$\lim_{X\to\infty} X^{-\frac{n+1}{2}}\sum_{\substack{x\in G_k\backslash V_k^{\mathrm{ss}}, x\approx\omega_S\\ \Delta_x\le X}} V(x) = \frac{2}{n+1}\mathfrak{R}_2\prod_{v\in S}\big(b_{\omega_v,v}^{-1}|P(\omega_v)|_v^{\frac{n+1}{2}}\big)\cdot\prod_{v\notin S} E_v\,.$$

Proof. In the following, sums over x will be understood to include the conditions $x \in G_k\backslash V_k^{\mathrm{ss}}$ and $x \approx \omega_S$ as well as any further conditions which may be explicitly imposed.

We have $\xi_{\omega_S,T}(s) = \sum_{m=1}^{\infty} c_m m^{-s}$ where

$$c_m = \sum_{x,n,\Delta_x d=m} V(x)a_{x,T,d}^*\,.$$

Applying the Tauberian theorem (Theorem I [17, p. 464]) to $\xi_{\omega_S,T}(s)$, we obtain

$$\lim_{X\to\infty} X^{-\frac{n+1}{2}}\sum_{x,d,\Delta_x d\le X} V(x)a_{x,T,d}^* = \tfrac{2}{n+1}\mathfrak{R}_2\big(\prod_{v\in S} b_{\omega_v,v}^{-1}|P(\omega_v)|_v^{\frac{n+1}{2}}\big)\cdot\prod_{v\in T\setminus S} E_v\,.$$

We shall denote the right hand side of this equation by $\mathfrak{L}_T$. Note that $\mathfrak{L} = \lim_{T\to\mathfrak{M}}\mathfrak{L}_T$ is the right hand side of the equation in the statement. Since $a_{x,T,d}^* \ge 0$ for all d and $a_{x,T,1}^* = 1$ we obtain

$$\limsup_{X\to\infty} X^{-\frac{n+1}{2}}\sum_{\Delta_x\le X} V(x) \le \mathfrak{L}_T$$

for all T, and so $\limsup_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\Delta_x \le X} V(x) \le \mathfrak{L}$. It follows that there is a constant C such that $\sum_{\Delta_x \le X} V(x) \le C X^{\frac{n+1}{2}}$ for all $X > 0$ (note that if $X < 1$ then the sum is 0). Furthermore,

$$\begin{aligned}
\sum_{\Delta_x \le X} V(x) &= \sum_{\Delta_x d \le X} V(x) a^*_{x,T,d} - \sum_{\Delta_x d \le X, d \ge 2} V(x) a^*_{x,T,d} \\
&\ge \sum_{\Delta_x d \le X} V(x) a^*_{x,T,n} - \sum_{\Delta_x d \le X, d \ge 2} V(x) l^*_{T,d} \\
&= \sum_{\Delta_x d \le X} V(x) a^*_{x,T,d} - \sum_{d=2}^{\infty} l^*_{T,d} \sum_{\Delta_x \le X/d} V(x) \\
&\ge \sum_{\Delta_x d \le X} V(x) a^*_{x,T,d} - C X^{\frac{n+1}{2}} \sum_{d=2}^{\infty} l^*_{T,d} d^{-\frac{n+1}{2}} \\
&= \sum_{\Delta_x d \le X} V(x) a^*_{x,T,d} - C X^{\frac{n+1}{2}} \left(L_T(\tfrac{n+1}{2}) - 1\right).
\end{aligned}$$

It follows that, for all $T \supseteq S$,

$$\liminf_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\Delta_x \le X} V(x) \ge \mathfrak{L}_T - C(L_T(\tfrac{n+1}{2}) - 1)$$

and letting $T \to \mathfrak{M}$ we obtain

$$\liminf_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\Delta_x \le X} V(x) \ge \mathfrak{L}$$

since $\lim_{T\to\mathfrak{M}} L_T(\frac{n+1}{2}) = 1$. ☐

Now we can remove the assumption $S \supseteq S_0$ from Theorem 4.29.

Theorem 4.30. *Let S be a finite set of places. Then the statement of Theorem 4.29 holds without the assumption $S \supseteq S_0$.*

Proof. Let $S_1 = S \cup S_0$ and choose an S_1-tuple, $\omega_{S_1} = (\omega_v)$ extending ω_S. According to Theorem 4.29,

$$\lim_{X\to\infty} X^{-\frac{n(n+1)}{2}} \sum_{\substack{x \approx \omega_{S_1} \\ \Delta_x \le X}} V(x) \tag{4.31}$$

exists and equals

$$\frac{2}{n+1} \mathfrak{R}_2 \prod_{v \in S_1} \left(b^{-1}_{\omega_v, v} |P(\omega_v)|_v^{\frac{n+1}{2}} \right) \cdot \prod_{v \notin S_1} E_v \,.$$

Taking the summation over all ω_{S_1} which extends ω_S, we obtain the statement of Theorem 4.29 without the assumption $S \supseteq S_0$. ☐

5. THE EXPLICIT FORM OF THE DENSITY THEOREM

In this section we shall make the constant in Theorem 4.30 more explicit. We remind the reader that we still assume that $n = 2r \geq 2$ is an even integer.

We first make the constant $\mathfrak{R}_2$ in Theorem 4.30 more explicit. By the definitions (2.4), (4.4) of $\mathfrak{V}_n$ and $Z_k(s)$,

$$\mathfrak{R}_2 = 2|\Delta_k|^{-\frac{n}{2}} \mathfrak{C}_k Z_k(2) \cdots Z_k(n)$$

Let $\Gamma_v(z)$ for $v \in \mathfrak{M}_\infty$ be as in (1.2). For $v \in \mathfrak{M}_{\mathrm{f}}$, we put $\Gamma_v(s) = (1 - q_v^{-s})^{-1}$. Then $Z_k(s) = |\Delta_k|^{\frac{s}{2}} \prod_{v \in \mathfrak{M}} \Gamma_v(s)$. So

$$\begin{aligned} \mathfrak{R}_2 &= 2|\Delta_k|^{-\frac{n}{2}+\frac{1}{2}(2+3+\cdots+n)} \mathfrak{C}_k \prod_{v \in \mathfrak{M}} \Gamma_v(2) \cdots \Gamma_v(n) \\ &= 2|\Delta_k|^{-\frac{(n-2)(n+1)}{4}} \mathfrak{C}_k \prod_{v \in \mathfrak{M}} \Gamma_v(2) \cdots \Gamma_v(n). \end{aligned} \tag{5.1}$$

For $v \in \mathfrak{M}_{\mathrm{f}}$ we put

$$\varepsilon_v(x) = b_{x,v}^{-1} |P(x)|_v^{\frac{n+1}{2}}, \quad E_v = \sum_x \varepsilon_v(x), \tag{5.2}$$

where $x \in V_{k_v}^{\mathrm{ss}}$ in the definition of $\varepsilon_v(x)$ and the sum is taken over all standard orbital representatives for $G_{k_v} \backslash V_{k_v}^{\mathrm{ss}}$ in the definition of E_v. By (5.1), the density in Theorem 4.29 is

$$m(\omega_S) = \frac{4|\Delta_k|^{-\frac{(n-2)(n+1)}{4}} \mathfrak{C}_k}{n+1} \prod_{v \in S} (\varepsilon_v(\omega_v) \Gamma_v(2) \cdots \Gamma_v(n)) \prod_{v \notin S} (E_v \Gamma_v(2) \cdots \Gamma_v(n)). \tag{5.3}$$

If $v \in \mathfrak{M}_{\mathrm{f}}$ then we put

$$\overline{\varepsilon}_v(x) = \sum_{y \asymp x} \varepsilon_v(y) \tag{5.4}$$

where y runs through all standard orbital representatives such that $y \asymp x$. Note that the notions $x \approx y$ and $x \asymp y$ differ only if x, y are of the type (rm). If x is an orbit of the type (rm) then we can compute $\varepsilon_v(x)$ if $v \notin \mathfrak{M}_{\mathrm{dy}}$. However, computing $\varepsilon_v(x)$ for individual orbits seems difficult if $v \in \mathfrak{M}_{\mathrm{dy}}$, and so we only use $\overline{\varepsilon}_v(x)$ in the density theorem. We put

$$\overline{m}(\omega_S) = \frac{4|\Delta_k|^{-\frac{(n-2)(n+1)}{4}} \mathfrak{C}_k}{n+1} \prod_{v \in S} (\overline{\varepsilon}_v(\omega_v) \Gamma_v(2) \cdots \Gamma_v(n)) \prod_{v \notin S} (E_v \Gamma_v(2) \cdots \Gamma_v(n)). \tag{5.5}$$

Lemma 5.6. *Let $v \in \mathfrak{M}_{\mathrm{f}}$ and $x \in V_{k_v}^{\mathrm{ss}}$. Then*

$$\varepsilon_v(x) = \mathrm{vol}(K_v x),$$

where the volume is with respect to the measure on V_{k_v} such that $\mathrm{vol}(V_{\mathcal{O}_v}) = 1$.

Proof. Let f be the characteristic function of K_v. We evaluate $\int_{G_{k_v}} f(g_v) dg_v$ using (4.6) as follows. By definition

$$\int_{G_{k_v}} f(g_v) dg_v = b_{x,v} \int_{G_{k_v}/G_{x\,k_v}^\circ} \left(\int_{G_{x\,k_v}^\circ} f(g'_{x,v} g''_{x,v}) dg''_{x,v} \right) dg'_{x,v}.$$

Index	$\overline{\varepsilon}_v(\omega_v)\Gamma_v(2)\cdots\Gamma_v(n)$
(sp)	$\frac{A(n,v)}{2}(1-q_v^{-1})(1+q_v^{-r})$
(in)	$\frac{A(n,v)}{2}(1-q_v^{-1})(1-q_v^{-r})$
(rm)	$A(n,v)q_v^{-1}(1-q_v^{-1})(1-q_v^{-2r})$
(dq)	$\frac{A(n,v)}{4}q_v^{-3}\frac{(1-q_v^{-1})^2(1-q_v^{-(r-1)})(1-q_v^{-2r})}{1-q_v^{-2}}$

TABLE 1. $\overline{\varepsilon}_v(x)$ for v finite

Let $g'_{x,v} \in G_{k_v}$. We first integrate $f(g'_{x,v}g''_{x,v})$ with respect to $g''_{x,v}$.

If $g'_{x,v} \in G_{k_v}$, $g''_{x,v} \in G^\circ_{x\,k_v}$ and $f(g'_{x,v}g''_{x,v}) \neq 0$ then $g'_{x,v}g''_{x,v} \in K_v$. So $g'_{x,v} \in K_v G^\circ_{x\,k_v}$. Suppose that $\kappa \in K_v$, $h \in G^\circ_{x\,k_v}$ and $g'_{x,v} = \kappa h$. If $g''_{x,v} \in G^\circ_{x\,k_v}$ then $g'_{x,v}g''_{x,v} \in K_v$ if and only if $g''_{x,v} \in h^{-1}(G^\circ_{x\,k_v} \cap K_v)$. We have chosen the measure on $G^\circ_{x\,k_v}$ so that the volume of $G^\circ_{x\,k_v} \cap K_v$ is 1. Since $dg''_{x,v}$ is an invariant measure, the volume of $h^{-1}(G^\circ_{x\,k_v} \cap K_v)$ is also 1. Therefore,

$$\int_{G^\circ_{x\,k_v}} f(g'_{x,v}g''_{x,v})dg''_{x,v} = \begin{cases} 1 & g'_{x,v} \in K_v G^\circ_{x\,k_v}, \\ 0 & \text{otherwise}. \end{cases}$$

By Proposition 3.14, $K_v G^\circ_{x\,k_v} = K_v G_{x\,k_v}$. So $\int_{G^\circ_{x\,k_v}} f(g'_{x,v}g''_{x,v})dg''_{x,v}$ is the characteristic function of $K_v G_{x\,k_v}$. If Φ is the characteristic function of $K_v x$ then $g'_{x,v} \to \Phi(g'_{x,v}x)$ is the characteristic function of $K_v G_{x\,k_v}$. Therefore,

$$1 = \int_{G_{k_v}} f(g_v)dg_v = b_{x,v}\int_{K_v G_{x\,k_v}/G^\circ_{x\,k_v}} dg'_{x,v} = b_{x,v}\int_{K_v x} |P(y)|_v^{-\frac{n+1}{2}}\,dy.$$

But $|P(y)|_v = |P(x)|_v$ for all $y \in K_v x$, and so

$$1 = b_{x,v}|P(x)|_v^{-\frac{n+1}{2}}\operatorname{vol}(K_v x).$$

So the lemma follows. □

Let $n = 2r$ and

$$A(n,v) = \frac{1}{\prod_{j=1}^{r}(1-q_v^{-2j})} = \Gamma_v(2)\Gamma_v(4)\cdots\Gamma_v(n). \tag{5.7}$$

We record the values of $\overline{\varepsilon}_v(\omega_v)\Gamma_v(2)\cdots\Gamma_v(n)$ for $v \in \mathfrak{M}_{\mathrm{f}}$ in Table 1. Note that the type (dq) occurs only if $n \geq 4$. The first column displays the type of the orbit and the second, $\overline{\varepsilon}_v(x)\Gamma_v(2)\cdots\Gamma_v(n)$, where x is the standard orbital representative for the orbit. The values of $\overline{\varepsilon}_v(x)\Gamma_v(2)\cdots\Gamma_v(n)$ will be computed in Section 6.

We recall that if $v \in \mathfrak{M}_{\mathbb{R}}$ then $\{w_{v,\mathbb{i}} \mid \mathbb{i} = 0,\ldots,r\}$ is the set of standard orbital representatives where $w_{v,\mathbb{i}}$ is defined in (3.13). If $v \in \mathfrak{M}_{\mathbb{C}}$ then $w_{v,\mathrm{sp}} = I_n$ is the only standard orbital representative.

We state the values of $\varepsilon_v(\omega_v)\Gamma_v(2)\cdots\Gamma_v(n)$ at infinite places in the following proposition.

Proposition 5.8. (1) *If* $v \in \mathfrak{M}_{\mathbb{R}}$ *then*

$$\varepsilon_v(w_{v,\mathbb{i}})\Gamma_v(2)\cdots\Gamma_v(n) = 2^{-n+\frac{\mathbb{i}(n-\mathbb{i}+1)}{2}} \prod_{1\le j\le \mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1\le j\le n-\mathbb{i}} \Gamma_{\mathbb{R}}(j).$$

(2) *If* $v \in \mathfrak{M}_{\mathbb{C}}$ *then*

$$\varepsilon_v(w_{v,\mathrm{sp}})\Gamma_v(2)\cdots\Gamma_v(n) = 2^{-3r}(4\pi)^r \prod_{1\le j\le n} \Gamma_{\mathbb{R}}(j).$$

There is no misprint in (2) and we did mean $\Gamma_{\mathbb{R}}(j)$. Proposition 5.8 will be proved in Section 9. We shall discuss that these values are compatible with Proposition 4.4 [1] if $n = 2$ at the end of Section 10.

The following proposition follows from Table 1 immediately.

Proposition 5.9. *Let* $v \in \mathfrak{M}_{\mathrm{f}}$.

(1) *If* $n \ge 4$ *then* $E_v\Gamma_v(2)\cdots\Gamma_v(n)$ *is equal to*

$$A(n,v)\left(1 - q_v^{-2} - q_v^{-2r-1} + q_v^{-2r-2} + \frac{1}{4}q_v^{-3}\frac{(1-q_v^{-1})^2(1-q_v^{-(r-1)})(1-q_v^{-2r})}{1-q_v^{-2}}\right).$$

(2) *If* $n = 2$ *then then* $E_v\Gamma_v(2)\cdots\Gamma_v(n)$ *is equal to*

$$A(2,v)(1-q_v^{-2}-q_v^{-3}+q_v^{-4}) = (1-q_v^{-2})^{-1}(1-q_v^{-2}-q_v^{-3}+q_v^{-4}).$$

Since $\prod_v \Gamma_v(2)\cdots\Gamma_v(n)$ converges absolutely, Proposition 5.9 implies that Condition 4.28 is satisfied. Therefore, if we can verify Conditions 4.20, 4.21 and 4.28 and carry out the computations of $b_{x,v}$, then we obtain the following theorem by Theorems 4.29, 4.30.

Theorem 5.10. *Let* $S \supset \mathfrak{M}_\infty$ *be a finite set of places of* k *and* ω_S *an* S*-tuple of standard orbital representatives. Then*

$$\lim_{X\to\infty} X^{-\frac{n+1}{2}} \sum_{\substack{x\asymp\omega_S\\ \Delta_x\le X}} V(x) = \overline{m}(\omega_S),$$

where $\overline{m}(\omega_S)$ *is defined in (5.5),* $\bar{\varepsilon}_v(x)\Gamma_v(2)\cdots\Gamma_v(n)$ *and* $E_v\Gamma_v(2)\cdots\Gamma_v(n)$ *for* $v\in\mathfrak{M}_f$ *are given in Table* 1 *and Proposition 5.9, and* $\bar{\varepsilon}_v(x)\Gamma_v(2)\cdots\Gamma_v(n)$ *for* $v\in\mathfrak{M}_\infty$ *is given in Proposition 5.8.*

Note that in Theorem 5.10, S does not have to contain S_0. The rest of this paper will be devoted to verifying Conditions 4.20, 4.21 and 4.28 and to the computation of $b_{x,v}$.

6. $\varepsilon_v(x)$ FOR FINITE PLACES

Throughout this section, $v \in \mathfrak{M}_{\mathrm{f}}$. In this section we shall compute $\overline{\varepsilon}_v(x)$ for all standard orbital representatives x. We continue to assume that $n = 2r$ is even. We denote $\mathcal{O}_v/\mathfrak{p}_v$ by $\mathbb{F}_v$ and express reduction modulo $\mathfrak{p}_v$ as $\overline{w}_{v,\mathrm{sp}}$, etc. In the following, if $x = (x_{ij}), y = (y_{ij}) \in V_{\mathcal{O}_v}$ and $d \in \mathbb{N}$ then we use the notation $x \equiv y \mod \mathfrak{p}_v^d$, etc., if $x_{ij} \equiv y_{ij} \mod \mathfrak{p}_v^d$ for all i, j. Note that we use the coordinate system (3.3), and 2's in the diagonal entries of (3.4) are not taken into the consideration.

Let $w_{n,v,\mathrm{sp}}$, etc., be as in (3.10), (3.11). The orders of $\mathrm{GO}(\overline{w}_{v,\mathrm{sp}})_{\mathbb{F}_v}$, etc., are known as follows (see [11, pp. 146–147] and Lemma 11.2 [8])

$$
\begin{aligned}
\#\mathrm{GO}(\overline{w}_{2r,v,\mathrm{sp}})_{\mathbb{F}_v} &= 2q_v^{2r^2-r+1}(1-q_v^{-1})(1+q_v^{-r})^{-1}\prod_{j=1}^{r}(1-q_v^{-2j}), \\
\#\mathrm{GO}(\overline{w}_{2r+1,v,\mathrm{sp}})_{\mathbb{F}_v} &= q_v^{2r^2+r+1}(1-q_v^{-1})\prod_{j=1}^{r}(1-q_v^{-2j}), \qquad (6.1)\\
\#\mathrm{GO}(\overline{w}_{2r,v,\mathrm{in}})_{\mathbb{F}_v} &= 2q_v^{2r^2-r+1}(1-q_v^{-1})(1-q_v^{-r})^{-1}\prod_{j=1}^{r}(1-q_v^{-2j}).
\end{aligned}
$$

Let $A(n, v)$ be as in (5.7). We recall that $|P(w_{v,\mathrm{sp}})|_v = |P(w_{v,\mathrm{in}})|_v = 1$.

Proposition 6.2. *Let $n = 2r$. Then*

$$
\begin{aligned}
\varepsilon_v(w_{v,\mathrm{sp}})\Gamma_v(2)\cdots\Gamma_v(n) &= \frac{A(n,v)}{2}(1-q_v^{-1})(1+q_v^{-r}), \\
\varepsilon_v(w_{v,\mathrm{in}})\Gamma_v(2)\cdots\Gamma_v(n) &= \frac{A(n,v)}{2}(1-q_v^{-1})(1-q_v^{-r}).
\end{aligned}
$$

Proof. Let $\mathbb{i} = \mathrm{sp}$ or in. Let $\mathcal{D} \subseteq V_{\mathcal{O}_v}$ be the set of elements x such that $x \equiv w_{v,\mathbb{i}} \mod \mathfrak{p}_v$. We first prove the following lemma.

Lemma 6.3. For $\mathbb{i} = \mathrm{sp}$ or in, $\mathcal{D} \subseteq K_v w_{v,\mathbb{i}}$.

Proof. Since the proof is similar, we only consider the case $\mathbb{i} = \mathrm{in}$, which is slightly more non-trivial than the case $\mathbb{i} = \mathrm{sp}$.

Let

$$
J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{M}(r-1, r-1)_{\mathcal{O}_v}, \quad w' = \begin{pmatrix} & & J \\ & A_{v,\mathrm{in}} & \\ J & & \end{pmatrix}
$$

Then it is easy to see that there exists $h \in K_v$ such that $w' = h w_{v,\mathrm{in}}$.

Suppose that we can show that $x \equiv w' \mod \mathfrak{p}_v$ implies $x \in K_v w'$. Then

$$
\begin{aligned}
x \equiv w_{v,\mathrm{in}} \mod \mathfrak{p}_v &\Longrightarrow hx \equiv w' \mod \mathfrak{p}_v \\
&\Longrightarrow hx \in K_v w' \\
&\Longrightarrow x \in h^{-1} K_v h w_{v,\mathrm{in}} = K_v w_{v,\mathrm{in}}.
\end{aligned}
$$

So it is enough to consider w' instead.

Let $n(u)$ be as in (4.1). Suppose that $x \equiv w' \mod \mathfrak{p}_v$. We first consider $y = (y_{ij}) = n(u)x$ and try to choose $u = (u_{ij}) \in \mathfrak{p}_v^{\frac{n(n+1)}{2}}$ so that $y_{ij} = 0$ for $i + j \geq n + 2$ except for $(i, j) = (r + 1, r + 1)$. Then we consider $z = (z_{ij}) = {}^t n(u')y$ and try to choose $u' = (u'_{ij}) \in \mathfrak{p}_v^{\frac{n(n+1)}{2}}$ so that $z_{ij} = 0$ for $i + j \leq n$ except for $(i, j) = (r, r)$.

Let $Y = (y_{ij})_{i+j \geq n+2, (i,j) \neq (r+1,r+1)}$ and $Z = (z_{ij})_{i+j \leq n, (i,j) \neq (r,r)}$. If $u_{ij}, u'_{ij} \in \mathfrak{p}_v$ for all i, j then Y, Z are elements of $\mathfrak{p}_v^{r(2r-1)-1}$. In both cases we apply Hensel's lemma in the form [7, p. 64]. So we only have to verify that the reduction modulo $\mathfrak{p}_v$ of the differentials of the maps $u \to Y$, $u' \to Z$ at the origin are surjective.

Since the situation is similar, we only consider the map $u \to Y$. Consider the ring of dual numbers $\mathbb{F}_v[\epsilon]/(\epsilon^2)$. Let $N = (N_{ij}) \in \mathrm{M}(n, n)_{\mathbb{F}_v}$ be the lower triangular matrix with diagonal entries 0 such that the (i, j)-entry is N_{ij} for $i > j$. The origin $u = 0$ corresponds to the unit matrix $n(0) = I_n$. So we identify tangent vectors at the origin $u = 0$ with elements of the form $I_n + \epsilon N$. Then

$$(I_n + \epsilon N)\overline{w}'{}^t(I_n + \epsilon N) = \overline{w}' + \epsilon(N\overline{w}' + \overline{w}'{}^t N)$$

($\overline{w}'$ is the reduction modulo $\mathfrak{p}$ of w').

Suppose that N is in the following form

$$N = \begin{pmatrix} N_1 & 0 & 0 \\ N_2 & 0 & 0 \\ N_3 & 0 & 0 \end{pmatrix}$$

where $N_1, N_3 \in \mathrm{M}(r-1, r-1)_{k_v}$, $N_2 \in \mathrm{M}(2, r-1)_{k_v}$ and N_1 is lower triangular with diagonal entries 0. Then

$$N\overline{w}' + \overline{w}'{}^t N = \begin{pmatrix} 0 & 0 & N_1 J \\ 0 & 0 & N_2 J \\ J^t N_1 & J^t N_2 & N_3 J + J^t N_3 \end{pmatrix}.$$

We choose N_3 so that $N_4 = (n'_{ij}) = N_3 J$ is lower triangular. Then $N_3 J + J^t N_3 = N_4 + {}^t N_4$ and this can be an arbitrary symmetric matrix. Note that the diagonal entries of $N_4 + {}^t N_4$ are $2n'_{11}, \dots, 2n'_{r-1r-1}$. As we discussed after (3.4), 2's in the coefficients are formal variables and so do not have any effect even if $\mathrm{ch}\, k = 2$.

Other parts are obvious and matrices of the above form exhaust matrices of the form

$$\begin{pmatrix} 0 & 0 & Y_1 \\ 0 & 0 & Y_2 \\ {}^t Y_1 & {}^t Y_2 & Y_3 \end{pmatrix}$$

where Y_3 is symmetric and entries of Y_1 above the anti-diagonal line are 0.

This proves that the reduction modulo $\mathfrak{p}_v$ of the differential of the map $u \to Y$ is surjective. The consideration is similar for the map $u' \to Z$. Therefore, we may assume that entries of x are 0 except for the anti-diagonal line and the central $(2, 2)$-block. This reduces the problem to the case $n = 2$.

Similarly as above, we consider

$$(I_2 + \epsilon N)\overline{A}_{v,\mathrm{in}}{}^t(I_2 + \epsilon N)$$

for $N \in \mathrm{M}(2, 2)_{\mathbb{F}_v}$. If $N = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & 0 \end{smallmatrix}\right)$ then

$$N\overline{A}_{v,\mathrm{in}} + \overline{A}_{v,\mathrm{in}}{}^t N = \begin{pmatrix} 2(2\overline{a}\alpha + b\beta) & \overline{b}\alpha + 2\overline{c}\beta + 2\overline{a}\gamma \\ \overline{b}\alpha + 2\overline{c}\beta + 2\overline{a}\gamma & 2\overline{b}\gamma \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} 2\overline{a} & \overline{b} & 0 \\ \overline{b} & 2\overline{c} & 2\overline{a} \\ 0 & 0 & \overline{b} \end{pmatrix} = (4\overline{ac} - \overline{b}^{\,2})\overline{b} \neq 0$$

in $\mathbb{F}_v$, $N\overline{A}_{v,\mathrm{in}} + \overline{A}_{v,\mathrm{in}}{}^t N$ can be an arbitrary symmetric matrix.

By Hensel's lemma, this completes the proof of the lemma. $\square$

We continue the proof of Proposition 6.2. If $g \in K_v$, $x, y \in \mathcal{D}$ and $gx = y$ then $g \mod \mathfrak{p}_v$ is an element of the stabilizer $G_{\overline{w}'\mathbb{F}_v}$, which is isomorphic to $\mathrm{GO}(\overline{w}_{v,\mathrm{in}})_{\mathbb{F}_v}$. Conversely, if $g \mod \mathfrak{p}_v$ is an element of $G_{\overline{w}'\mathbb{F}_v}$, then $g\mathcal{D} \subseteq \mathcal{D}$. Therefore,

$$\mathrm{vol}(K_v w_{v,\mathrm{in}}) = q_v^{-\frac{n(n+1)}{2}} \frac{\# G_{\mathbb{F}_v}}{\# \mathrm{GO}(\overline{w}_{v,\mathrm{in}})_{\mathbb{F}_v}}.$$

Then Proposition 6.2 follows from Lemma 5.6, the third formula in (6.1) and the well-known formula for $\# G_{\mathbb{F}_v}$.

The computation of $\mathrm{vol}(K_v w_{v,\mathrm{sp}})$ is similar and slightly easier. $\square$

The following corollary is obvious.

Corollary 6.4. $(\varepsilon_v(w_{v,\mathrm{sp}}) + \varepsilon(w_{v,\mathrm{in}}))\Gamma_v(2) \cdots \Gamma_v(n) = A(n,v)(1 - q_v^{-1})$.

We next consider the type (rm). Let $w_{v,(\mathrm{rm},1)}, \ldots, w_{v,(\mathrm{rm},\lambda_v)}$ be the standard orbital representatives of the type (rm) as in Proposition 3.12.

Proposition 6.5. *We have*

$$\sum_{j=1}^{\lambda_v} \varepsilon_v(w_{v,(\mathrm{rm},j)})\Gamma_v(2) \cdots \Gamma_v(n) = A(n,v) q_v^{-1}(1 - q_v^{-1})(1 - q_v^{-n}).$$

Moreover, if $v \notin \mathfrak{M}_{\mathrm{dy}}$ *then*

$$\varepsilon_v(w_{v,(\mathrm{rm},j)})\Gamma_v(2) \cdots \Gamma_v(n) = \frac{1}{2} A(n,v) q_v^{-1}(1 - q_v^{-1})(1 - q_v^{-n})$$

for $j = 1, 2$.

Proof. Let $\mathcal{D}$ be the set of symmetric matrices of the form

$$x = \begin{pmatrix} & & & x_1 \\ & X & & \vdots \\ & & & x_{n-1} \\ x_1 & \cdots & x_{n-1} & 2x_n \end{pmatrix} \tag{6.6}$$

where $X \equiv w_{n-1,v,\mathrm{sp}} \mod \mathfrak{p}_v$, $x_1, \ldots, x_{2r-1} \in \mathfrak{p}_v$ and $\mathrm{ord}_v(x_n) = 1$.

Let $\mathcal{H}$ be the subgroup of K_v consisting of elements of the form

$$\left(t_0, \begin{pmatrix} & & & a_1 \\ & h & & \vdots \\ & & & a_{n-1} \\ b_1 & \cdots & b_{n-1} & c \end{pmatrix} \right)$$

where $t_0 h w_{n-2,v,\mathrm{in}} \equiv w_{n-2,v,\mathrm{in}} \mod \mathfrak{p}_v$, $b_1, \ldots, b_{n-1} \in \mathfrak{p}_v$ and $c \in \mathcal{O}_v^{\times}$.

Clearly, $w_{v,(\mathrm{rm},1)}, \ldots, w_{v,(\mathrm{rm},\lambda_v)} \in \mathcal{D}$.

Lemma 6.7. $\mathcal{D} \subseteq \cup_{j=1}^{\lambda_v} K_v w_{v,(\mathrm{rm},j)}$.

Proof. Consider elements of $\mathcal{D}$ in the form (6.6). By Lemma 11.5 [9], we may assume that $X = w_{n-1,v,\mathrm{sp}}$. Then it is easy find $g \in K_v$ so that

$$gx = \begin{pmatrix} H & & & & \\ & \ddots & & & \\ & & H & & \\ & & & 2 & y_{n-1} \\ & & & y_{n-1} & y_n \end{pmatrix}$$

and $y_{n-1} \in \mathfrak{p}_v$, $\mathrm{ord}_v(y_n) = 1$.

This reduces the consideration to the case $n = 2$. Then the proof is known (see [1, p. 222]), but we include the proof for the reader's sake.

Suppose that $n = 2$ and α is a root of the Eisenstein polynomial $z^2 + x_1 z + x_2 = 0$. By assumption, there exists i such that if π_v is a root of $z^2 + a_i z + b_i = 0$ then $k_v(\alpha) = k_v(\pi_v)$. Since both α, π_v are uniformizers, $\alpha = c\pi_v + d$ where $c \in \mathcal{O}_v^\times, d \in \mathfrak{p}_v$. Then

$$\begin{pmatrix} 2 & x_1 \\ x_1 & 2x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -d & c \end{pmatrix} \begin{pmatrix} 2 & a_i \\ a_i & 2b_i \end{pmatrix} {}^t\begin{pmatrix} 1 & 0 \\ -d & c \end{pmatrix}.$$

□

We continue the proof of Proposition 6.5. It is easy to see that if $g \in \mathcal{H}$ then $g\mathcal{D} \subseteq \mathcal{D}$. Conversely, suppose that $x, y \in \mathcal{D}$, $g \in K_v$ and $gx = y$. Let g be in the following form

$$g = \left(t_0, \begin{pmatrix} h & a \\ b & c \end{pmatrix}\right)$$

where $h \in \mathrm{M}(n-1, n-1)_{\mathcal{O}_v}$, $a \in \mathrm{M}(n-1, 1)_{\mathcal{O}_v}$, $b \in \mathrm{M}(1, n-1)_{\mathcal{O}_v}$ and $c \in \mathcal{O}_v$. By reduction modulo $\mathfrak{p}_v$, $\overline{g}\,\overline{x} = \overline{y}$. Since

$$\overline{x} = \overline{y} = \begin{pmatrix} \overline{w}_{n-1,v,\mathrm{sp}} & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$\overline{g}\,\overline{x} = \bar{t}_0 \begin{pmatrix} \overline{h}\overline{w}_{n-1,v,\mathrm{sp}}{}^t\overline{h} & \overline{w}_{n-1,v,\mathrm{sp}}{}^t\overline{b} \\ \bar{b}\overline{w}_{n-1,v,\mathrm{sp}}{}^t\overline{h} & \bar{b}\overline{w}_{n-1,v,\mathrm{sp}}{}^t\overline{b} \end{pmatrix}. \tag{6.8}$$

Therefore, $\bar{b} = 0$, which implies that $c \in \mathcal{O}_v^\times$. Hence, $g \in \mathcal{H}$.

By the above consideration,

$$\sum_{j=1}^{\lambda_v} \mathrm{vol}(K_v w_{v,(\mathrm{rm},j)}) = q_v^{-\frac{n(n-1)}{2}} q_v^{-(n-1)} q_v^{-1}(1 - q_v^{-1}) \frac{\#\mathrm{GL}(1)_{\mathbb{F}_v} \#\mathrm{GL}(n)_{\mathbb{F}_v}}{\#\mathrm{GO}(\overline{w}_{n-1,v,\mathrm{sp}})_{\mathbb{F}_v} q_v^{n-1}(q_v - 1)}.$$

In the above formula, $q_v^{-\frac{n(n-1)}{2}}$ is the volume of the set of $X \equiv w_{n-1,v,\mathrm{sp}}$, $q_v^{-(n-1)}$, $q_v^{-1}(1 - q_v^{-1})$ are the volumes of the sets of $(x_1, \ldots, x_{n-1}) \in \mathfrak{p}_v^{n-1}$, $x_n \in \mathfrak{p}_v \backslash \mathfrak{p}_v^2$ respectively and q_v^{n-1}, $(q_v - 1)$ are the number of possibilities for b, c modulo $\mathfrak{p}_v$ respectively. Then the first statement of Proposition 6.5 follows from Lemma 5.6 and the second formula in (6.1).

Suppose that $v \notin \mathfrak{M}_{\mathrm{dy}}$. Let π_v be a uniformizer and μ a non-square unit. There are exactly two orbits of the type (rm). They are orbits of elements of the form (6.6) where $x_n/\pi_v = 1, \mu$. It can easily be verified that whether or not x_n/π_v is a square does not change by the action of the group $\mathcal{H}$. So instead of $\mathcal{D}$, we define subsets $\mathcal{D}_1, \mathcal{D}_2$ according as x_n/π_v is a square or not. Since the volume of the set of $x_n \in \mathfrak{p}_v \backslash \mathfrak{p}_v^2$ such that x_n/π_v is a square is half the volume of $\mathfrak{p}_v \backslash \mathfrak{p}_v^2$, we obtain the second statement of Proposition 6.5. $\square$

We finally consider the type (dq).

Proposition 6.9. *If* $n = 2r \geq 4$ *then*

$$\varepsilon_v(w_{v,\mathrm{dq}})\Gamma_v(2)\cdots\Gamma_v(n) = A(n,v)\frac{q_v^{-3}(1-q_v^{-1})^2(1-q_v^{-(r-1)})(1-q_v^{-2r})}{4(1-q_v^{-2})}.$$

Proof. Let $\mathcal{D}$ be the set of symmetric matrices of the form

$$x = \begin{pmatrix} & & & x_1 & y_1 \\ & X & & \vdots & \vdots \\ & & & x_{n-2} & y_{n-2} \\ x_1 & \cdots & x_{n-2} & \multicolumn{2}{c}{\multirow{2}{*}{αY}} \\ y_1 & \cdots & y_{n-2} & & \end{pmatrix} \tag{6.10}$$

where $X \equiv w_{n-2,v,\mathrm{in}}$, $Y \equiv A_{v,\mathrm{in}} \mod \mathfrak{p}_v$, $x_i, y_i \in \mathfrak{p}_v$ for all i and $\mathrm{ord}_v(\alpha) = 1$.

Let $\mathcal{H}$ be the subgroup of K_v consisting of elements of the form

$$h = \left(t_0, \begin{pmatrix} & & & a_1 & b_1 \\ & h_1 & & \vdots & \vdots \\ & & & a_{n-2} & b_{n-2} \\ c_1 & \cdots & c_{n-2} & \multicolumn{2}{c}{\multirow{2}{*}{h_2}} \\ d_1 & \cdots & d_{n-2} & & \end{pmatrix} \right) \tag{6.11}$$

where $t_0 h_1 w_{n-2,v,\mathrm{in}} \equiv w_{n-2,v,\mathrm{in}} \mod \mathfrak{p}_v$, $c_i, d_i \in \mathfrak{p}_v$ for all i and $h_2 A_{v,\mathrm{in}} \equiv \gamma A_{v,\mathrm{in}} \mod \mathfrak{p}_v$ for some $\gamma \in \mathcal{O}_v^\times$.

It is easy to see that $\mathcal{H}\mathcal{D} \subseteq \mathcal{D}$.

Lemma 6.12. $\mathcal{D} \subseteq K_v w_{v,\mathrm{dq}}$.

Proof. Suppose that $x \in \mathcal{D}$ is in the form (6.10). By Lemma 6.3 (for $n-2$), we may assume that $X = w_{n-2,v,\mathrm{in}}$. Then it is easy to find an element $h \in \mathcal{H}$ so that if we replace x by hx then $x_i = y_i = 0$ for all i and $\alpha = \pi_v$.

Then we have to find $h \in \mathrm{GL}(2)_{\mathcal{O}_v}$ such that $hY = w_{v,\mathrm{in}}$. This was already carried out at the end of the proof of Proposition 6.5. $\square$

We continue the proof of Proposition 6.9. Suppose that $x, x' \in \mathcal{D}$, $h \in K_v$ and $hx = x'$. By reduction modulo $\mathfrak{p}_v$, $\overline{h}\,\overline{x} = \overline{x}'$. By a similar computation as in (6.8), h is in the form (6.11) and $t_0 h_1 w_{n-2,v,\mathrm{in}} \equiv w_{n-2,v,\mathrm{in}} \mod \mathfrak{p}_v$, $c_i, d_i \in \mathfrak{p}_v$ for all i. Then $h_2 \in \mathrm{GL}(2)_{\mathcal{O}_v}$.

By computation, if x is in the form (6.10) then the last $(2,2)$-block of hx is congruent modulo $\mathfrak{p}_v^2$ to $t_0 \alpha h_2 Y$. Suppose that the last $(2,2)$-block of x' is $\alpha' Y'$ and that $Y \equiv \gamma A_{v,\mathrm{in}}$, $Y' \equiv \gamma' A_{v,\mathrm{in}} \mod \mathfrak{p}_v$ with $\gamma, \gamma' \in \mathcal{O}_v^\times$. Then

$$t_0 \alpha h_2 A_{v,\mathrm{in}} \equiv t_0 \alpha h_2 Y \equiv \alpha' Y' \equiv \alpha' A_{v,\mathrm{in}} \mod \mathfrak{p}_v^2.$$

Since $\mathrm{ord}_v(\alpha) = \mathrm{ord}_v(\alpha') = 1$, there exists $\gamma'' \in \mathcal{O}_v^\times$ such that

$$h_2 A_{v,\mathrm{in}} \equiv \gamma'' A_{v,\mathrm{in}} \mod \mathfrak{p}_v.$$

Therefore, $h \in \mathcal{H}$.

By the above consideration,

$$\begin{aligned} \mathrm{vol}(K_v w_{v,\mathrm{dq}}) &= \mathrm{vol}(\mathcal{D})\#(K_v/\mathcal{H}) \\ &= q_v^{-\frac{(n-1)(n-2)}{2}} q_v^{-2(n-2)}(q_v-1)q_v^{-6} \\ &\quad \times \frac{\#\mathrm{GL}(1)_{\mathbb{F}_v}\#\mathrm{GL}(n)_{\mathbb{F}_v}}{\#\mathrm{GO}(\overline{w}_{n-2,v,\mathrm{in}})_{\mathbb{F}_v}\#\mathrm{GO}(\overline{w}_{2,v,\mathrm{in}})_{\mathbb{F}_v} q_v^{2(n-2)}}. \end{aligned}$$

Note that $q_v^{-\frac{(n-1)(n-2)}{2}}$ is the volume of the set of $X \equiv w_{n-2,v,\mathrm{in}}$ and that $q_v^{-2(n-2)}$ is the volume of the set of $x_1, \ldots, y_{n-2} \in \mathfrak{p}_v$. Since αY is determined modulo $\mathfrak{p}_v^2$ and there are $q_v - 1$ possibilities for α modulo $\mathfrak{p}_v^2$, the volume of the set of αY is $(q_v-1)q_v^{-6}$. Then Proposition 6.9 follows from Lemma 5.6 and the third formula in (6.1). □

7. The constant term of the q-expansion

In this section we formulate a way to prove that Condition 4.20 is satisfied. Since our argument works in a fairly general situation, we make enough definitions to state Proposition 7.3 below. We consider general prehomogeneous vector spaces in what follows.

Let $(\widetilde{G}, V)$ be an irreducible regular prehomogeneous vector space defined over k, $v \in \mathfrak{M}_{\mathrm{f}}$ and $\widetilde{K}_v \subseteq \widetilde{G}_{k_v}$ an open compact subgroup. Suppose that $P(x) \in k[V]$ is a non-constant polynomial, $\chi(g)$ a rational character of $\widetilde{G}$ and and that $P(gx) = \chi(g)P(x)$ for $g \in \widetilde{G}$ and $x \in V$. Since P is not a constant, χ is not the trivial character. Let $m = \dim V$ and $N = \deg P$. If (G, V) is the pair (3.1) then $m = \frac{n(n+1)}{2}$ and $N = n$.

Suppose that $x \in V_{k_v}^{\mathrm{ss}} = \{y \in V_{k_v} \mid P(y) \neq 0\}$. Since $(\widetilde{G}, V)$ is a regular prehomogeneous vector space, $\widetilde{G}_x$ is reductive (or one can make this the definition of the regularity) and so $\widetilde{G}_{x\,k_v}$ has a unimodular measure. We choose an invariant measure $d\widetilde{g}_v$ (resp. $d\widetilde{g}''_{x,v}$) on $\widetilde{G}_{k_v}$ (resp. $\widetilde{G}^\circ_{x\,k_v}$) so that $\mathrm{vol}(\widetilde{K}_v) = 1$ ($\mathrm{vol}(\widetilde{K}_v \cap \widetilde{G}^\circ_{x\,k_v}) = 1$).

Let $d\widetilde{g}'_{x,v}$ be the measure on $\widetilde{G}_{k_v}/\widetilde{G}^\circ_{x\,k_v}$ such that if f is a measurable function on $\widetilde{G}_{k_v}x$ then

$$\int_{\widetilde{G}_{k_v}/\widetilde{G}^\circ_{x\,k_v}} f(\widetilde{g}'_{x,v}x)d\widetilde{g}'_{x,v} = \int_{\widetilde{G}_{k_v}x} f(y)|P(y)|_v^{-\frac{m}{N}}dy.$$

There exists a constant $b_{x,v} > 0$ such that

$$d\widetilde{g}_v = b_{x,v}d\widetilde{g}'_{x,v}d\widetilde{g}''_{x,v}. \tag{7.1}$$

For $\Phi \in \mathcal{S}(V_{k_v})$ and $s \in \mathbb{C}$, we define

$$Z_{x,v}(\Phi, s) = b_{x,v}\int_{\widetilde{G}_{k_v}/\widetilde{G}^\circ_{x\,k_v}} |\chi(\widetilde{g}'_{x,v})|^s \Phi(\widetilde{g}'_{x,v}x)d\widetilde{g}'_{x,v}$$

We fix an integral structure on V_{k_v}. If Φ is the characteristic function of $V_{\mathcal{O}_v}$ then we denote $Z_{x,v}(\Phi, s)$ by $Z_{x,v}(s)$. We consider the following condition.

Condition 7.2. If $g \in \widetilde{G}_{k_v}$ and $gx \in V_{\mathcal{O}_v}$ then $\chi(g) \in \mathcal{O}_v$. Moreover, if $\chi(g) \in \mathcal{O}_v^\times$ then $g \in \widetilde{K}_v\widetilde{G}^\circ_{x\,k_v}$.

Since $\mathrm{ord}_v(\chi(g'_{x,v})) \in \mathbb{Z}$, we can express $Z_{x,v}(s)$ as $Z_{x,v}(s) = \sum_{d=-\infty}^{\infty} a_{x,d}q^{-ds}$.

Proposition 7.3. *If Condition 7.2 is satisfied then* $a_{x,d} = 0$ *if* $d < 0$ *and* $a_{x,0} = 1$.

Proof. Let $X_d = \{\widetilde{g}_v \in \widetilde{G}_{k_v} \mid \widetilde{g}_v x \in V_{\mathcal{O}_v},\ \mathrm{ord}_v(\chi(\widetilde{g}_v)) = d\}$. Then X_d is clearly right $\widetilde{G}_{x\,k_v}$-invariant. It is easy to see that $a_{x,d} = b_{x,v} \int_{X_d/\widetilde{G}^{\circ}_{x\,k_v}} d\widetilde{g}'_{x,v}$. Condition 7.2(2) implies that $X_d = \emptyset$ if $d < 0$, and so $a_{x,d} = 0$ if $d < 0$. Also $X_0 = \widetilde{K}_v \widetilde{G}^{\circ}_{x\,k_v}$. So

$$a_{x,0} = b_{x,v} \int_{\widetilde{K}_v \widetilde{G}^{\circ}_{x\,k_v}/\widetilde{G}^{\circ}_{x\,k_v}} d\widetilde{g}'_{x,v}.$$

Let $f(\widetilde{g}_v)$ be the characteristic function of $\widetilde{K}_v$. We integrate f on $\widetilde{G}_{k_v}$ using the decomposition $d\widetilde{g}_v = b_{x,v} d\widetilde{g}'_{x,v} d\widetilde{g}''_{x,v}$. The first part of the proof of Lemma 5.6 works and so

$$\int_{\widetilde{G}^{\circ}_{x\,k_v}} f(\widetilde{g}'_{x,v}\widetilde{g}''_{x,v}) d\widetilde{g}''_{x,v} = \begin{cases} 1 & \widetilde{g}''_{x,v} \in \widetilde{K}_v \widetilde{G}^{\circ}_{x\,k_v}, \\ 0 & \text{otherwise}. \end{cases}$$

So the above integral is the characteristic function of $\widetilde{K}_v \widetilde{G}^{\circ}_{x\,k_v}$. Therefore,

$$1 = \int_{\widetilde{G}_{k_v}} f(\widetilde{g}_v) d\widetilde{g}_v = b_{x,v} \int_{\widetilde{K}_v \widetilde{G}^{\circ}_{x\,k_v}/\widetilde{G}^{\circ}_{x\,k_v}} d\widetilde{g}'_{x,v} = a_{x,0}.$$

□

We now return to the prehomogeneous vector space (1.1). Proposition 7.3 can be used to show that the constant term of the q-expansion of the standard local orbital zeta function is 1. Proposition 5.23 [8] and Proposition 3.14 imply that Condition 7.2 is indeed satisfied. Therefore, we have the following proposition.

Proposition 7.4. *If* $x \in V^{\mathrm{ss}}_{k_v}$ *is a standard orbital representative then Condition 7.2 is satisfied. Therefore, Condition 4.20 is satisfied for* x *also.*

Remark 7.5. As pointed out at the end of Section 3 of [14], the constant term of the local orbital zeta function may not necessarily be 1. However, if the constant term is 1, then we expect that Condition 7.2 is satisfied. So Proposition 7.3 is a place to start if one tries to prove that the constant term is 1. In [12] the notion of omega sets was used to carry out this task. In general we no longer have to construct omega sets, but if we can find them, then it may be more efficient to use omega sets than to use Proposition 7.3.

8. A UNIFORM ESTIMATE OF THE LOCAL ORBITAL ZETA FUNCTIONS

The purpose of this section is to verify Condition 4.21. So we assume that $v \in \mathfrak{M} \setminus S_0$ and $x \in V^{\mathrm{ss}}_{k_v}$. We continue to assume that $n = 2r \geq 2$ is even.

Let $X_x = \{g_v \in G_v \mid g_v x \in V_{\mathcal{O}_v}\}$. By Proposition 7.4,

$$\Xi_{x,v}(s) - 1 = b_{x,v} \int_{\substack{X_x/G^{\circ}_{w_{v,i}\,k_v} \\ \chi(g'_{x,v}) \in \mathfrak{p}_v}} |\chi(g'_{x,v})|_v^s dg'_{x,v}.$$

Since n is even, $\mathrm{ord}_v(\chi(g))$ is always even. So if $y = g'_{x,v}x$ and $\chi(g'_{x,v}) \in \mathfrak{p}_v$ then $\mathrm{ord}_v(P(y)) \geq \mathrm{ord}_v(P(x)) + 2$ and $\mathrm{ord}_v(P(y)) - \mathrm{ord}_v(P(x))$ is even. Let

$$Y_x = \{y \in G_{k_v}x \cap V_{\mathcal{O}_v} \mid \mathrm{ord}_v(P(y)) \geq \mathrm{ord}_v(P(x)) + 2\}. \tag{8.1}$$

In the following, we very often use Y_x as the limit of integrals. It is easy to see that

$$\Xi_{x,v}(s) - 1 = b_{x,v}|P(x)|_v^{-s} \int_{Y_x} |P(y)|_v^{s-\frac{n+1}{2}} dy.$$

Since $v \notin S_0$, we have determined $b_{x,v}$ for all orbits in Propositions 6.2, 6.5 and 6.9. Let $A(n,v)$ be as in (5.7). We put $B(n,v) = \Gamma_v(2)\cdots\Gamma_v(n)A(n,v)^{-1}$ and

$$\alpha_{x,v} = \begin{cases} \frac{1}{2}(1-q_v^{-1})(1+q_v^{-r}) & \text{(sp)}, \\ \frac{1}{2}(1-q_v^{-1})(1-q_v^{-r}) & \text{(in)}, \\ \frac{1}{2}(1-q_v^{-1})(1-q_v^{-2r}) & \text{(rm)}, \\ \frac{1}{4}\frac{(1-q_v^{-1})^2(1-q_v^{-(r-1)})(1-q_v^{-2r})}{1-q_v^{-2}} & \text{(dq)}, \end{cases} \qquad \beta_{x,v} = \begin{cases} 1 & \text{(sp), (in)}, \\ q_v & \text{(rm)}, \\ q_v^3 & \text{(dq)}. \end{cases}$$

Then

$$b_{x,v} = \varepsilon_v(x)^{-1}|P(x)|_v^{\frac{n+1}{2}} = B(n,v)\alpha_{x,v}^{-1}\beta_{x,v}|P(x)|_v^{\frac{n+1}{2}}$$

in all cases. Therefore,

$$\Xi_{x,v}(s) - 1 = B(n,v)\alpha_{x,v}^{-1}\beta_{x,v}|P(x)|_v^{-s+\frac{n+1}{2}} \int_{Y_x} |P(y)|_v^{s-\frac{n+1}{2}} dy.$$

We choose a small number $\delta > 0$. Suppose that $\mathrm{Re}(s) > \frac{n}{2} + \delta$. Since $\mathrm{ord}_v(P(y)) - \mathrm{ord}_v(P(x))$ is even,

$$\begin{aligned} &\Xi_{x,v}(s) - 1 \\ &= B(n,v)\alpha_{x,v}^{-1}\beta_{x,v}|P(x)|_v^{-s+\frac{n+1}{2}} \int_{Y_x} |P(y)|_v^{s-\frac{n}{2}-\delta}|P(y)|_v^{-\frac{1}{2}+\delta} dy \\ &\preccurlyeq B(n,v)\alpha_{x,v}^{-1}\beta_{x,v}|P(x)|_v^{-s+\frac{n+1}{2}} \int_{Y_x} \sum_{j=1}^{\infty} |P(x)|_v^{s-\frac{n}{2}-\delta} q_v^{-2j(s-\frac{n}{2}-\delta)}|P(y)|_v^{-\frac{1}{2}+\delta} dy \\ &= B(n,v)\alpha_{x,v}^{-1}\beta_{x,v}|P(x)|_v^{\frac{1}{2}-\delta} \frac{q_v^{-2(s-\frac{n}{2}-\delta)}}{1-q_v^{-2(s-\frac{n}{2}-\delta)}} \int_{Y_x} |P(y)|_v^{-\frac{1}{2}+\delta} dy. \end{aligned}$$

Since we would like to avoid unnecessary complications arising from the introduction of the number $\delta > 0$, we introduce the following notation.

Definition 8.2. Let $f_1(\delta), f_2(\delta) \geq 0$ be real functions of δ. If there exist constants $C_{1,\delta}, C_2 > 0$ (C_2 does not depend on δ) independent of the place v such that $f_1(\delta) \leq C_{1,\delta}q_v^{C_2\delta}f_2(\delta)$, then we use the notation

$$f_1(\delta) \lhd_\delta f_2(\delta).$$

We devote the rest of this section to proving the following proposition.

Proposition 8.3. *If $v \notin S_0$ and x is a standard orbital representative then*

$$\beta_{x,v}|P(x)|_v^{\frac{1}{2}-\delta} \int_{Y_x} |P(y)|_v^{-\frac{1}{2}+\delta} dy \lhd_\delta q_v^{-1}.$$

If we assume this proposition then we can verify Condition 4.21 as follows.

Proposition 8.4. *Condition 4.21 is satisfied.*

Proof. By assumption, there exist constants $C_{1,\delta} > 0, C_2$ such that

$$\Xi_{x,v}(s) - 1 \preccurlyeq C_{1,\delta} q_v^{-1+C_2\delta} B(n,v)\alpha_{x,v}^{-1} \frac{q_v^{-2(s-\frac{n}{2}-\delta)}}{1-q_v^{-2(s-\frac{n}{2}-\delta)}}$$

It is easy to see that there exists a constant C independent of v such that $B(n,v)\alpha_{x,v}^{-1} \leq C$. By absorbing C into $C_{1,\delta}$,

$$\Xi_{x,v}(s) - 1 \preccurlyeq C_{1,\delta} q_v^{-1+C_2\delta} \frac{q_v^{-2(s-\frac{n}{2}-\delta)}}{1-q_v^{-2(s-\frac{n}{2}-\delta)}}.$$

The right hand side no longer depends on x, and so we denote it by $L_{\delta,v}(s)$.

Suppose that $\mathrm{Re}(s) \geq \frac{n}{2} + (C_2+1)\delta$. It is easy to see that

$$|L_{\delta,v}(s)| \leq \frac{C_{1,\delta}}{1-q_v^{-2C_2\delta}} q_v^{-1-C_2\delta}.$$

Therefore, $\prod_{v\notin S_0}(1+|L_{\delta,v}(s)|)$ converges absolutely and uniformly for $\mathrm{Re}(s) \geq \frac{n}{2} + (C_2+1)\delta$. Since $\delta > 0$ is small, Condition 4.21 is satisfied. □

Before proving Proposition 8.3, we recall results on local zeta functions in [10].

Let ν be a large integer and $R_\nu = \mathcal{O}_v/\mathfrak{p}_v^\nu$. Let $S_n(R_\nu)$ be the set of symmetric matrices x with entries in R_ν. Let $n = n_1 + \cdots + n_m$ be a partition of n. We denote this partition by $\{n_i\}$. For each i let t_i be a non-negative integers satisfying

$$0 \leq t_1 < t_2 < t_2 \cdots < t_m.$$

We denote this sequence by $\{t_i\}$. For such $\{n_i\}$ and $\{t_i\}$, we define $S_n^0(R_\nu, \{n_i\}, \{t_i\})$ to be the subset of $S_n(R_\nu)$ consisting of elements of the form

$$x = \begin{pmatrix} \pi_v^{t_1}x_1 & 0 & \cdots & 0 \\ 0 & \pi_v^{t_2}x_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \pi_v^{t_m}x_m \end{pmatrix}$$

where $x_i \in S_{n_i}(R_\nu)$ and $\det x_i \in R_\nu^\times$. It is known [16] that any element of $S_n(R_\nu)$ is equivalent by the action of $\mathrm{GL}(n)_{R_\nu}$ to an element of $S_n^0(R_\nu, \{n_i\}, \{t_i\})$, and that $\{n_i\}, \{t_i\}$ depends only on x. Note that since $v \notin \mathfrak{M}_{\mathrm{dy}}$, 2's in the diagonal entries do not cause any problem.

Let

$$S_n(R_\nu, \{n_i\}, \{t_i\}) = \mathrm{GL}(n)_{R_\nu} S_n^0(R_\nu, \{n_i\}, \{t_i\}),$$

i.e., the set of elements which are equivalent to elements of $S_n^0(R_\nu, \{n_i\}, \{t_i\})$. Then

$$S_n(R_\nu) = \cup S_n(R_\nu, \{n_i\}, \{t_i\}).$$

For $x \in S_n^0(R_\nu, \{n_i\}, \{t_i\})$, let $\mathcal{H}_x$ be the subgroup of $\mathrm{GL}(n)_{R_\nu}$ consisting of elements g such that $gx \in S_n^0(R_\nu, \{n_i\}, \{t_i\})$.

Remark 8.5. We have been considering $\mathrm{GL}(1)\times\mathrm{GL}(n)$-equivalence throughout this paper and so the reader may find it strange that we now consider $\mathrm{GL}(n)_{R_\nu}$-equivalence. All we need is the orders of the sets $S_n^0(R_\nu, \{n_i\}, \{t_i\})$, $S_n(R_\nu, \{n_i\}, \{t_i\})$ in Lemma 8.6 and Proposition 8.7 (or more specifically (8.9)) below. It is possible to use

$\mathrm{GL}(1) \times \mathrm{GL}(n)$-equivalence and obtain the same result. However, since $\mathrm{GL}(n)_{R_\nu}$-equivalence was used in [10, p. 1110], we chose to use the formulation in [10] for this part.

The following lemma is Lemma 3.2 [10, p. 1110].

Lemma 8.6. *The order* $\#\mathcal{H}_x$ *depends only on* $\{n_i\}, \{t_i\}$ *and*

$$\frac{\#\mathrm{GL}(n)_{R_\nu}}{\#\mathcal{H}_x} = \frac{\#\mathrm{GL}(n)_{R_\nu}}{\prod_{i=1}^m \#\mathrm{GL}(n_i)_{R_\nu}} \times q_v^{-\nu \sum_{1 \le i < j \le m} n_i n_j - \sum_{1 \le i < j \le m} t_i n_i n_j}$$

if ν *is sufficiently large.*

Let $p_\nu : V_{\mathcal{O}} \to S_n(R_\nu)$ be the natural projection. Given $\{n_i\}, \{t_i\}$, we choose ν sufficiently large and define

$$\begin{aligned} \mathcal{D}_n^0(\{n_i\}, \{t_i\}) &= p_\nu^{-1}(S_n^0(R_\nu, \{n_i\}, \{t_i\})), \\ \mathcal{D}_n(\{n_i\}, \{t_i\}) &= p_\nu^{-1}(S_n(R_\nu, \{n_i\}, \{t_i\})). \end{aligned}$$

For any positive integer l, we define

$$e(l, v) = \begin{cases} (1 - q_v^{-1})(1 - q_v^{-3}) \cdots (1 - q_v^{-(2r+1)}) & l = 2r + 1, \\ (1 - q_v^{-1})(1 - q_v^{-3}) \cdots (1 - q_v^{-(2r-1)}) & l = 2r. \end{cases}$$

We put

$$\begin{aligned} f(\{n_i\}) &= \frac{(1 - q_v^{-1}) \cdots (1 - q_v^{-n}) \prod_{i=1}^m e(n_i, v)}{\prod_{i=1}^m (1 - q_v^{-1}) \cdots (1 - q_v^{-n_i})}, \\ Q(\{n_i\}, \{t_i\}) &= \sum_{1 \le i \le m} t_i \frac{n_i(n_i + 1)}{2} + \sum_{1 \le i < j \le m} t_i n_i n_j. \end{aligned}$$

The following proposition is essentially used (but not explicitly) in [10]. We include its proof for the reader's sake.

Proposition 8.7. $\mathrm{vol}(\mathcal{D}_n(\{n_i\}, \{t_i\})) = f(\{n_i\}) q_v^{-Q(\{n_i\}, \{t_i\})}$.

Proof. We first prove the following lemma without assuming that n is even.

Lemma 8.8. *For all* n, $\#V_{\mathbb{F}_v}^{\mathrm{ss}} = e(n, v) q_v^{\frac{n(n+1)}{2}}$.

Proof. Any element of $V_{\mathbb{F}_v}^{\mathrm{ss}}$ is in the form $\overline{x}$ where $x \in V_{\mathcal{O}_v}^{\mathrm{ss}}$ and $\overline{x}$ is the reduction modulo $\mathfrak{p}_v$. Let $w_{v,\mathbb{i}}$ be the standard orbital representative in the orbit of x. Then $x \in G_{k_v} w_{v,\mathbb{i}} \cap V_{\mathcal{O}_v}$. By Proposition 5.23 [8], $\mathrm{ord}_v(P(x)) \geqq \mathrm{ord}_v(P(w_{v,\mathbb{i}}))$. Since $P(\overline{x}) \ne 0$, $\mathrm{ord}_v(P(x)) = 0$. Therefore, $\mathrm{ord}_v(P(w_{v,\mathbb{i}})) = 0$ and so $\mathbb{i} = \mathrm{sp}$ or in. The case $\mathbb{i} = \mathrm{in}$ occurs only if n is even.

If $x = g w_{v,\mathrm{sp}}$ or $g w_{v,\mathrm{in}}$ then $\chi(g) \in \mathcal{O}_v^\times$ and so $g \in K_v G_{x\, k_v}$. So we may assume that $g \in K_v$. By reduction modulo $\mathfrak{p}_v$, $\overline{x} \in G_{\mathbb{F}_v} w_{v,\mathrm{sp}}$ or $G_{\mathbb{F}_v} w_{v,\mathrm{in}}$ Therefore, we only have to count elements in the $G_{\mathbb{F}_v}$-orbits of $w_{v,\mathrm{sp}}, w_{v,\mathrm{in}}$. If n is odd then $\#V_{\mathbb{F}_v}^{\mathrm{ss}} = \#G_{\mathbb{F}_v} / G_{w_{v,\mathrm{sp}}\, \mathbb{F}_v}$. If n is even then $\#V_{\mathbb{F}_v}^{\mathrm{ss}} = \#G_{\mathbb{F}_v} / G_{w_{v,\mathrm{sp}}\, \mathbb{F}_v} + \#G_{\mathbb{F}_v} / G_{w_{v,\mathrm{in}}\, \mathbb{F}_v}$. In both cases the lemma follows from (6.1). □

To consider $\pi_v^{t_i} x_i \mod \mathfrak{p}_v^{\nu}$ is the same as to consider $x \mod \mathfrak{p}_v^{\nu - t_i}$. Therefore, by Lemma 8.8,

$$\#S_n^0(R_\nu, \{n_i\}, \{t_i\}) = \prod_{i=1}^{m} e(n_i, v) q_v^{(\nu - t_i)\frac{n_i(n_i+1)}{2}}. \tag{8.9}$$

Since $\mathrm{GL}(n)_{\mathcal{O}_v}$ surjects to R_ν,

$$\begin{aligned}
\mathrm{vol}(\mathcal{D}_n(\{n_i\}, \{t_i\})) &= \frac{\#\mathrm{GL}(n)_{R_\nu}}{\#\mathcal{H}_x} \mathrm{vol}(\mathcal{D}_n^0(\{n_i\}, \{t_i\})) \\
&= \frac{\#\mathrm{GL}(n)_{R_\nu}}{\#\mathcal{H}_x} q_v^{-\nu \frac{n(n+1)}{2}} \#S_n^0(R_\nu, \{n_i\}, \{t_i\})) \\
&= q_v^{-\nu \frac{n(n+1)}{2}} \frac{\#\mathrm{GL}(n)_{R_\nu}}{\prod_{i=1}^m \mathrm{GL}(n_i)_{R_\nu}} \times q_v^{-\nu \sum_{1 \le i < j \le m} n_i n_j - \sum_{1 \le i < j \le m} t_i n_i n_j} \\
&\quad \times \prod_{i=1}^{m} e(n_i, v) q_v^{(\nu - t_i)\frac{n_i(n_i+1)}{2}} \\
&= q_v^{-\nu \frac{n(n+1)}{2}} q_v^{\nu(n^2 - \sum_i n_i^2)} \frac{(1 - q_v^{-1}) \cdots (1 - q_v^{-n})}{\prod_{i=1}^m (1 - q_v^{-1}) \cdots (1 - q_v^{-n_i})} \\
&\quad \times q_v^{-\nu \sum_{1 \le i < j \le m} n_i n_j - \sum_{1 \le i < j \le m} t_i n_i n_j} \prod_{i=1}^{m} e(n_i, v) q_v^{(\nu - t_i)\frac{n_i(n_i+1)}{2}}.
\end{aligned}$$

Simplifying the result, we obtain the formula of the proposition. □

Now we are ready to prove Proposition 8.3.

Proof of Proposition 8.3. Note that if $y \in \mathcal{D}_n(\{n_i\}, \{t_i\})$ then

$$|P(y)|_v = q_v^{-\sum_{i=1}^m n_i t_i}.$$

Let x be a standard orbital representative. Obviously,

$$\int_{Y_x} |P(y)|_v^{-\frac{1}{2}+\delta} dy \le \sum_{\{n_i\}, \{t_i\}} \int_{Y_x \cap \mathcal{D}_n(\{n_i\}, \{t_i\})} |P(y)|_v^{-\frac{1}{2}+\delta} dy.$$

By the above remark, the right hand side is equal to

$$\sum_{Y_x \cap \mathcal{D}_n(\{n_i\}, \{t_i\}) \neq \emptyset} \mathrm{vol}(\mathcal{D}_n(\{n_i\}, \{t_i\})) q_v^{(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i}.$$

Since the number of partitions of n is finite, there is a constant C independent of v such that $f(\{n_i\}) \le C$. Therefore, we may ignore this factor. So

$$\begin{aligned}
&\beta_{x,v} |P(x)|_v^{\frac{1}{2}-\delta} \int_{Y_x} |P(y)|_v^{-\frac{1}{2}+\delta} dy \\
&\lhd_\delta \ \beta_{x,v} |P(x)|_v^{\frac{1}{2}-\delta} \sum_{Y_x \cap \mathcal{D}_n(\{n_i\}, \{t_i\}) \neq \emptyset} q_v^{-Q(\{n_i\}, \{t_i\}) + (\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i}.
\end{aligned} \tag{8.10}$$

We now consider each case. Since the number of partitions is finite, we may fix $\{n_i\}$.

(sp), (in) Let $x = w_{v,\mathrm{sp}}$ or $w_{v,\mathrm{in}}$.

In these cases $\beta_{x,v} = 1$ and $|P(x)|_v = 1$. So the right hand side of (8.10) is equal to

$$\sum_{Y_x \cap \mathcal{D}_n(\{n_i\},\{t_i\}) \neq \emptyset} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i}. \tag{8.11}$$

If $y \in Y_x$ then $\mathrm{ord}_v(P(y)) \geq 2$. The exponent of q_v is

$$-\sum_i \frac{n_i^2}{2} t_i - \sum_{i<j} t_i n_i n_j - \delta \sum_i n_i t_i \leq -\sum_i \frac{1}{2} t_i. \tag{8.12}$$

Since $(1 - q_v^{-\frac{1}{2}})^{-1} \leq 10$ for all v,

$$\sum_{\text{some } t_i \geq 2} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i} \leq \frac{q_v^{-1}}{(1 - q_v^{-\frac{1}{2}})^m} \lhd_\delta \; q_v^{-1}.$$

Therefore, we only have to consider $\{t_i\}$ such that $t_i \leq 1$ for all i. So the number of terms we have to consider is finite, and we may consider individual terms in the sum (8.11) without taking the sum.

If $i \neq j$ and $t_i, t_j > 0$ then $-\sum_i \frac{1}{2} t_i \leq -1$, and so

$$q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i} \lhd_\delta \; q_v^{-1}.$$

So the only case we have to consider is $m = 2$, $t_1 = 0$, $t_2 = 1$. However, $\mathrm{ord}_v(P(y)) = 1$ if $y \in \mathcal{D}_n(\{n_i\},\{t_i\})$, which contradicts to the assumption $\mathrm{ord}_v(P(y)) \geq 2$.

This completes the consideration for the cases (sp), (in).

(rm) We consider the cases $x = w_{v,(\mathrm{rm},1)}$ or $w_{v,(\mathrm{rm},2)}$ next.

In these cases $\beta_{x,v} = q_v$, $|P(x)|_v = q_v^{-1}$. So the right hand side of (8.10) is equal to

$$q_v^{\frac{1}{2}+\delta} \sum_{Y_x \cap \mathcal{D}_n(\{n_i\},\{t_i\}) \neq \emptyset} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i}. \tag{8.13}$$

If $y \in Y_x$ then $\mathrm{ord}_v(P(y)) \geq 3$. Using (8.12), we only have to consider terms such that $t_i \leq 2$ for all i. If $i \neq j$ and $t_i, t_j > 0$ then the left hand side of (8.12) is at most $-\frac{1}{2} - \frac{1}{2} - 1 \leq -2$. Therefore, since δ is small,

$$q_v^{\frac{1}{2}+\delta} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i} \leq q_v^{-\frac{3}{2}+\delta} \leq q_v^{-1}.$$

Therefore, we only consider the case where $m = 2$, $t_1 = 0$ and $t_2 = 1, 2$. Since $\mathrm{ord}_v(P(y)) = n_2 t_2 \geq 3$, $n_2 \geq 2$ if $t_2 = 2$ and $n_2 \geq 3$ if $t_2 = 1$. Then $-\frac{n_2^2}{2} t_2 \leq -4$. Therefore,

$$q_v^{\frac{1}{2}+\delta} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i} \leq q_v^{-\frac{7}{2}+\delta} \leq q_v^{-1}.$$

This completes the consideration for the cases (rm).

(dq) We finally consider the case $x = w_{v,\mathrm{dq}}$ (which means $n \geq 4$).

In this case $\beta_{x,v} = q_v^3$, $|P(x)|_v = q_v^{-2}$. So the right hand side of (8.10) is equal to

$$q_v^{2+2\delta} \sum_{Y_x \cap \mathcal{D}_n(\{n_i\},\{t_i\}) \neq \emptyset} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i}. \tag{8.14}$$

If $y \in Y_x$ then $\mathrm{ord}_v(P(y)) \geq 4$. By a similar argument as above, we only have to consider terms such that $t_i \leq 5$ for all i. If $t_i > 0$ and $n_i \geq 3$ then $-\frac{n_i^2}{2} t_i \leq -\frac{9}{2}$. So we may assume that $n_i \leq 2$ for all i. If there are three indices $i_1 < i_2 < i_3$ such that $t_{i_1}, t_{i_2}, t_{i_3} > 0$ then $t_{i_1} \geq 1, t_{i_2} \geq 2, t_{i_3} \geq 3$. So

$$-\sum_i \frac{n_i^2}{2} t_i \leq -3,$$

which implies that

$$q_v^{2+2\delta} q_v^{-Q(\{n_i\},\{t_i\})+(\frac{1}{2}-\delta)\sum_{i=1}^m n_i t_i} \leq q_v^{-1+2\delta} \lhd_\delta q_v^{-1}. \tag{8.15}$$

So we may assume that there are at most two indices i such that $t_i > 0$.

We consider terms such that there are exactly two indices $i < j$ such that $t_i, t_j > 0$. Then $t_j \geq 2$. Unless $n_i = n_j = 1$,

$$-\frac{n_i^2}{2} t_i - \frac{n_j^2}{2} t_j - n_i n_j t_i \leq -\frac{n_i^2}{2} - n_j^2 - n_i n_j \leq -3.$$

So (8.15) holds also.

Therefore, we may assume that $n_i = n_j = 1$. Then $\mathrm{ord}_v(P(y)) = t_i + t_j \geq 4$. So

$$-\frac{n_i^2}{2} t_i - \frac{n_j^2}{2} t_j - n_i n_j t_i = -\frac{t_i + t_j}{2} - t_i \leq -3.$$

Therefore, (8.15) holds also.

Now we are left with the terms such that there is exactly one i such that $t_i > 0$. Then i must be m. If $n_m = 1$ then $\mathrm{ord}_v(P(y)) = t_m$ is even. If we apply the element

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \pi_v^{-\frac{t_m}{2}} \end{pmatrix}$$

to y then y becomes an integral symmetric matrix such that $\det y \in \mathcal{O}_v^{\times}$.

We have shown in the proof of Lemma 8.8 that the type of y is (sp) or (in), which is a contradiction. Therefore, we may assume that $n_m = 2$. Since $\mathrm{ord}_v(P(y)) = n_m t_m = 2t_m \geq 4$, $t_m \geq 2$. Then $-\frac{n_m^2}{2} t_m \leq -4$. Hence, (8.15) holds also.

This completes the proof of Proposition 8.3, and so the proof of Proposition 8.4 also. □

9. $\varepsilon_v(x)$ FOR INFINITE PLACES

We assume that $v \in \mathfrak{M}_\infty$ throughout this section. In this section we compute $\varepsilon_v(x)$ for standard representatives x. It will turn out that the computation of $\varepsilon_v(x)$ has essentially been done in part II.

Let $x \in V_{k_v}^{\mathrm{ss}}$ be a standard representative. The canonical measure $dg''_{x,v}$ on $G^{\circ}_{x\,k_v}$ was defined in Definition 4.17 [9] and the Tamagawa measure $d\mu''_{x,v}$ on $G^{\circ}_{x\,k_v}$ was defined in Definition 9.7 [8]. There is a constant $c''_{v,x}$ such that $dg''_{x,v} = c''_{v,x} d\mu''_{x,v}$. The value of $c''_{v,x}$ was computed in Propositions 5.20, 5.34 [9] as follows.

Proposition 9.1. *Let* $n = 2r$.

(1) *If* $v \in \mathfrak{M}_{\mathbb{R}}$ *and* $x = w_{v,\mathbb{i}}$ *where* $0 \leqq \mathbb{i} \leqq r$ *then*

$$c''_{v,x} = 2^{-n+\frac{\mathbb{i}(n-\mathbb{i}+1)}{2}+1} \prod_{1 \leq j \leq \mathbb{i}} \Gamma_{\mathbb{R}}(j) \prod_{1 \leq j \leq n-\mathbb{i}} \Gamma_{\mathbb{R}}(j).$$

(2) *If* $v \in \mathfrak{M}_{\mathbb{C}}$ *and* $x = w_{v,\mathrm{sp}}$ *then*

$$c''_{v,x} = 2^{-3r+1}(4\pi)^r \prod_{1 \leq j \leq n} \Gamma_{\mathbb{R}}(j)$$

This proposition and the following proposition proves Proposition 5.8.

Proposition 9.2. *If* $v \in \mathfrak{M}_\infty$ *and* x *is a standard representative then*

$$\varepsilon_v(x)\Gamma_v(2)\cdots\Gamma_v(n) = \frac{c''_{v,x}}{2}$$

Proof. We chose the measure $dg'_{x,v}$ so that if Φ is a measurable function on $G_{k_v}x$ then

$$\int_{G_{k_v}/G^\circ_{x\,k_v}} \Phi(g'_{x,v}x)dg'_{x,v} = \int_{G_{k_v}x} \Phi(y)|P(y)|_v^{-\frac{n+1}{2}}dy$$

By Proposition 3.14, there exists an element ν_x of $K_v \cap G_{x\,k_v}$ not in $G^\circ_{x\,k_v}$ such that $\nu_x^2 = 1$. Let $f(g_v)$ be a measurable function on G_{k_v} such that $f(g_v) = f(g_v\nu_x)$. It is easy to see that we can choose such f so that it is integrable on G_{k_v} and that $\int_{G_{k_v}} f(g_v)dg_v \neq 0$. We define a function $F(g'_{x,v})$ of $g'_{x,v} \in G_{k_v}$ as follows

$$F(g'_{x,v}) = \int_{G^\circ_{x\,k_v}} f(g'_{x,v}g''_{x,v})dg''_{x,v}.$$

Since $f(g_v) = f(g_v\nu_x)$,

$$F(g'_{x,v}) = c''_{v,x}\int_{G^\circ_{x\,k_v}} f(g'_{x,v}g''_{x,v})d\mu(g''_{x,v}) = \frac{c''_{v,x}}{2}\int_{G_{x\,k_v}} f(g'_{x,v}g''_{x,v})d\mu(g''_{x,v})$$

This implies that there exists a function Φ on $G_{k_v}x$ such that $F(g'_{x,v}) = \Phi(g'_{x,v}x)$.

It is explained in Lemma 5.1 that

$$dg_v = \Gamma_v(2)\cdots\Gamma_v(n)d\mu_{1,v}d\mu_{n,v}.$$

Note that $\Gamma_v(1) = 1$. If $x \in V^{\mathrm{ss}}_{k_v}$ is a standard representative then $|P(x)|_v = 1$ and so $\varepsilon_v(x) = b^{-1}_{x,v}$.

Therefore,

$$\begin{aligned}
&\varepsilon_v(x)\Gamma_v(2)\cdots\Gamma_v(n)\int_{G_{k_v}} f(g_v)d\mu_{1,v}d\mu_{n,v}\\
&= b_{x,v}^{-1}\int_{G_{k_v}} f(g_v)dg_v\\
&= \int_{G_{k_v}/G^{\circ}_{x\,k_v}}\left(\int_{G^{\circ}_{x\,k_v}} f(g'_{x,v}g''_{x,v})dg''_{x,v}\right)dg'_{x,v}\\
&= \int_{G_{k_v}/G^{\circ}_{x\,k_v}} \Phi(g'_{x,v}x)dg'_{x,v}\\
&= \int_{G_{k_v}x} \Phi(y)|P(y)|_v^{-\frac{n+1}{2}}dy\\
&= \int_{G_{k_v}/G_{x\,k_v}} F(g'_{x,v})d\mu'(g'_{x,v})\\
&= \frac{c''_{v,x}}{2}\int_{G_{k_v}/G_{x\,k_v}}\left(\int_{G_{x\,k_v}} f(g'_{x,v}g''_{x,v})d\mu''(g''_{x,v})\right)d\mu'(g'_{x,v})\\
&= \frac{c''_{v,x}}{2}\int_{G_{k_v}} f(g_v)d\mu_{1,v}d\mu_{n,v}.
\end{aligned}$$

This proves the proposition. □

10. THE CASE $k=\mathbb{Q}$, $n=2$

In this section we interpret out result in the case $k=\mathbb{Q}$ and $n=2$. We assume that $k=\mathbb{Q}$ throughout this section.

Let $\zeta(s)$ be the Riemann zeta function. We put

$$E_p=\begin{cases}1-p^{-2}-p^{-2r-1}+p^{-2r-2}+\frac{1}{4}p^{-3}\frac{(1-p^{-1})^2(1-p^{-(r-1)})(1-p^{-2r})}{1-p^{-2}} & n\geqq 4,\\ 1-p^{-2}-p^{-3}+p^{-4} & n=2.\end{cases}$$

Let $S_{n,\mathbb{i}}$ be the subset of S_n consisting of groups of the form $\mathrm{PSO}(x)$ such that x is a quadratic form with signature $(\mathbb{i}, n-\mathbb{i})$. Then Theorem 5.10 states

$$\begin{aligned}
&\lim_{X\to\infty} X^{-\frac{n+1}{2}}\sum_{x\in S_{n,\mathbb{i}},\Delta_x<X}\mathrm{vol}(\mathrm{PSO}(x)_{\mathbb{A}}/\mathrm{PSO}(x)_k)\\
&=\frac{2^{-n+\frac{\mathbb{i}(n-\mathbb{i}+1)}{2}+2}}{n+1}\zeta(2)\zeta(4)\cdots\zeta(n)\prod_{1\le j\le\mathbb{i}}\Gamma_{\mathbb{R}}(j)\prod_{1\le j\le n-\mathbb{i}}\Gamma_{\mathbb{R}}(j)\prod_p E_p
\end{aligned}$$

which gives the statement in Introduction.

Now we specialize to the case $n=2$. In this case $\mathbb{i}=0,1$. It is easy to see that $\Gamma_{\mathbb{R}}(1)=1$, $\Gamma_{\mathbb{R}}(2)=\pi^{-1}$. If $\mathbb{i}=0$ then the above constant is

$$\frac{1}{3}\zeta(2)\Gamma_{\mathbb{R}}(1)\Gamma_{\mathbb{R}}(2)\prod_p E_p=\frac{\pi}{18}\prod_p(1-p^{-2}-p^{-3}+p^{-4}).$$

If $\mathbb{i} = 1$ then the above constant is

$$\frac{2}{3}\zeta(2)\Gamma_{\mathbb{R}}(1)^2 \prod_p E_p = \frac{\pi^2}{9}\prod_p (1 - p^{-2} - p^{-3} + p^{-4}).$$

If $\mathbb{i} = 1$ then our measure on $G^{\circ}_{x\,\mathbb{R}}$ coincides with that in [1, p. 224]. If $\mathbb{i} = 0$ the choice of the measure on $G^{\circ}_{x\,\mathbb{R}}$ is slightly different from that in [1, p. 224]. In this situation $G^{\circ}_{x\,\mathbb{R}} \cong \mathbb{C}^{\times}$ and its maximal compact subgroup is $\mathbb{C}^1$ which can be identified with the classical special orthogonal group SO(2). In [1, p. 224], the measure dh'' on $\mathbb{C}^{\times}$ was defined using the measure $d\kappa$ on SO(2) such that the volume of SO(2) is 2π, i.e., if f is a measurable function on $\mathbb{C}^{\times}$ then

$$\int_{\mathbb{C}^{\times}} f(h'')dh'' = \int_{\mathbb{C}^1 \times \mathbb{R}^{\times}} f(\kappa\lambda)d\kappa|\lambda|^{-1}d\lambda = 2\int_{\mathbb{C}^1 \times \mathbb{R}_+} f(\kappa\lambda)d\kappa d^{\times}\lambda.$$

Therefore, this is the measure $d^{\times}t_{\mathbb{C}}$ on $\mathbb{C}^{\times}$ regarded as the algebraic group $\mathbb{C}^{\times}$ over the ground field $\mathbb{C}$. We defined our measure $dg''_{x,\mathbb{R}}$ so that the volume of SO(2) is 1 (see Definition 4.17 [9]). Therefore, $dh'' = 2\pi dg''_{x,\mathbb{R}}$. So if we use the measure in [1], then the answer for this case has to be multiplied by 2π.

If $n = 2$ then $G_{\mathbb{Q}} \backslash V^{\mathrm{ss}}_{\mathbb{Q}}$ is in bijective correspondence with extensions of $\mathbb{Q}$ of degree at most two. When we count them we may ignore the trivial extension $\mathbb{Q}$. If $F/\mathbb{Q}$ is a quadratic extension which corresponds to $x \in V^{\mathrm{ss}}_{\mathbb{Q}}$ then $\mathbb{i} = 1$ (resp. $\mathbb{i} = 0$) at the infinite place means that F is a real (resp. imaginary) quadratic extension of $\mathbb{Q}$. It is proved in Proposition 5.1 [1, p. 229] that

$$\mathrm{vol}(\widetilde{G}^{\circ}_{x\,\mathbb{A}}/\widetilde{G}^{\circ}_{x\,\mathbb{Q}}) = 2\mathfrak{C}_F.$$

Therefore, we obtain

$$\lim_{X\to\infty} X^{-\frac{3}{2}} \sum_{0<\pm\Delta_F<X} \mathfrak{C}_F = \frac{\pi^2}{18}\prod_p (1 - p^{-2} - p^{-3} + p^{-4})$$

($\frac{\pi}{18} \times (2\pi)/2 = \frac{\pi^2}{18}$ of course).

If F is a real (resp. imaginary) quadratic field then $\mathfrak{C}_F = 4h_F R_F e_F^{-1}$ (resp. $2\pi h_F R_F e_F^{-1} = 2\pi h_F e_F^{-1}$). Except for a finite number of quadratic fields, $e_F = 2$. If $e_F = 2$ then $\mathfrak{C}_F = 2h_F R_F$ if F is real and $\mathfrak{C}_F = \pi h_F$ if F is imaginary. Therefore,

$$\lim_{X\to\infty} X^{-\frac{3}{2}} \sum_{0<\Delta_F<X} h_F R_F = \frac{\pi^2}{36}\prod_p (1 - p^{-2} - p^{-3} + p^{-4}),$$
$$\lim_{X\to\infty} X^{-\frac{3}{2}} \sum_{0<-\Delta_F<X} h_F = \frac{\pi}{18}\prod_p (1 - p^{-2} - p^{-3} + p^{-4}).$$

Thus, we obtain the theorem of Goldfeld–Hoffstein.

If $n = 2$ and $v \in \mathfrak{M}_{\mathbb{R}}$ then $\mathbb{i} = 0, 1$. Proposition 9.1 states that $c''_{v,x} = (2\pi)^{-1}$ if $\mathbb{i} = 0$ and $c''_{v,x} = 1$ if $\mathbb{i} = 1$. Since $b_{x,v} = \varepsilon_v(x)^{-1} = 2\Gamma_{\mathbb{R}}(2){c''_{v,x}}^{-1} = 2/(\pi c''_{v,x})$,

$$b_{x,v} = \begin{cases} 4 & \mathbb{i} = 0, \\ \frac{2}{\pi} & \mathbb{i} = 1. \end{cases}$$

If we use the measure in [1], the value for the case $\mathbb{i} = 0$ has to be divided by 2π and so $b_{x,v} = \frac{2}{\pi}$. This of course is compatible with Proposition 4.4 [1, p. 224].

Suppose that $n = 2$ and $v \in \mathfrak{M}_{\mathbb{C}}$. In this case our measure on $G^{\circ}_{x\,k_v}$ coincides with that in [1]. Then Proposition 9.1 states that $c''_{v,x} = 2^{-2}(4\pi)\Gamma_{\mathbb{R}}(2) = 1$. It is easy to see that $\Gamma_{\mathbb{C}}(1) = 1,\ \Gamma_{\mathbb{C}}(2) = (2\pi)^{-1}$. So

$$b_{x,v} = \varepsilon_v(x)^{-1} = 2\Gamma_{\mathbb{C}}(2){c''_{v,x}}^{-1} = \frac{1}{\pi}.$$

This also is compatible with Proposition 4.4 [1, p. 224].

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