A counterexample to generalizations of the Milnor-Bloch-Kato conjecture

By Michael Spieß and Takao Yamazaki

Abstract

We construct an example of a torus T over a field K for which the Galois symbol $K(K;T,T)/nK(K;T,T) \to H^2(K, T[n] \otimes T[n])$ is not injective for some n. Here $K(K;T,T)$ is the Milnor K-group attached to T introduced by Somekawa. We show also that the motive $M(T \times T)$ gives a counterexample to another generalization of the Milnor-Bloch-Kato conjecture (proposed by Beilinson).

1 Introduction

Let K be a field, m a positive integer and n an integer prime to the characteristic of K. The Milnor-Bloch-Kato conjecture asserts that the Galois symbol

$$K_m^M(K)/nK_m^M(K) \quad \longrightarrow \quad H^m(K, \mathbb{Z}/n\mathbb{Z}(m)) \tag{1}$$

from Milnor K-groups to Galois cohomology is bijective. Recently, Rost and Voevodsky have announced a proof (special cases have been obtained earlier by Merkurjev-Suslin, Rost and Voevodsky).

In [So], Somekawa has introduced certain *generalized Milnor K-groups* $K(K; A_1, \ldots, A_m)$ attached to semi-abelian varieties $A_1, \ldots, A_m$. If $A_1 = \ldots = A_m = \mathbb{G}_m$ is the one-dimensional split torus they agree with the usual $K_m^M(K)$. If $m = 2$, $A_1 = \mathrm{Jac}_X$ and $A_2 = \mathrm{Jac}_Y$ are the Jacobians of smooth, projective and connected curves X and Y over K having a K-rational point, then $K(K; A_1, A_2)$ is the kernel of the Albanese map $\mathrm{CH}_0(X \times Y)_{\deg=0} \to \mathrm{Alb}_{X\times Y}(K)$.

Somekawa has defined a Galois symbol

$$K(K; A_1, \ldots, A_m)/nK(K; A_1, \ldots, A_m) \longrightarrow H^m(K, A_1[n] \otimes \ldots \otimes A_m[n]) \tag{2}$$

and conjectured that it is always injective. In this note we present a counterexample (see section 2). Let us describe it briefly. Let L/K be a cyclic extension of degree n and σ a generator of the Galois group $\mathrm{Gal}(L/K)$. Let T

be the kernel of the norm map $\mathrm{Res}_{L/K}\,\mathbb{G}_m \to \mathbb{G}_m$. We show that the norm $K(L;T,T) \to K(K;T,T)$ induces an isomorphism $K(L;T,T)/(1-\sigma) \to K(K;T,T)$. On the other hand, the corresponding map of Galois cohomology groups $H^2(L,T[n]\otimes T[n])/(1-\sigma) \to H^2(K,T[n]\otimes T[n])$ is neither injective nor surjective (for a suitable choice of L/K). Note that, since T is split over L, the Galois symbol $K(L;T,T)/n \to H^2(L,T[n]\otimes T[n])$ is bijective. Consequently, $K(K;T,T)/n \to H^2(K,T[n]\otimes T[n])$ is in general not injective.

In the section 3 we show that the motive $M(T\times T)$ gives a counterexample to another generalization of the Milnor-Bloch-Kato conjecture (proposed by Beilinson). This section is largely independent of the previous section.

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2 Counterexample to Somekawa's conjecture

Algebraic groups as Mackey-functors Let K be a field. For a finite field extension L/K and commutative algebraic groups G over K and H over L we denote by G_L the base change of G to L and by $\mathrm{Res}_{L/K}\,H$ the Weil restriction of H. The functor $G \mapsto G_L$ is left and right adjoint to $H \mapsto \mathrm{Res}_{L/K}\,H$. In particular there are adjunction homomorphisms $\iota_{L/K}: G \to \mathrm{Res}_{L/K}\,G_L$ and $N_{L/K}: \mathrm{Res}_{L/K}\,G_L \to G$. When L/K is a Galois extension, the Galois group $\mathrm{Gal}(L/K)$ acts canonically on $\mathrm{Res}_{L/K}\,G_L$. The following simple result, whose proof will be left to the reader, will be used later.

Lemma 1 *Let L/K be a cyclic Galois extension of degree n, σ a generator of $\mathrm{Gal}(L/K)$ and let G be a commutative algebraic group over K. Let G' be the kernel of $N_{L/K}: \mathrm{Res}_{L/K}\,G_L \to G$ so that $G'_L \cong G_L^{n-1}$. Then the map*

$$\mathrm{Res}_{L/K}(G_L)^{n-1} \cong \mathrm{Res}_{L/K}\,G'_L \xrightarrow{N_{L/K}} G' \hookrightarrow \mathrm{Res}_{L/K}\,G_L$$

is given on the i-th summand by $1-\sigma^i$.

We denote by $\mathcal{C}_K$ the category of finite reduced K-schemes. Thus each object of $\mathcal{C}_K$ is isomorphic to $\mathrm{Spec}(E_1\times\ldots\times E_r)$ where $E_1,\ldots,E_r/K$ are finite field extensions. A commutative algebraic group G over K defines a *cohomological Mackey-functor*, i.e. a co- and contravariant functor $G:\mathcal{C}_K \to$

$\mathrm{Mod}_{\mathbb{Z}}$ satisfying (i), (ii) and (iii) below. (The terminology *cohomological* refers to the property (iii).) If $f : X \to Y$ is a morphism we denote by $f_* : G(X) \to G(Y)$ and $f^* : G(Y) \to G(X)$ the homomorphisms induced by co- and contravariant functoriality respectively.

(i) If $X = X_1 \coprod X_2 \in \mathrm{Obj}(\mathcal{C}_K)$ then $G(X) = G(X_1) \oplus G(X_2)$.

(ii) If

$$\begin{array}{ccc} X' & \xrightarrow[f']{} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a cartesian square in $\mathcal{C}_K$ then $g^* f_* = (f')_* \deg(f;g)_{\mathrm{insep}} (g')^*$. Here the map $\deg(f;g)_{\mathrm{insep}} : G(X') \to G(X')$ is defined as follows. For a point $x \in X'$ let m be the length of the Artinian local ring $\mathcal{O}_{X \times_Y Y', x}$ (note that $X' = (X \times_Y Y')_{\mathrm{red}}$). Then $\deg(f;g)_{\mathrm{insep}}$ is given on the summand $G(\{x\})$ of $G(X') = \bigoplus_{x' \in X'} G(\{x'\})$ by multiplication with m.

(iii) If $f : X \to Y$ is a morphism in $\mathcal{C}_K$ with X, Y connected, then $f_* f^*$ is the multiplication by the degree of f.

If $K \subseteq E_1 \subseteq E_2$ are two finite field extensions and $f : \mathrm{Spec}\, E_2 \to \mathrm{Spec}\, E_1$ the corresponding map in $\mathcal{C}_K$ then f^* (resp. f_*) is given by $\iota_{E_2/E_1} : G(E_1) \to G(E_2)$ (resp. $N_{E_2/E_1} : G(E_2) \to G(E_1)$).

Remark 2 If K is perfect then a cohomological Mackey functor is nothing else than a presheaf with transfers on $\mathcal{C}_K$ (compare [Vo1, MVW]). We briefly recall this notion. Let Cor_K denote the additive category of finite correspondences ([MVW], 1.1). The objects of Cor_K are smooth separated K-schemes of finite type and for $X, Y \in \mathrm{Obj}(Cor_K)$ the group of morphisms $Cor_K(X,Y)$ is the free abelian group generated by integral closed subschemes W of $X \times Y$ which are finite and surjective over X. We denote by Cor_K^0 the full subcategory of Cor_K generated by finite étale K-schemes. A *presheaf with transfers on* $\mathcal{C}_K$ is a contravariant functor $\mathcal{F} : Cor_K^0 \to \mathrm{Mod}_{\mathbb{Z}}$ which additionally satisfies property (i) above.

Local symbols We recall also the notion of a *local symbol* ([Se] and [So]) for G. Let $C \to \mathrm{Spec}\, K$ be a proper non-singular algebraic curve (note that we do not assume that C is connected). Let $K(C)$ denote the ring of rational functions on C and $|C|$ the set of closed points of C. For $P \in |C|$ we denote by K_P the quotient field of the completion $\widehat{\mathcal{O}}_{C,P}$ of $\mathcal{O}_{C,P}$, by $v_P : K_P \to \mathbb{Z} \cup \{\infty\}$ the normalized valuation and by $K(P)$ the residue field of K_P. The local symbol at P is a homomorphism $\partial_P : (K_P)^* \otimes G(K_P) \to G(K(P))$. It is characterized by the following properties:

(i) If $f \in (K_P)^*$ and $g \in G(\widehat{\mathcal{O}}_{C,P})$ then $\partial_P(f \otimes g) = v_P(f)g(P)$. Here $g(P)$ is the image of g under the canonical map $G(\widehat{\mathcal{O}}_{C,P}) \to G(K(P))$.

(ii) For $f \in K(C)^*$ and $g \in G(K(C))$ we have $\sum_{P\in|C|} N_{K(P)/K}(\partial_P(f \otimes g)) = 0$.

Milnor K-groups attached to commutative algebraic groups Let $G_1, \ldots, G_m$ be commutative algebraic groups over K. In [So] Somekawa has introduced the Milnor K-group $K(K; G_1, \ldots, G_m)$ (actually Somekawa considered only the case of semiabelian varieties though his construction works for arbitrary commutative algebraic groups). It is given as

$$K(K; G_1, \ldots, G_m) = \left(\bigoplus_X G_1(X) \otimes \ldots \otimes G_m(X) \right) /R$$

where X runs through all objects of $\mathcal{C}_K$ and where the subgroup R is generated by the following elements:

(R1) If $f : X \to Y$ is a morphism in $\mathcal{C}_K$ and if $x_{i_0} \in G_{i_0}(Y)$ for some i_0 and $x_i \in G_i(X)$ for $i \neq i_0$, then

$$x_1 \otimes \ldots \otimes f_*(x_{i_0}) \otimes \ldots \otimes x_m - f^*(x_1) \otimes \ldots \otimes x_{i_0} \otimes \ldots \otimes f^*(x_m) \in R.$$

(R2) Let $C \to \operatorname{Spec} K$ be a proper non-singular curve, $f \in K(C)^*$ and $g_i \in G_i(K(C))$. Assume that for each $P \in |C|$ there exists $i(P)$ such that $g_i \in G_i(\widehat{\mathcal{O}}_{C,P})$ for all $i \neq i(P)$. Then

$$\sum_{P\in|C|} g_1(P) \otimes \ldots \otimes \partial_P(f \otimes g_{i(P)}) \otimes \ldots \otimes g_m(P) \in R.$$

For $X \in \mathcal{C}_K$ and $x_i \in G_i(X)$ for $i = 1, \ldots, m$ we write $\{x_1, \ldots, x_m\}_{X/K}$ for the image of $x_1 \otimes \ldots \otimes x_m$ in $K(K; G_1, \ldots, G_m)$ (elements of this form will be referred to as *symbols*).

A sequence of algebraic groups $G' \to G \to G''$ over K will be called *Zariski exact* if $G'(E) \to G(E) \to G''(E)$ is exact for every extension E/K. The proof of the following result is straightforward; hence will be omitted.

Lemma 3 *Let m be a positive integer and let $i \in \{1, \ldots, m\}$. Let $G_1, \ldots, G_m$ be commutative algebraic groups over K and let $G'_i \to G_i \to G''_i \to 1$ be a Zariski exact sequence of commutative algebraic groups over K. Then the sequence*

$$K(K; G_1, \ldots, G'_i, \ldots) \to K(K; G_1, \ldots, G_i, \ldots) \to K(K; G_1, \ldots, G''_i, \ldots) \to 0$$

is exact as well.

The norm map Let $G_1, \ldots, G_m$ be commutative algebraic groups over K and let L/K be a finite extension. Set $K(L; G_1, \ldots, G_m) \colon = K(L; (G_1)_L, \ldots, (G_m)_L)$. Then we have the norm map [So]

$$N_{L/K} : K(L; G_1, \ldots, G_m) \longrightarrow K(K; G_1, \ldots, G_m) \tag{3}$$

defined on symbols by $N_{L/K}(\{x_1, \ldots, x_m\}_{X/L}) = \{x_1, \ldots, x_m\}_{X/K}$ for any $X \in \mathcal{C}_L$ and $x_i \in G_i(X)$ $(i = 1, \ldots, m)$. We give another interpretation of (3) below when L/K is separable. It is based on the following result.

Lemma 4 *Let L/K be a finite separable extension and let i, m be positive integers with $i \leq m$. Let $G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_m$ be commutative algebraic groups over K and let G_i be a commutative algebraic group over L. Then, we have an isomorphism*

$$K(K; G_1, \ldots, \operatorname{Res}_{L/K} G_i, \ldots, G_m) \cong K(L; (G_1)_L, \ldots, G_i, \ldots, (G_m)_L).$$

Proof. To simplify the notation we assume that $i = m$. We denote by $\pi^{-1} : \mathcal{C}_K \to \mathcal{C}_L$ and $\pi : \mathcal{C}_L \to \mathcal{C}_K$ the functors

$$\pi^{-1}(X \to \operatorname{Spec} K) \colon = (X \otimes_K L \to \operatorname{Spec} L),$$

$$\pi(Y \to \operatorname{Spec} L) \colon = (Y \to \operatorname{Spec} L \to \operatorname{Spec} K).$$

π is left adjoint to π^{-1}. For $X \in \mathcal{C}_K$ and $Y \in \mathcal{C}_L$ let

$$p_X : X \otimes_K L \longrightarrow X, \quad \iota_Y : Y \longrightarrow Y \otimes_K L.$$

be the adjunction morphisms. We define homomorphisms

$$\phi : K(K; G_1, \ldots, G_{m-1}, \operatorname{Res}_{L/K} G_m) \longrightarrow K(K; (G_1)_L, \ldots, (G_{m-1})_L, G_m),$$

$$\psi : K(L; (G_1)_L, \ldots, (G_{m-1})_L, G_m) \longrightarrow K(K; G_1, \ldots, G_{m-1}, \operatorname{Res}_{L/K} G_m).$$

as follows. For $X \in \mathcal{C}_K$, $x_1 \in G_1(X), \ldots, x_{m-1} \in G_{m-1}(X)$ and $x_m \in G_m(X \otimes_K L)$ we put

$$\phi(\{x_1, \ldots, x_m\}_{X/K}) = \{p_X^*(x_1), \ldots, p_X^*(x_{m-1}), x_m\}_{(X \otimes_K L)/L}.$$

Conversely, for $Y \in \mathcal{C}_L$ and $y_1 \in G_1(Y), \ldots, y_m \in G_m(Y)$ let

$$\psi(\{y_1, \ldots, y_{m-1}, y_m\}_{Y/L}) = \{y_1, \ldots, y_{m-1}, (\iota_Y)_*(y_m)\}_{Y/K}.$$

One can easily verify that these maps are well-defined and mutually inverse to each other. □

Let $G_1, \ldots, G_m$ be commutative algebraic groups over K and let L/K be a finite separable extension. Take any $i \in \{1, \ldots, m\}$. The map $N_{L/K}$:

$\mathrm{Res}_{L/K}(G_i)_L \to G_i$ induces a map $K(K; G_1, \dots, \mathrm{Res}_{L/K}(G_i)_L, \dots, G_m) \longrightarrow K(K; G_1, \dots, G_m)$, and the composition of it with the isomorphism ψ above coincides with the norm map (3). When L/K is a Galois extension, the action of $\mathrm{Gal}(L/K)$ on $\mathrm{Res}_{L/K}(G_i)_L$ induces its action on the right hand side of

$$K(L; G_1, \dots, G_m) \cong K(K; G_1, \dots, \mathrm{Res}_{L/K}(G_i)_L, \dots, G_m),$$

which is compatible with the natural action on the left hand side. This action does not depend on the choice of i, and we have $N_{L/K} \circ \sigma = N_{L/K}$ for all $\sigma \in \mathrm{Gal}(L/K)$.

Lemma 5 *Let L/K be a cyclic Galois extension and let $\sigma \in \mathrm{Gal}(L/K)$ be a generator. Suppose that for two different $i \in \{1, \dots, m\}$ the sequence*

$$\mathrm{Res}_{L/K}(G_i)_K \overset{N_{L/K}}{\longrightarrow} G_i \longrightarrow 1 \tag{4}$$

is Zariski exact. Then the sequence of abelian groups

$$K(L; G_1, \dots, G_m) \overset{1-\sigma}{\longrightarrow} K(L; G_1, \dots, G_m) \overset{N_{L/K}}{\longrightarrow} K(K; G_1, \dots, G_m) \longrightarrow 0$$

is exact.

Proof. Suppose that (4) is exact for $i = m-1, m$. Let $G'_m := \mathrm{Ker}(N_{L/K} : \mathrm{Res}_{L/K}(G_m)_L \to G_m)$. By Lemmas 3 and 4 there are exact sequences

$$K(K; G_1, \dots, G'_m) \to K(L; G_1, \dots, G_m) \overset{N_{L/K}}{\longrightarrow} K(K; G_1, \dots, G_m) \to 0 \tag{5}$$

$$K(L; G_1, \dots, G_{m-1}, G'_m) \overset{N_{L/K}}{\longrightarrow} K(K; G_1, \dots, G_{m-1}, G'_m) \longrightarrow 0. \tag{6}$$

Since $(G'_m)_L \cong (G_m)_L^{n-1}$ ($n := [L : K]$) we can replace the first group of (6) by $K(L; G_1, \dots, G_m)^{n-1}$. By Lemma 1 the composite

$$K(L; G_1, \dots, G_m)^{n-1} \to K(K; G_1, \dots, G_{m-1}, G'_m) \to K(L; G_1, \dots, G_m)$$

is given on the i-th summand by $1 - \sigma^i$. The assertion follows. □

Galois symbol Let $G_1, \dots, G_m$ be connected commutative algebraic groups over K, and let n be an integer prime to the characteristic of K. For any finite extension L/K, we have a homomorphism [So]

$$h_L : K(L; G_1, \dots, G_m)/n \longrightarrow H^m(L, G_1[n] \otimes \dots \otimes G_m[n]) \tag{7}$$

called the *Galois symbol*. This is characterized by the following properties.

(i) If $x_i \in G_i(L)$ for $i = 1, \ldots, m$, then $h_L(\{x_1, \ldots, x_m\}_{L/L}) = (x_1) \cup \ldots \cup (x_m)$. Here we write by (x_i) for the image of x_i in $H^1(L, G_i[n])$ by the connecting homomorphism associated to the exact sequence $1 \to G_i[n] \to G_i \xrightarrow{n} G_i \to 1$.

(ii) If $M/L/K$ is a tower of finite extensions and if M/L is separable (resp. purely inseparable), then the diagram

$$\begin{array}{ccc} K(M; G_1, \ldots, G_m)/n & \xrightarrow[h_M]{} & H^m(M, G_1[n] \otimes \ldots \otimes G_m[n]) \\ \downarrow N_{M/L} & & \downarrow \\ K(L; G_1, \ldots, G_m)/n & \xrightarrow[h_L]{} & H^m(L, G_1[n] \otimes \ldots \otimes G_m[n]) \end{array}$$

is commutative, where the right vertical map is the corestriction (resp. the multiplication by $[M : L]$ under the identification $H^m(M, G_1[n] \otimes \ldots \otimes G_m[n]) \cong H^m(L, G_1[n] \otimes \ldots \otimes G_m[n])$).

Property (i) implies in particular that (7) coincides with the usual Galois symbol (1) in the case $G_1 = \ldots = G_m = \mathbb{G}_m$. In [So] Remark 1.7, Somekawa conjectured that the Galois symbol associated to semiabelian varieties should be injective.

Galois cohomology of cyclic extensions Let L/K be a cyclic Galois extension of degree n and let σ be a generator of $G := \mathrm{Gal}(L/K)$. For a discrete G_K-module M, tensoring the short exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[G] \xrightarrow{1-\sigma} \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0 \tag{8}$$

with M yields a distinguished triangle

$$M[1] \xrightarrow{\alpha} C^{\cdot}(M) \xrightarrow{\beta} M \xrightarrow{\gamma} M[2] \tag{9}$$

in the derived category $D(G_K)$. Here we denote by $C^{\cdot}(M)$ the complex

$$\mathrm{Res}_{L/K} M \xrightarrow{1-\sigma} \mathrm{Res}_{L/K} M$$

concentrated in degree -1 and 0. The spectral sequence

$$E_1^{p,q} = H^q(K, C^p(M)) \Longrightarrow E^{p+q} = H^{p+q}(K, C^{\cdot}(M))$$

induces short exact sequences

$$0 \to H^q(L, M)_G \to H^q(K, C^{\cdot}(M)) \to H^{q+1}(L, M)^G \to 0. \tag{10}$$

It is easy to see that the composite

$$H^{q+1}(K, M) \xrightarrow{\alpha} H^q(K, C^{\cdot}(M)) \to H^{q+1}(L, M)^G$$

is the restriction and

$$H^q(L,M)_G \to H^q(K, C^{\cdot}(M)) \xrightarrow{\beta} H^q(K,M)$$

is induced by the corestriction. In particular we have $\gamma(H^q(K,M)) \subseteq \mathrm{Ker}(\mathrm{res} : H^{q+2}(K,M) \to H^{q+2}(L,M))$ hence

$$n\gamma(H^q(K,M)) = 0. \tag{11}$$

For an integer m prime to $\mathrm{char}\, K$ and $r \in \mathbb{N}$ we write $\mathbb{Z}/m\mathbb{Z}(r) \colon = \mu_m^{\otimes r}$ and

$$H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) \colon = \mathrm{Ker}(H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \xrightarrow{\mathrm{res}} H^3(L, \mathbb{Z}/m\mathbb{Z}(2))).$$

By restricting $\alpha : H^3(K, \mathbb{Z}/m\mathbb{Z}(2)) \to H^2(K, C^{\cdot}(\mathbb{Z}/m\mathbb{Z}(2)))$ to the subgroup $H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2))$ and composing it with the inverse of the first map in (10) we obtain a map

$$H^3(L/K, \mathbb{Z}/m\mathbb{Z}(2)) \to \mathrm{Ker}(H^2(L, \mathbb{Z}/m\mathbb{Z}(2))_G \overset{\mathrm{cor}}{\to} H^2(K, \mathbb{Z}/m\mathbb{Z}(2))). \tag{12}$$

Lemma 6 *Assume that n is prime to* $\mathrm{char}\, K$ *and $\mu_{n^2}(\overline{K}) \subset K$. Then the homomorphism (12) is injective for $m = n$.*

Proof. It is enough to show that $\gamma : H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) \to H^3(K, \mathbb{Z}/n\mathbb{Z}(2))$ is zero. Consider the commutative diagram

$$\begin{array}{ccc} H^1(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow & H^1(K, \mathbb{Z}/n^2\mathbb{Z}(2)) \\ \big\downarrow \gamma & & \big\downarrow \gamma \\ H^3(K, \mathbb{Z}/n\mathbb{Z}(2)) & \longrightarrow & H^3(K, \mathbb{Z}/n^2\mathbb{Z}(2)) \end{array}$$

induced by the canonical injection $\mathbb{Z}/n\mathbb{Z}(2) \to \mathbb{Z}/n^2\mathbb{Z}(2)$. The assumption $\mu_{n^2}(\overline{K}) \subset K$ implies that the upper horizontal map can be identified with

$$K^*/(K^*)^n \longrightarrow K^*/(K^*)^{n^2}, x(K^*)^n \mapsto x^n(K^*)^{n^2}.$$

In particular the image is contained in $nH^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))$. By (11) it is mapped under γ to $n\gamma(H^1(K, \mathbb{Z}/n^2\mathbb{Z}(2))) = 0$. On the other hand it is a simple consequence of the Merkurjev-Suslin theorem [MS] that the lower horizontal map is injective. Hence $\gamma(H^1(K, \mathbb{Z}/n(2))) = 0$. □

The counterexample Let L/K be as in the last section and let $T \colon = \mathrm{Ker}(N_{L/K} : \mathrm{Res}_{L/K}\, \mathbb{G}_m \to \mathbb{G}_m)$. We make the following assumptions

(13) n is prime to $\mathrm{char}\, K$ and $\mu_{n^2}(\overline{K}) \subset K$,

(14) $H^3(L/K, \mathbb{Z}/n\mathbb{Z}(2))) \neq 0$.

Proposition 7 *The Galois symbol* $K(K;T,T)/n \to H^2(K, T[n] \otimes T[n])$ *is not injective.*

Proof. Let σ be a generator of $G\colon = \mathrm{Gal}(L/K)$. The exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Res}_{L/K}\,\mathbb{G}_m \stackrel{1-\sigma}{\longrightarrow} \mathrm{Res}_{L/K}\,\mathbb{G}_m \stackrel{N_{L/K}}{\longrightarrow} \mathbb{G}_m \longrightarrow 1$$

yields two short exact sequences

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Res}_{L/K}\,\mathbb{G}_m \longrightarrow T \longrightarrow 1, \tag{15}$$

$$1 \longrightarrow T \longrightarrow \mathrm{Res}_{L/K}\,\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1. \tag{16}$$

Correspondingly, (8) induces two short exact sequences

$$0 \to \mathbb{Z} \to \mathbb{Z}[G] \to X \to 0, \qquad 0 \to X \to \mathbb{Z}[G] \to \mathbb{Z} \to 0 \tag{17}$$

where X denotes the cocharacter group of T. Note that the sequence (15) is Zariski exact by Hilbert 90. Since the map $\mathrm{Res}_{L/K}\,\mathbb{G}_m \to T$ factors through $\mathrm{Res}_{L/K}\,\mathbb{G}_m \to \mathrm{Res}_{L/K}\,T \to T$ the sequence $\mathrm{Res}_{L/K}\,T \to T \to 1$ is Zariski exact as well. By Lemma 5 the upper horizontal map in the diagram

$$\begin{array}{ccc} (K(L;T,T)/n)_G & \stackrel{N_{L/K}}{\longrightarrow} & K(K;T,T)/n \\ \downarrow & & \downarrow \\ H^2(L, T[n] \otimes T[n])_G & \stackrel{\mathrm{cor}}{\longrightarrow} & H^2(K, T[n] \otimes T[n]) \end{array}$$

is an isomorphism. The vertical maps are Galois symbols. Since T_L is a split torus the left vertical map is an isomorphism by the Merkurjev-Suslin theorem [MS]. Thus to finish the proof it remains to show that the lower horizontal arrow is not injective. Note that $T[n] \cong \mathbb{Z}/n\mathbb{Z}(1) \otimes X$. Hence the assertion follows from Lemma 6 and Lemma 8 below.

Lemma 8 *There exist homomorphisms of G-modules* $e : \mathbb{Z} \to X \otimes_{\mathbb{Z}} X$ *and* $f : X \otimes_{\mathbb{Z}} X \to \mathbb{Z}$ *such that* $f \circ e : \mathbb{Z} \to \mathbb{Z}$ *is multiplication by* $n-1$.

Proof. For a G-module M we write $M^\vee$ for the G-module $\mathrm{Hom}(M, \mathbb{Z})$. Let $(\ ,\ ) : \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G] \longrightarrow \mathbb{Z}$ be the symmetric pairing given by

$$(g, g') \quad = \quad \begin{cases} 1 & \text{if } g = g', \\ 0 & \text{if } g \neq g'. \end{cases} \tag{18}$$

It yields an isomorphism $\mathbb{Z}[G] \to \mathbb{Z}[G]^\vee$. For a submodule $M \subseteq \mathbb{Z}[G]$ let

$$M^\perp = \{x \in \mathbb{Z}[G] \mid (x, m) = 0 \ \forall\, m \in M\}.$$

Then we have $X^\perp = \mathbb{Z}S$ and $(\mathbb{Z}S)^\perp = X$ where $S = \sum_{i=0}^{n-1} \sigma^i$. Thus (18) yields an isomorphism $X \cong (\mathbb{Z}[G]/\mathbb{Z}S)^\vee$. By (17) we have $\mathbb{Z}[G]/\mathbb{Z}S \cong X$, hence

$$X \otimes_{\mathbb{Z}} X \quad \cong \quad X \otimes_{\mathbb{Z}} X^\vee \quad \cong \quad \mathrm{Hom}(X, X)$$

Thus it suffices to prove the assertion for $\mathrm{Hom}(X, X)$. Obviously, for the two maps $e : \mathbb{Z} \to \mathrm{Hom}(X, X), m \mapsto m\,\mathrm{id}_X$ and $f : \mathrm{Hom}(X, X) \to \mathbb{Z}, \tau \mapsto \mathrm{Tr}(\tau)$ we have $f \circ e = \mathrm{rank}(X) = n - 1$. □

Remark 9 It is easy to construct examples where the assumptions (13) and (14) above are satisfied. For instance if K is a 2-local field satisfying property (13) and L/K is any cyclic extension of degree n then (14) holds by [Ka].

3 Counterexample to a conjecture of Beilinson

We first introduce some notation and recall a few facts from [Vo1] and [MVW]. Let K be a field of characteristic zero. Let Cor_K denote the additive category of finite correspondences ([MVW], 1.1; see also Remark 2 above) and let $D^-(Shv_{\mathrm{Nis}}(Cor_K))$ (resp. $D^-(Shv_{\mathrm{et}}(Cor_K))$) denote the derived category of complexes of Nisnevich (resp. étale) sheaves with transfers bounded from above.

The category of effective motivic complexes $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$ (resp. étale effective motivic complexes $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{et}}(K)$) is the full subcategory of $D^-(Shv_{\mathrm{Nis}}$ $(Cor_K))$ (resp. $D^-(Shv_{\mathrm{et}}(Cor_K))$) which consists of complexes $C^\star$ with homotopy invariant cohomology sheaves $H^i(C^\star)$ for all i (see [Vo1], 3.1 or [MVW], 14.1, resp. 9.2). $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$ and $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{et}}(K)$ are triangulated tensor categories. They are equipped with the t-structure induced from the standard t-structure on $D^-(Shv_{\mathrm{Nis}}(Cor_K))$ (resp. $D^-(Shv_{\mathrm{et}}(Cor_K))$). There is a covariant functor $M : Cor_K \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K), X \mapsto M(X)$ and we have $M(X \times Y) = M(X) \otimes M(Y)$. There is also the "change of topology" functor $\alpha^* : \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{et}}(K)$. It is a tensor functor which admits a right adjoint $R\alpha_* : \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{et}}(K) \to \mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$.

Beilinson [Be] has proposed the following generalization of (a part of) the Milnor-Bloch-Kato conjecture: For any smooth affine K-scheme X the adjunction morphism $M(X) \to R\alpha_*\alpha^* M(X)$ induces an isomorphism on cohomology in degrees ≤ 0, i.e. the map

$$(19) \qquad a_X : M(X) \longrightarrow t_{\leq 0} R\alpha_*\alpha^* M(X)$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$.

If $X = (\mathbb{G}_m)^d = \mathbb{G}_m \times \ldots \times \mathbb{G}_m$ (d-fold product of $\mathbb{G}_m$) we have $M(X) \cong (\mathbb{Z} \oplus \mathbb{Z}(1)[1])^d$. Thus a_X is an isomorphism if and only if

$$\mathbb{Z}(n) \longrightarrow t_{\leq n} R\alpha_* \alpha^* \mathbb{Z}(n) \tag{20}$$

is an isomorphism for all $n \leq d$, which implies the injectivity of the Galois symbol (1). It is known (compare [SV]) that the Milnor-Bloch-Kato conjecture is equivalent to a stronger assertion that

$$\mathbb{Z}(n) \longrightarrow t_{\leq n+1} R\alpha_* \alpha^* \mathbb{Z}(n)$$

is an isomorphism for all $n \geq 0$.

Let L/K be a separable quadratic extension and let $T \colon = \mathrm{Ker}(N_{L/K} : \mathrm{Res}_{L/K} \mathbb{G}_m \to \mathbb{G}_m)$. We shall show that (19) is in general not an isomorphism for $X = T^n$ for $n \geq 2$. By ([HK], 7.3) there exists a canonical decomposition $M(T) = \mathbb{Z} \oplus \mathbb{Z}(L/K, 1)[1]$ where $\mathbb{Z}(L/K, 1)$ is the cone of the morphism $\mathbb{Z}(1) \to \mathrm{Res}_{L/K} \mathbb{Z}(1)$.

Remarks 10 (a) Here is a more explicit description of the motive $\mathbb{Z}(L/K, 1)$. The torus T defines a homotopy invariant étale (hence Nisnevich) sheaf with transfers and therefore an element of $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$. We have

$$\mathbb{Z}(L/K, 1) \quad \cong \quad T[-1].$$

This can be deduced from the corresponding statement for $\mathbb{G}_m$ ([MVW], 4.1) and the exactness of (15) (as a sequence in $\mathit{Shv}_{\mathrm{Nis}}(\mathit{Cor}_K)$).

(b)[1] Let $A_1, \ldots, A_n$ be semi-abelian varieties over K. Recently, B. Kahn proved that the generalized Milnor K-group $K(K; A_1, \ldots, A_n)$ surjects onto a Hom-group in $\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)$. For that we view $A_1, \ldots, A_n$ again as elements in $\mathit{Shv}_{\mathrm{Nis}}(\mathit{Cor}_K)$. Then he showed that there exists a surjection

$$K(K; A_1, \ldots, A_n) \to \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff},-}_{\mathrm{Nis}}(K)}(\mathbb{Z}, A_1 \otimes \ldots \otimes A_n).$$

A conjecture of Somekawa [So, introduction] can be interpreted as the bijectivity of this homomorphism. If $A_1 = \ldots = A_n = \mathbb{G}_m$ this is proved in ([MVW], lecture 5).

For $p, q \geq 0$ and $n = p + q$ we define

$$\mathbb{Z}(L/K, p, q) \colon = \mathbb{Z}(L/K, 1)^{\otimes p} \otimes \mathbb{Z}(q)$$

and denote by $C(p, q)$ the cone of $\mathbb{Z}(L/K, p, q) \longrightarrow R\alpha_* \alpha^* \mathbb{Z}(L/K, p, q)$. Note that $\mathbb{Z}(L/K, p, q)[n]$ is a direct summand of $M(T^p \times (\mathbb{G}_m)^q)$. We also put $C(n) \colon = C(0, n)$. We have

$$H^k(C(n)) = 0 \ (k \leq n+1), \quad H^{n+2}(C(n))(K) \cong H^{n+1}(K, \mathbb{Q}/\mathbb{Z}(n)). \tag{21}$$

[1] This remark has been communicated to us by B. Kahn in a private correspondence.

This follows from the Milnor-Bloch-Kato conjecture (in fact for our purpose we need (21) only after localization at the prime 2 where it follows from the Milnor conjecture [Vo2]).

Tensoring $\mathbb{Z}(1) \to \mathrm{Res}_{L/K}\,\mathbb{Z}(1) \to \mathbb{Z}(L/K,1) \to \mathbb{Z}(1)[1]$ with $\mathbb{Z}(L/K,p-1,q)$ (for $p \geq 1, q \geq 0$) yields a distinguished triangle

$$\mathbb{Z}(L/K,p-1,q+1) \to \mathrm{Res}_{L/K}\,\mathbb{Z}(n) \to \mathbb{Z}(L/K,p,q) \to \mathbb{Z}(L/K,p-1,q+1)[1]$$

hence also a triangle

$$(22) \qquad C(p-1,q+1) \to \mathrm{Res}_{L/K}\,C(n) \to C(p,q) \to C(p-1,q+1)[1].$$

The following Lemma follows easily by induction on p using (21) and (22).

Lemma 11 *Let $p \geq 1, q \geq 0$ and $n = p+q$. Then we have $H^k(C(p,q)) = 0$ for $k < q+2$ and*

$$H^{q+2}(C(p,q))(K) \cong H^{n+1}(L/K,\mathbb{Q}/\mathbb{Z}(n))$$

where $H^{n+1}(L/K,\mathbb{Q}/\mathbb{Z}(n)) \colon = \mathrm{Ker}(H^{n+1}(K,\mathbb{Q}/\mathbb{Z}(n)) \xrightarrow{\mathrm{res}} H^{n+1}(L,\mathbb{Q}/\mathbb{Z}(n)))$.

Since $[L:K]=2$ we have

$$\begin{aligned} H^{n+1}(L/K,\mathbb{Q}/\mathbb{Z}(n)) &\cong H^{n+1}(L/K,\mathbb{Q}_2/\mathbb{Z}_2(n)) \cong H^{n+1}(L/K,\mathbb{Z}/2\mathbb{Z}(n)) \\ &\cong H^{n+1}(L/K,\mathbb{Z}/2\mathbb{Z}) \end{aligned}$$

(the second isomorphism is a consequence of the Milnor conjecture). Now the following Proposition follows by applying Lemma 11 for $(p,q)=(2,0)$ and $(n,0)$.

Proposition 12 *(a) There exists a short exact sequence*

$$0 \longrightarrow H^0(M(T\times T))(K) \longrightarrow R^0\alpha_*\alpha^* M(T\times T)(K) \longrightarrow H^3(L/K,\mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

In particular if $H^3(L/K,\mathbb{Z}/2\mathbb{Z}) \neq 0$ then (19) is not an isomorphism for $X = T \times T$.

(b) More generally let n be an integer ≥ 2 and assume that $H^{n+1}(L/K,\mathbb{Z}/2\mathbb{Z}) \neq 0$. Then the map (19) is not an isomorphism for $X = T^n$. More precisely either the map

$$H^{2-n}(M(X)) \longrightarrow R^{2-n}\alpha_*\alpha^* M(X)$$

is not surjective or

$$H^{3-n}(M(X)) \longrightarrow R^{3-n}\alpha_*\alpha^* M(X)$$

is not injective.

An n-local field K of characteristic 0 provides an example where the above assumption holds. In fact by [Ka] we have $H^{n+1}(L/K,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for such fields.

References

[Be] A.A. Beilinson, Talk at Fields Institute at Toronto on 21 March 2007.

[HK] A. Huber, B. Kahn, *The slice filtration and mixed Tate motives.* Compos. Math. **142** (2006), 907–936.

[Ka] K. Kato, *A generalization of local class field theory by using K-groups I*, J. Fac. Sci. Univ. Tokyo **26** (1979), 303–376.

[MVW] C. Mazza, V. Voevodsky and C. Weibel, *Lecture notes on motivic cohomology.* Clay Mathematics Monographs **2**. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.

[MS] A.S. Merkurjev, A.A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.* Math. USSR Izvestiya **21** (1983), 307–340.

[Se] J.-P. Serre, *Algebraic groups and class fields.* Graduate Texts in Mathematics **117**. Springer Verlag 1988.

[So] M. Somekawa, *On Milnor K-groups attached to semi-abelian varieties.* K-theory **4** (1990), 105–119.

[SV] A.A. Suslin,V. Voevodsky, *Bloch-Kato conjecture and motivic cohomology with finite coefficients.* In: The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

[Vo1] V. Voevodsky, *Triangulated categories of motives over a field.* In: V. Voevodsky, A. Suslin, E.M. Friedlander: Cycles, Transfers, and Motivic Homology Theories. Annals of Math. Studies **143**, Princeton University Press, 2000, 188–238.

[Vo2] V. Voevodsky, *Motivic cohomology with $\mathbb{Z}/2$-coefficients.* Publ. Math., Inst. Hautes Étud. Sci. 98, 59-104 (2003).

Michael Spieß
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld, Germany
mspiess@math.uni-bielefeld.de

Takao Yamazaki
Mathematical Institute
Tohoku University
Aoba
Sendai 980-8578, Japan
ytakao@math.tohoku.ac.jp