

MILNOR K -GROUP ATTACHED TO A TORUS AND BIRCH-TATE CONJECTURE

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To Professor Tatsuo Kimura on the occasion of his sixtieth birthday

ABSTRACT. We formulate (and prove under a certain assumption) a conjecture relating the order of Somekawa's Milnor K -group attached to a torus T and the value of the Artin L -function attached to the cocharacter group of T (regarded as an Artin representation) at $s = -1$. The case $T = \mathbb{G}_m$ reduces to the classical Birch-Tate conjecture.

1. INTRODUCTION

The Birch-Tate conjecture states that, for a totally real number field K , the following equality should hold:

$$|\zeta_K(-1)| = \frac{|K_2(O_K)|}{|W_2(K)|}.$$

Here $\zeta_K(s)$ is the Dedekind zeta function of K , $K_2(O_K)$ is the second K -group of the ring O_K of integers in K , and $W_2(K) = H^0(K, \mathbb{Q}/\mathbb{Z}(2))$. This equality is proved up to a power of 2 by Wiles [21].

We shall formulate a conjecture with coefficients in a torus T , which reduces to the Birch-Tate conjecture recalled above when $T = \mathbb{G}_m$. Let T be a torus over a number field K , and let $X(T) = \text{Hom}(\mathbb{G}_m, T)$ be the cocharacter group. Then $X \otimes \mathbb{C}$ is a finite dimensional representation of the absolute Galois group G_K of K with finite image, i.e. an Artin representation. Let $L_K(X(T), s)$ be the Artin L -function attached to it. We set $W^T(K) = H^0(K, X(T) \otimes \mathbb{Q}/\mathbb{Z}(2))$. We write $K^T(K)$ for the Milnor K -group $K(K; T, \mathbb{G}_m)$ attached to T and \mathbb{G}_m introduced by Somekawa [17]. (We will recall the definition of $K^T(K)$ in §2.) In §4, we define a subgroup $K^T(O_K)$ of $K^T(K)$. When $T = \mathbb{G}_m$, we have identities

$$L_K(X(T), s) = \zeta_K(s), \quad W^T(K) \cong W_2(K), \quad K^T(O_K) \cong K_2(O_K).$$

We propose the following generalization of the Birch-Tate conjecture, which we shall prove for a certain class of tori.

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Conjecture 1.1. *Let T be a torus over a totally real number field K . Assume T is split by a totally real field. Then the equality*

$$|L_K(X(T), -1)| = \frac{|K^T(O_K)|}{|W^T(K)|}$$

should hold.

Remark 1.2. The assumption that T is split by a totally real number field implies that $L_K(X(T), -1)$ is a non-zero (rational) number (cf. proof of Theorem 4.8).

In §2, we introduce a condition for a torus (over an arbitrary field) to ‘admit a motivic interpretation’, and prove the following.

Proposition 1.3. *A torus split by a meta-cyclic extension admits a motivic interpretation. (A finite Galois extension E/F of fields is called meta-cyclic if all Sylow subgroups of $\text{Gal}(E/F)$ are cyclic.)*

We also have some examples of tori which admits a motivic interpretation without being split by a meta-cyclic extension (see Remark 2.10). Our main result is the following.

Theorem 1.4. *Let L/K be an extension of totally real fields, and let T be a torus over K split by L . If T admits a motivic interpretation, then the equality*

$$|L_K(X(T), -1)| = \frac{|K^T(O_K)|}{|W^T(K)|}$$

holds up to a power of 2.

This result will be proved in §4, where we also prove an analogous result for a torus over a global field of positive characteristic. In §3, we study $K^T(k)$ for a torus T over a local field k .

1.1. Conventions. For a field F , we fix an algebraic closure \bar{F} , and all algebraic extension of F is supposed to be a subfield of \bar{F} . We write G_F for the absolute Galois group of F . For a torus T over a F , we write $X(T) = \text{Hom}(T, \mathbb{G}_m)$ for the cocharacter group of T .

Let A be an abelian group. For a non-zero integer n , we write $A[n]$ and A/n for the kernel and cokernel of the map $n : A \rightarrow A$. We define $A_{\text{Tor}} = \cup_n A[n]$ (resp. $A_{\text{div}} = \text{Im}(\text{Hom}(\mathbb{Q}, A) \rightarrow \text{Hom}(\mathbb{Z}, A) = A)$) to be the subgroup of torsion elements (resp. the maximal divisible subgroup) in A . For a prime number p , we define $A[p^\infty] = \cup_n A[p^n]$ (resp. $A_{p\text{-div}} = \text{Im}(\text{Hom}(\mathbb{Z}[\frac{1}{p}], A) \rightarrow \text{Hom}(\mathbb{Z}, A) = A)$) to be the subgroup of p -primary torsion elements (resp. the maximal p -divisible subgroup) in A . We write $A/\text{div} = A/A_{\text{div}}$ and $A/p\text{-div} = A/A_{p\text{-div}}$. When a group G acts on A , we write A^G and A_G for the invariants and coinvariants of A by G .

2. MILNOR K -GROUP ATTACHED TO A TORUS

In this section, F will be an arbitrary field.

2.1. Definition and basic properties. Somekawa [17] has introduced the Milnor K -group $K(F; G_1, \dots, G_r)$ attached to a family of semi-abelian varieties G_1, \dots, G_r over F . In this paper, we only need a special case where $G_1 = T$ is a torus, $G_2 = \mathbb{G}_m$ and $r = 2$. To ease the notation, we put

$$K^T(F) = K(F; T, \mathbb{G}_m).$$

It is defined as a quotient

$$(1) \quad K^T(F) = \bigoplus_{E/F} T(E) \otimes E^* / R,$$

where E runs all finite extensions of F , and R is the group generated by the elements of the following form:

- (Projection formula) Let $E_1/E_2/F$ be a tower of finite extensions, and let $a \in T(E_1), b \in E_2^*$. Then

$$N_{E_2}^{E_1}(a) \otimes b - a \otimes R_{E_2}^{E_1}(b)$$

is a generator of R . Here $N_{E_2}^{E_1} : T(E_1) \rightarrow T(E_2)$ and $R_{E_2}^{E_1} : E_2^* \hookrightarrow E_1^*$ are the norm and restriction maps respectively.

- (Weil reciprocity) Let $F(C)$ be a function field of one variable over F , and let S be the set of all normalized discrete valuation on $F(C)$ over F . For $v \in S$, we write O_v (resp. F_v) for the valuation ring (resp. the residue field). Let $a \in T(F(C))$ and $b, c \in F(C)^*$. Set $S(b) = \{v \in S \mid v(b) \neq 0\}$. Assume that $a \in T(O_{v_i})$ if $v \notin S(b)$. Then

$$\sum_{v \in S(b)} a(v) \otimes \partial_v(b, c) + \sum_{v \in S \setminus S(b)} \partial_v(a, c) \otimes b(v)$$

is a generator of R . Here ∂_v is the local symbol defined in [15], while $a(v) \in T(F_v)$ and $b(v) \in F_v^*$ denote the the reduction of a and b respectively. (Recall that $\partial_v(b, c)$ is the usual tame symbol.)

The class of $a \otimes b \in T(E) \otimes E^*$ in $K^T(F)$ is written by $\{a, b\}_{E/F}$. We recall some properties of this group.

Lemma 2.1 ([17] Theorem 1.4.). *The correspondence $\{a, b\}_{E/F} \mapsto N_F^E \{a, b\}$ defines a canonical isomorphism*

$$K^{\mathbb{G}_m}(F) \cong K_2(F),$$

where the right hand side is the usual second K -group, and N_F^E is the norm map.

We often identify $K^{\mathbb{G}_m}(F)$ with $K_2(F)$ by this isomorphism.

Lemma 2.2 ([17] Proposition 1.5). *Let T be a torus over F , and let n be a natural number invertible in F . Then we have a homomorphism*

$$h_F^T : K^T(F)/n \rightarrow H^2(F, T[n] \otimes \mu_n)$$

called the Galois symbol. This map satisfies that $h_F^T(\{a, b\}_{E/F}) = \text{Cor}_F^E((a) \cup (b))$ for any finite extension E/F , $a \in T(E)$ and $b \in E^*$. Here $(a) \in H^1(E, T[n])$ denotes the image of a by the connecting homomorphism associated to the exact sequence $1 \rightarrow T[n] \rightarrow T \xrightarrow{n} T \rightarrow 1$, and similarly for $(b) \in H^1(E, \mu_n)$.

Remark 2.3. By the Merkurjev-Suslin theorem [9], the Galois symbol h_F^T above is bijective when $T = \mathbb{G}_m$. It is conjectured by Somekawa [17] that h_F^T should be always injective. In Proposition 2.11 below, we shall show the injectivity of h_F^T under a certain assumption on T . However, the surjectivity does not hold in general. (For example, see Proposition 3.4). See [16] for a related result.

Lemma 2.4 ([16] Lemma 3). *Let E/F be a finite separable extension of fields. Let T be a torus over E , and let $S = \text{Res}_F^E T$ be the Weil restriction. Then, we have an isomorphism*

$$K^T(E) \cong K^S(F).$$

A sequence of algebraic groups $G' \rightarrow G \rightarrow G''$ over F is called *Zariski exact* if $G'(E) \rightarrow G(E) \rightarrow G''(E)$ is exact for any extension E/F .

Lemma 2.5 ([16] Lemma 2). *Let $R \rightarrow S \rightarrow T \rightarrow 0$ be a Zariski exact sequence of tori over F . Then the sequence*

$$K^R(F) \rightarrow K^S(F) \rightarrow K^T(F) \rightarrow 0$$

is exact as well.

2.2. Motivic interpretation. We recall Lichtenbaum's weight two motivic complex.

Review 2.6. Let $\mathbb{Z}(2)$ be the weight two motivic complex, which is a two-term complex of discrete G_F -modules (concentrated on degrees one and two) constructed by Lichtenbaum [6, 7]. We recall some properties of $\mathbb{Z}(2)$.

- (1) There is a canonical isomorphism $H^2(F, \mathbb{Z}(2)) \cong K_2(F)$.
- (2) We have $H^3(F, \mathbb{Z}(2)) = 0$.
- (3) If $n \in \mathbb{Z}$ is invertible in F , then we have a triangle $\mathbb{Z}(2) \xrightarrow{n} \mathbb{Z}(2) \rightarrow \mu_n^{\otimes 2} \rightarrow \mathbb{Z}(2)[1]$.
- (4) If the characteristic p of F is positive, then we have a triangle $\mathbb{Z}(2) \xrightarrow{p^s} \mathbb{Z}(2) \rightarrow \nu_s(2)[-2] \rightarrow \mathbb{Z}(2)[1]$, where $\nu_s(2)$ is the second logarithmic Hodge-Witt sheaf of level s .
- (5) There is a product map $\mathbb{Z}(1) \otimes^{\mathbb{L}} \mathbb{Z}(1) \rightarrow \mathbb{Z}(2)$. (Here $\mathbb{Z}(1) \cong \mathbb{G}_m[-1]$.)

Let T be a torus over F , and let $X = X(T)$ be the cocharacter group of T . For a finite extension E/F , we have a homomorphism

$$(2) \quad T(E) \otimes E^* \cong H^1(E, X \otimes \mathbb{Z}(1)) \otimes H^1(E, \mathbb{Z}(1)) \xrightarrow{\cup} H^2(E, X \otimes \mathbb{Z}(2)) \xrightarrow{N_F^E} H^2(F, X \otimes \mathbb{Z}(2))$$

deduced by the product and norm maps.

Definition 2.7. We say T admits a motivic interpretation if the homomorphism (2) induces, via eq. (1), an isomorphism

$$K^T(F) \xrightarrow{\cong} H^2(F, X \otimes \mathbb{Z}(2)).$$

We expect any torus admits a motivic interpretation. In the next subsection, we prove this under a certain assumption.

Remark 2.8. (1) It follows from Lemma 2.4 and Shapiro's lemma that, if a torus T over F admits a motivic interpretation, then the base change $T \otimes_F E$ of T by a finite separable extension E/F admits a motivic interpretation as well.

(2) In order to prove that the map (2) factors through $K^T(F)$, one has to show that it kills R in eq. (1). There is no difficulty in proving this for the projection formula. As for the Weil reciprocity, it seems that a natural way to prove this is to use the weight three motivic complex $\mathbb{Z}(3)$ (and to show the vanishing of $H^3(F(C), X \otimes \mathbb{Z}(3)) \rightarrow \bigoplus_v H^2(F(v), X \otimes \mathbb{Z}(2)) \xrightarrow{\text{sum}} H^2(F, X \otimes \mathbb{Z}(2))$ where $F(C)$ is the function field of an irreducible smooth proper curve C over F , and v runs all closed points of C). If one used Voevodsky's definition of $\mathbb{Z}(r)$, this would follow from the Gysin sequence [22]. However, Voevodsky's theory is developed under the assumption of the resolution of singularity. Because we will also consider the global fields of positive characteristic, we avoid the use of Voevodsky's theory. See also [10] for a related result.

2.3. Tori split by a meta-cyclic extension. We recall some facts from [3]. A torus P over F is called quasi-trivial if P is isomorphic to $\bigoplus_i \text{Res}_F^{E_i} \mathbb{G}_m$, where E_i runs a family of finite extensions of F . A torus Q over F is called flasque if $H^1(E, X(Q)) = 0$ for all finite extension E/F . A torus I over F is called invertible if there exists a torus I' over F such that $I \oplus I'$ is quasi-trivial. We have implications 'quasi-trivial \Rightarrow invertible \Rightarrow flasque'.

If T is a torus over F split by E , then there exists an exact sequence

$$(3) \quad 0 \rightarrow Q \rightarrow P \rightarrow T \rightarrow 0,$$

where P (resp. Q) is a quasi-trivial (resp. flasque) torus over F split by E . We call (3) a flasque resolution of T . A flasque resolution (3) is unique up to a direct summand of a quasi-trivial torus in P and Q .

Proposition 2.9. *Let T be a torus over F , and let (3) be a flasque resolution of T . If Q is invertible, then T admits a motivic interpretation.*

Proof. Review 2.6 (1) and Lemma 2.1 show that a split torus admits a motivic interpretation. By Lemma 2.4 and Shapiro's lemma, the same holds for a quasi-trivial torus, hence also for an invertible torus.

Assume a torus T admits a flasque resolution (3). If Q is invertible, then $H^1(F', Q) = H^3(F', X(Q) \otimes \mathbb{Z}(2)) = 0$ for any extension F'/F by Hilbert 90 and Review 2.6 (2). This

in particular implies that (3) is Zariski exact, and we have by Lemma 2.5 a commutative diagram with exact rows

$$\begin{array}{ccccccc} K^Q(F) & \rightarrow & K^P(F) & \rightarrow & K^T(F) & \rightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ H^2(F, X(Q) \otimes \mathbb{Z}(2)) & \rightarrow & H^2(F, X(P) \otimes \mathbb{Z}(2)) & \rightarrow & H^2(F, X(T) \otimes \mathbb{Z}(2)) & \rightarrow & 0, \end{array}$$

showing the well-definedness and bijectivity of the right vertical map. \square

Proof of Proposition 1.3. It follows from a result of Endo-Miyata [4] (see also [3]) that a flasque torus split by a meta-cyclic extension is always invertible. Now Proposition 1.3 is a consequence of Proposition 2.9. \square

Remark 2.10. We give a few examples of a torus T which satisfies the assumption of Proposition 2.9 without being split by a meta-cyclic extension.

- (1) Let E/F be a finite Galois extension which is not meta-cyclic. Let T be the kernel of the norm map $\text{Res}_F^E \mathbb{G}_m \rightarrow \mathbb{G}_m$. Then the dual torus \check{T} of T satisfies the assumption of Proposition 2.9, since it fits into an exact sequence $0 \rightarrow \mathbb{G}_m \rightarrow \text{Res}_F^E \mathbb{G}_m \rightarrow \check{T} \rightarrow 0$.
- (2) Let C be an integral proper curve over F whose normalization is isomorphic to the projective line \mathbb{P}^1 . Assume that all singular points on C are of coordinate axes type (cf. [20]). Then the generalized Jacobian variety T of C is a torus satisfying the assumption of Proposition 2.9. Indeed, such T fits into an exact sequence

$$0 \rightarrow \bigoplus_s \text{Res}_F^{F_s} \mathbb{G}_m \rightarrow \bigoplus_s \bigoplus_{t \in S(s)} \text{Res}_F^{F_t} \mathbb{G}_m \rightarrow T \rightarrow 0,$$

where s runs all singular points of C , and $S(s)$ is the inverse image of s by the normalization map.

2.4. A few auxiliary results.

Proposition 2.11. *Let T be a torus over F and let $X = X(T)$. Assume that T admits a motivic interpretation, and that $n \in \mathbb{Z}$ is invertible in F . Then we have an isomorphism*

$$H^0(F, T[n] \otimes \mu_n) \cong H^1(F, X \otimes \mathbb{Z}(2))[n]$$

and exact sequences

$$\begin{aligned} 0 \rightarrow H^1(F, X \otimes \mathbb{Z}(2))/n \rightarrow H^1(F, T[n] \otimes \mu_n) \rightarrow K^T(F)[n] \rightarrow 0 \\ 0 \rightarrow K^T(F)/n \rightarrow H^2(F, T[n] \otimes \mu_n) \rightarrow H^3(F, X \otimes \mathbb{Z}(2))[n] \rightarrow 0. \end{aligned}$$

Proof. This follows from the distinguished triangle $X \otimes \mathbb{Z}(2) \xrightarrow{n} X \otimes \mathbb{Z}(2) \rightarrow T[n] \otimes \mu_n \rightarrow X \otimes \mathbb{Z}(2)[1]$ deduced from Review 2.6 (3). \square

Corollary 2.12. *Let T be a torus over F and let $X = X(T)$. Assume T admits a motivic interpretation. Let p be a prime different from the characteristic of F . For a natural number r , we put $\hat{X}_p(r) = \varprojlim X \otimes \mu_p^{\otimes r}$. Then we have isomorphisms*

$$K^T(F)[p^\infty]/\text{div} \cong H^2(F, \hat{X}_p(2))_{\text{Tor}} \cong H^1(F, X \otimes \mathbb{Q}_p/\mathbb{Q}_p(2))/\text{div}.$$

(Here $H^2(F, \hat{X}_p(2))$ denotes the continuous Galois cohomology.)

Proof. This proof is almost identical to [19] Theorem 3.5. Set $M = T[p] \otimes \mu_p$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & K^T(F)[p] & \rightarrow & K^T(F) & \xrightarrow{p} & K^T(F) & \rightarrow & K^T(F)/p & \rightarrow & 0 \\ & & \uparrow \text{surj} & & \downarrow h & & \downarrow h & & \downarrow \text{inj} & & \\ & & H^1(F, M) & \rightarrow & H^2(F, \hat{X}_p(2)) & \rightarrow & H^2(F, \hat{X}_p(2)) & \rightarrow & H^2(F, M) & & \end{array}$$

where the left and right vertical arrows are the maps in Proposition 2.11, and h is the ‘continuous symbol’ defined by the same way as Tate [19]. Since $H^2(F, \hat{X}_p(2))$ has no p -divisible subgroup (cf. [19] Proposition 2.1), we see $\ker(h) = K^T(F)_{p\text{-div}}$ and $\text{coker}(h)_{\text{Tor}} = 0$. This implies that $K^T(F)[p^\infty] \rightarrow H^2(F, \hat{X}_p(2))_{\text{Tor}}$ is a surjection whose kernel is $K^T(F)[p^\infty]_{\text{div}} = K^T(F)_{\text{div}}[p^\infty]$. This proves the first identity. The second isomorphism is given by [19] Proposition 2.3. \square

Lemma 2.13. *Let T be a torus over F . Then $H^3(F, X(T) \otimes \mathbb{Z}(2))$ is a torsion group of finite exponent. Moreover we have $H^3(F, X(T) \otimes \mathbb{Z}(2))[p^\infty] = 0$ if $\text{cd}_p(F) \leq 2$ for a prime $p \neq \text{Char}(F)$, or if $[F : F^p] \leq p$ for $p = \text{Char}(F)$.*

Proof. We set $X(2) = X(T) \otimes \mathbb{Z}(2)$. We take a finite separable extension E/F which splits T . Then we know $H^3(E, X(2)) = 0$ by Review 2.6 (2). By the norm argument, we see that $H^3(F, X(2))$ is annihilated by $n = [L : K]$. To prove the second assertion, we write $n = p^k m$ with $(p, m) = 1$. By Review 2.6 (3) (resp. (4)), $H^3(F, X(2))[p^\infty]$ injects to $H^3(F, T[p^k] \otimes \mu_{p^k})$ (resp. $H^1(F, X(T) \otimes \nu_k(2))$) when $p \neq \text{Char}(F)$ (resp. $p = \text{Char}(F)$), which is trivial by assumption. \square

Lemma 2.14. *Let T be a torus over a field F of positive characteristic p . Assume $[F : F^p] \leq p$. Then $K^T(F)$ and $K^T(F)[p^\infty]$ are p -divisible.*

Proof. (Cf. [18] p. 205.) It is enough to show the p -divisibility of $K^T(F)$. We take $x \in T(E), y \in E^*$ where E/F is a finite extension. Because the norm maps $(E^{1/p})^* \rightarrow E^*$ and $T(E^{1/p}) \rightarrow T(E)$ are bijective, there exist $x' \in T(E^{1/p}), y' \in (E^{1/p})^*$ such that $N(x') = x, N(y') = y$. Then we have $\{x, y\}_{E/F} = p\{x', y'\}_{E^{1/p}, F}$, and we are done. \square

3. LOCAL FIELD

When k is a local field, we can prove that any torus over k admits a motivic interpretation. This is an immediate consequence of Theorem 1.3 if $k = \mathbb{R}$ (or $k = \mathbb{C}$).

3.1. Non-archimedean local field.

Lemma 3.1. *Let T be a torus over a non-archimedean local field k .*

- (1) $K^T(k)$ is the direct sum of a finite group and a uniquely divisible group.
- (2) If $p = \text{Char}(k) > 0$, then $K^T(k)[p^\infty] = 0$.
- (3) Let k_1/k be a finite extension. Then the norm map $N_k^{k_1} : K^T(k_1) \rightarrow K^T(k)$ is surjective.
- (4) Let m be a natural number invertible in k . Then, the Galois symbol

$$K^T(k)/m \rightarrow H^2(k, T[m] \otimes \mu_m)$$

is bijective.

- (5) Let $\hat{X}(r) = \varprojlim X(T) \otimes \mu_n^{\otimes r}$ and $\mathbb{Q}/\mathbb{Z}(r)' = \varinjlim \mu_n^{\otimes r}$, where n runs through natural numbers prime to the characteristic of k . Then we have isomorphisms

$$K^T(k)_{\text{Tor}} \cong K^T(k)/\text{div} \cong H^2(k, \hat{X}(2)) \cong H^1(k, X \otimes \mathbb{Q}/\mathbb{Z}(2)')/\text{div} \cong \hat{X}(1)_{G_k}.$$

Proof. We take a finite Galois extension k'/k that splits T . It is proved in [8] that $K_2(k')$ is the direct sum of a finite group and a uniquely divisible group. By the norm argument, this shows that $K^T(k)$ is the direct sum of a uniquely divisible group and a torsion group of finite exponent. If $\text{Char}(k) = p > 0$, then $K^T(k)[p^\infty]$ is both divisible (by Lemma 2.14) and of finite exponent, hence trivial. This proves (2).

We prove (3). When $\text{Char}(k) = 0$, this is proved in [23] Proposition 3.1. The same proof works as well when $p = \text{Char}(k) > 0$, if $[k_1 : k]$ is prime to p . The general case can be reduced to this case. Indeed, the map $K^T(k_1)_{\text{div}} \rightarrow K^T(k)_{\text{div}}$ induced by $N_k^{k_1}$ is surjective by the norm argument. Thus it suffices to show the surjectivity of $K^T(k_1)/\text{div} \rightarrow K^T(k)/\text{div}$, which is equivalent to that of $K^T(k_1)[n] \rightarrow K^T(k)[n]$, where n is the exponent of $K^T(k)_{\text{Tor}}$. By (2), we know that n is prime to p . Hence we are reduced to the case $[k_1 : k]$ is prime to p by the norm argument.

We prove (4) and (5). We have a commutative diagram

$$\begin{array}{ccccc} K^T(k')/m & \cong & H^2(k', T[m] \otimes \mu_m) & \cong & T[m]_{G_{k'}} \\ \downarrow N & & \downarrow \text{Cor}_k^{k'} & & \downarrow \text{proj.} \\ K^T(k)/m & \xrightarrow{h} & H^2(k, T[m] \otimes \mu_m) & \cong & T[m]_{G_k}. \end{array}$$

The upper horizontal map is bijective by the Merkurjev-Suslin Theorem [9]. The right vertical map is surjective because it is induced by the identity map on $T[m]$. This shows the surjectivity of h . This also shows that the kernel of $h \circ N$ is $\sum_{\sigma \in \text{Gal}(k'/k)} (1 - \sigma)K^T(k')$, which is killed by N due to the ‘projection formula’ relation. In view of the surjectivity of N proved in (3), this shows (4). Now (5) is an immediate consequence.

Lastly, we prove (1). If n is the exponent of $K^T(k)_{\text{Tor}}$, we have

$$K^T(k)_{\text{Tor}} \cong K^T(k)/n \cong H^2(k, T[n] \otimes \mu_n),$$

by (4). Since the right hand side is a finite group, we see that $K^T(k)_{\text{Tor}}$ is finite. This completes the proof. \square

Theorem 3.2. *Let T be a torus over a non-archimedean local field k . Then T admits a motivic interpretation.*

Proof. We take a finite Galois extension k'/k which splits T . We set $X(2) = X(T) \otimes \mathbb{Z}(2)$. We are going to show that (2) induces the homomorphism ρ fitting into the commutative diagram

$$\begin{array}{ccc} K^T(k') & \cong & H^2(k', X(2)) \\ \downarrow N & & \downarrow \\ K^T(k) & \xrightarrow{\rho} & H^2(k, X(2)). \end{array}$$

The right vertical map is surjective. Indeed, setting $T' = \ker[\text{Res}_k^{k'} T \rightarrow T]$, we have a distinguished triangle

$$X(T') \otimes \mathbb{Z}(2) \rightarrow \text{Res}_k^{k'} X(2) \rightarrow X(2) \rightarrow X(T') \otimes \mathbb{Z}(2)[1],$$

but we have $H^3(k, X(T') \otimes \mathbb{Z}(2)) = 0$ by Lemma 2.13. The left vertical map N is also surjective by Lemma 3.1 (3). Lemma 3.1 (5) shows that the kernel of N is generated by the elements of the form $x - \sigma(x)$ with $x \in K^T(k')$ and $\sigma \in \text{Gal}(k'/k)$. Such an element is killed in $H^2(k, X \otimes \mathbb{Z}(2))$ as well. This shows the existence and surjectivity of ρ .

Since $K^T(k)_{\text{div}}$ is uniquely divisible, one sees that $\rho|_{K^T(k)_{\text{div}}}$ is injective by the norm argument. On the other hand, $\rho|_{K^T(k)_{\text{Tor}}}$ is also injective as the composition

$$K^T(k)_{\text{Tor}} \cong K^T(k)/n \rightarrow H^2(k, X(2))/n \rightarrow H^2(k, T[n] \otimes \mu_n)$$

(here n is the exponent of $K^T(k)_{\text{Tor}}$) is bijective by Lemma 3.1 (4). Now the theorem follows from Lemma 3.1 (1). \square

Remark 3.3. If p is a prime different from the residue characteristic of k , then $H^1(k, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{div}} = 0$. If further T has good reduction T_v over the residue field \mathbb{F} , then $H^1(k, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong T_v(\mathbb{F})[p^\infty]$.

3.2. Archimedean local field. Because $K_2(\mathbb{C})$ is uniquely divisible, $K^T(\mathbb{C})$ is uniquely divisible for any torus T over \mathbb{C} . Any torus T over \mathbb{R} admits a motivic interpretation by Theorem 1.3. We see that $K^T(\mathbb{R})$ is the direct sum of the finite group $K^T(\mathbb{R})_{\text{Tor}}$ and the uniquely divisible group $K^T(\mathbb{R})_{\text{div}}$. We need to know the structure of $K^T(\mathbb{R})_{\text{Tor}}$. Note that any torus over \mathbb{R} is isomorphic to a direct sum of copies of tori appearing in the following proposition.

Proposition 3.4. *We have $K^T(\mathbb{R})_{\text{Tor}} \cong \mathbb{Z}/2\mathbb{Z}$ (resp. 0) if $T = \mathbb{G}_m$ (resp. if $T = \text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m$ or $\ker[\text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \rightarrow \mathbb{G}_m]$). Moreover, for any even natural number n , the exact sequence*

$$0 \rightarrow K^T(\mathbb{R})/n \rightarrow H^2(\mathbb{R}, T[n] \otimes \mu_n) \rightarrow H^3(\mathbb{R}, X(T) \otimes \mathbb{Z}(2))$$

is isomorphic to the following sequence:

$$\begin{array}{ll}
0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0 & \text{if } T = \mathbb{G}_m \\
0 \rightarrow 0 \rightarrow 0 \rightarrow 0 & \text{if } T = \text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \\
0 \rightarrow 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} & \text{if } T = \ker[\text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \rightarrow \mathbb{G}_m].
\end{array}$$

Proof. The case $T = \mathbb{G}_m$ is well-known. The other cases can be deduced from Lemma 2.4 and the exact sequence $1 \rightarrow \ker[\text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \rightarrow \mathbb{G}_m] \rightarrow \text{Res}_{\mathbb{R}}^{\mathbb{C}} \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$. \square

4. GLOBAL FIELD

Let K be a global field. For a place v of K , we write K_v for the completion of K with respect to v . For a finite place v of K , we write \mathbb{F}_v (resp. K_v^{nr}) for the residue field of v (resp. the maximal unramified extension of K_v). When K is a number field, we write O_K for the ring of integers in K , and set $C = \text{Spec}(O_K)$. When K is of positive characteristic, we assume K is the function field of a smooth projective irreducible curve C over a finite field \mathbb{F} .

4.1. Bloch-Moore exact sequence. We recall some known results.

Theorem 4.1. (1) (Somekawa [17]) *Let T be a torus over K . Set $X = X(T)$ and $\hat{X}(r) = \varprojlim X \otimes \mu_n^{\otimes r}$ where n runs through natural numbers invertible in K . Let m be the order of the finite group $\hat{X}(1)_{G_K}$. Then we have an exact sequence*

$$K^T(K) \rightarrow \left(\bigoplus_{v|\infty} \hat{X}(1)_{G_{K_v}} \right) \oplus \left(\bigoplus_{v|\infty} K^T(K_v)/m \right) \rightarrow \hat{X}(1)_{G_K} \rightarrow 0.$$

(2) (Moore [11], Garland [5]) *There exists an exact sequence*

$$0 \rightarrow WK_2(K) \rightarrow K_2(K) \rightarrow \bigoplus_{v:\text{non complex}} \mu(K_v) \rightarrow \mu(K) \rightarrow 0,$$

where $WK_2(K)$ is a finite group (so-called wild kernel). In particular, $K_2(K)$ is a torsion group without any p -divisible subgroup for any prime p . In the number field case, this also implies the finiteness of

$$K_2(C) = \ker[K_2(K) \rightarrow \bigoplus_{v|\infty} \mathbb{F}_v^*].$$

(In the function field case, we have $WK_2(K) = K_2(C)$.)

We shall prove the finiteness of the kernel of the first map in (1) when T admits a motivic interpretation in Proposition 4.6 below.

4.2. Definition of $K^T(O_K)$ and $K^T(C)$. Let T be a torus over K , and let $X = X(T)$. By Theorem 4.1 and the norm argument, we see that $K^T(K)$ is a torsion group without any p -divisible subgroup for any prime p . Hence, by Lemma 2.14 we have $K^T(K)[p^\infty] = 0$ if $p = \text{Char}(K) > 0$.

Definition 4.2. For each prime $p \neq \text{Char}(K)$, we define

$$K^T(C)[p^\infty] = \ker[K^T(K)[p^\infty] \rightarrow \prod_{v|\not p^\infty} H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}}].$$

Here the v -component of the map is the composition

$$K^T(K)[p^\infty] \rightarrow K^T(K_v)[p^\infty] \cong H^1(K_v, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}},$$

where the second isomorphism is given by Theorem 3.2 and Remark 3.3. We then define

$$K^T(C) = \bigoplus_{p \neq \text{Char}(K)} K^T(C)[p^\infty].$$

In the number field case, we also write $K^T(C) = K^T(O_K)$.

Remark 4.3. Let p be a prime different from $\text{Char}(K)$.

- (1) When T admits a motivic interpretation, we have an isomorphism $K^T(K)[p^\infty] \cong H^1(K, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))/\text{div}$ by Corollary 2.12. The corank of $H^1(K, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))$ is $r_2 \dim T$, where r_2 is the number of complex places on K by [19] Corollary to Theorem 6.5. Hence, if further K is totally real or of positive characteristic, then we have

$$K^T(K)[p^\infty] \cong H^1(K, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

- (2) When T has good reduction at a finite place v , we have

$$H^1(K_v, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong T_v(\mathbb{F}_v)[p^\infty]$$

(cf Remark 3.3). The map $K^T(K)[p^\infty] \rightarrow T_v(\mathbb{F}_v)[p^\infty]$ can be interpreted by the analogous way as the Hilbert symbol (cf. [17] §3). In particular, we see $K_2(C) = K^T(C)$ if $T = \mathbb{G}_m$.

- (3) Summarizing, if K is totally real or of positive characteristic, and if T admits a motivic interpretation, then $K^T(O_K)[p^\infty]$ is isomorphic to the kernel of

$$H^1(K, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)) \rightarrow \bigoplus_{v \notin S, v|\not p} T_v(\mathbb{F}_v)[p^\infty] \oplus \bigoplus_{v \in S, v|\not p} H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}},$$

where S is a finite set of places of K including all infinite places and all places where T has bad reduction.

4.3. Hasse principle and the finiteness of $K^T(C)$. In Proposition 4.6 below, we prove the finiteness of $K^T(C)$ when T is a torus which admits a motivic interpretation. In the proof, we need the following result.

Proposition 4.4 (Hasse principle). *Let T be torus over K , and let $X(2) = X(T) \otimes \mathbb{Z}(2)$.*

(1) *For all $i \geq 3$, we have an isomorphism*

$$H^i(K, X(2)) \cong \bigoplus_{v|\infty} H^i(K_v, X(2)).$$

(2) *Suppose that T admits a motivic interpretation. Let L/K be a finite Separable extension. For each infinite place v of K , we choose a place $w(v)$ of L above v . Then we have an isomorphism of finite groups*

$$K^T(K)/N_K^L K^T(L) \cong \bigoplus_{v|\infty} K^T(K_v)/N_{K_v}^{L_{w(v)}} K^T(L_{w(v)}).$$

(When $\text{Char}(K) > 0$, both statements mean that the left hand sides are trivial.)

Remark 4.5. It is possible to compute the finite group appearing in (2) explicitly by using Proposition 3.4. When $T = \mathbb{G}_m$, Proposition 4.4 (2) is proved in [1, 2]. See also [23] Proposition 4.1 for a related result.

Proof. Firstly, we claim that $H^i(K, X(2))$ is a torsion group of finite exponent for all $i \geq 3$. This is reduced to the case $T = \mathbb{G}_m$ by the norm argument. By Review 2.6 (2), we have $H^3(K, \mathbb{Z}(2)) = 0$. We also see $H^4(K, \mathbb{Z}(2)) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is a torsion group of exponent at most 2. For $i \geq 5$, the claim follows from the spectral sequence $E_2^{m,n} = H^m(K, \mathbb{H}^n(\mathbb{Z}(2))) \Rightarrow H^{m+n}(K, \mathbb{Z}(2))$, together with the fact that $\mathbb{H}^n(\mathbb{Z}(2)) = 0$ unless $n = 1, 2$.

Let n_i be the exponent of $H^i(K, X(2))$ for $i \geq 3$. We set n to be the prime to $\text{Char}(K)$ -part of $n_i n_{i+1}$ and put $M = T[n] \otimes \mu_n$. The distinguished triangle $X(2) \xrightarrow{n} X(2) \rightarrow M \rightarrow X(2)[1]$ induces a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \rightarrow & H^i(K, X(2)) & \rightarrow & H^i(K, M) & \rightarrow & H^{i+1}(K, X(2)) & \rightarrow 0 \\ & \downarrow f_i & & \downarrow \cong & & \downarrow f_{i+1} & \\ 0 \rightarrow & \bigoplus_{v|\infty} H^i(K_v, X(2)) & \rightarrow & \bigoplus_{v|\infty} H^i(K, M) & \rightarrow & \bigoplus_{v|\infty} H^{i+1}(K, X(2)) & \rightarrow 0 \end{array}$$

Here the middle vertical map is an isomorphism by the Poitou-Tate theorem (cf. [14] §6.3 Théorème B). This shows the injectivity of f_i for all $i \geq 3$ and (using the injectivity of f_4 thus obtained) the surjectivity of f_i for all $i \geq 3$ as well. When $\text{Char}(K) = p > 0$, a similar argument using Review 2.6 (4) shows that $H^i(K, X(2))[p^\infty] = 0$ for all $i \geq 3$. This completes the proof of (1).

We prove (2). Let S be the kernel of the norm map $\text{Res}_K^L T \rightarrow T$. We set $Y = X(S)$ and $Y(2) = Y \otimes \mathbb{Z}(2)$. Then we have a distinguished triangle $Y(2) \rightarrow \text{Res}_K^L X(2) \rightarrow X(2) \rightarrow$

$Y(2)[1]$. By the assumption that T admits a motivic interpretation, this induces the exact sequence at the upper row in the following commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow & K^T(K)/K^T(L) & \rightarrow & H^3(K, Y(2)) & \rightarrow & H^3(L, X(2)) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \bigoplus_{v|\infty} N_{K_v}^{L_w(v)} K^T(L_w(v)) & \rightarrow & \bigoplus_{v|\infty} H^3(K_v, Y(2)) & \rightarrow & \bigoplus_{v|\infty} \bigoplus_{w|v} H^3(L_w, X(2)) & \end{array}$$

The exact sequence at the lower row in the diagram is deduced in a similar way, by noting the following facts (v is a place of K): (i) The base change of $\text{Res}_K^L T$ to K_v is isomorphic to $\bigoplus_{w|v} \text{Res}_{K_v}^{L_w}(T \otimes_K K_v)$. (ii) The image of the norm map $N_{K_v}^{L_w} : K^T(L_w) \rightarrow K^T(K_v)$ is the same for all w over v (because L/K is separable). (iii) When v is a finite place, the norm map $N_{K_v}^{L_w} : K^T(L_w) \rightarrow K^T(K_v)$ is surjective by Lemma 3.1 (3). Now the assertion follows since the middle and right vertical maps are bijective by (1). \square

Proposition 4.6. *If T is a torus over K which admits a motivic interpretation, then $K^T(C)$ is a finite group.*

Proof. We take a finite Galois extension L/K which splits T . Let $G = \text{Gal}(L/K)$. For a prime $p \neq \text{Char}(K)$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} K^T(C_L)[p^\infty]_G & \rightarrow & K^T(L)[p^\infty]_G & \rightarrow & \bigoplus_{v|\not p^\infty} [\bigoplus_{w|v} H^1(L_w^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_w}}]_G & \rightarrow & 0 \\ & & \downarrow f & & \downarrow & & \\ 0 \rightarrow & K^T(C)[p^\infty] & \rightarrow & K^T(K)[p^\infty] & \rightarrow & \bigoplus_{v|\not p^\infty} H^1(K_v^{nr}, X \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))^{G_{\mathbb{F}_v}} & \rightarrow 0. \end{array}$$

If T has good reduction at v , then the v -component of the right vertical map is an isomorphism since it is isomorphic to

$$(\bigoplus_{w|v} \hat{X}_p(1)_{G_{L_w}})_G \cong \hat{X}_p(1)_{G_{K_v}}.$$

Hence the kernel of the right vertical map is finite. By Theorem 4.1, $K^T(C_L)$ is a finite group. By Proposition 4.4 (2), the cokernel of f is a finite group which is trivial if $p \neq 2$ or $\text{Char}(K) > 0$. This completes the proof. \square

4.4. Isogeny. We write $\mathbb{Q}/\mathbb{Z}(2)' = \varinjlim \mu_n^{\otimes 2}$ where n runs natural numbers prime to $\text{Char}(K)$. For a torus T over K , we set $W^T(K) = H^0(K, X(T) \otimes \mathbb{Q}_p/\mathbb{Z}_p(2))$.

Proposition 4.7. *Let T_1, T_2 be tori over K admitting a motivic interpretation. We assume T_1 and T_2 are isogenous. If $\text{Char } K > 0$, then the equality*

$$\frac{|K^{T_1}(C)|}{|W^{T_1}(K)|} = \frac{|K^{T_2}(C)|}{|W^{T_2}(K)|}$$

holds. When K is a number field, the same equality holds up to a power of 2, if T_1 and T_2 are split by a totally real field.

Proof. We only prove the number field case. (The function field case is easier.) Let S be a finite set of places of K including all infinite places and all places where T_1 or T_2 have bad reduction. We fix an odd prime p , and set $M_i = X(T_i) \otimes \mathbb{Q}_p/\mathbb{Z}_p(2)$ for $i = 1, 2$. We consider a commutative diagram with exact rows for $i = 1, 2$

$$\begin{array}{ccccccc} 0 \rightarrow & K^{T_i}(O_K)[p^\infty] & \rightarrow & H^1(K, M_i) & \rightarrow & \bigoplus_{v \notin S} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} & \rightarrow 0 \\ & \downarrow & & \parallel & & \downarrow & \\ 0 \rightarrow & H^1(O_K[\frac{1}{pS}], M_i) & \rightarrow & H^1(K, M_i) & \rightarrow & \bigoplus_{v \notin S} T_{i,v}(\mathbb{F}_v)[p^\infty] & \rightarrow 0, \end{array}$$

where the lower row is the localization sequence of the étale cohomology, and the surjectivity of the upper right horizontal map is due to Theorem 4.1 (1), which implies the surjectivity of the lower right horizontal map as well. Thus we have an exact sequence

$$0 \rightarrow K^{T_i}(O_K)[p^\infty] \rightarrow H^1(O_K[\frac{1}{pS}], M_i) \rightarrow \bigoplus_{v \notin S} H^1(K_v^{nr}, M_i)^{G_{\mathbb{F}_v}} \rightarrow 0.$$

The localization sequence also implies that $W^{T_i}(K) = H^0(O_K[\frac{1}{pS}], M_i)$, $H^2(O_K[\frac{1}{pS}], M_i) = H^2(K, M_i) = 0$. (The last group is trivial because $H^2(K, M_i)$ is a torsion group of finite exponent (by the norm argument) and $H^2(K, M)$ is p -divisible for any p -primary torsion divisible group M).

Now we are reduced to showing the following equalities:

$$\begin{aligned} (i) \quad & \frac{|H^1(O_K[\frac{1}{pS}], M_1)|}{|H^0(O_K[\frac{1}{pS}], M_1)|} = \frac{|H^1(O_K[\frac{1}{pS}], M_2)|}{|H^0(O_K[\frac{1}{pS}], M_2)|}, \\ (ii) \quad & |H^1(K_v^{nr}, M_1)^{G_{\mathbb{F}_v}}| = |H^1(K_v^{nr}, M_2)^{G_{\mathbb{F}_v}}|. \end{aligned}$$

The isogeny $f : T_1 \rightarrow T_2$ implies an exact sequence

$$0 \rightarrow C \otimes \mu_n \rightarrow M_1 \rightarrow M_2 \rightarrow 0,$$

where $C = \ker(f)$ and n is the p -power part of the order of C . Then we have an exact sequence

$$0 \rightarrow \ker(a) \rightarrow H^1(K_v^{nr}, M_1) \xrightarrow{a} H^1(K_v^{nr}, M_2) \rightarrow 0,$$

in which $\ker(a)$ is a quotient of a finite group $H^1(\mathbb{F}_v, C \otimes \mu_n)$. Since $H^1(\mathbb{F}_v, H^1(K_v^{nr}, M_i)) \cong H^2(K_v, M_i) = K^T(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$, we get an exact sequence

$$0 \rightarrow \ker(a)^{G_{\mathbb{F}_v}} \rightarrow H^1(K_v^{nr}, M_1)^{G_{\mathbb{F}_v}} \rightarrow H^1(K_v^{nr}, M_2)^{G_{\mathbb{F}_v}} \rightarrow H^1(\mathbb{F}_v, \ker(a)) \rightarrow 0.$$

Since $\ker(a)$ is a finite $G_{\mathbb{F}_v}$ -module, (ii) follows.

In order to prove (i), it suffices to show

$$\frac{|H^0(O_K[\frac{1}{pS}], C \otimes \mu_n)| \cdot |H^2(O_K[\frac{1}{pS}], C \otimes \mu_n)|}{|H^1(O_K[\frac{1}{pS}], C \otimes \mu_n)|} = 1.$$

By [12] Theorem 8.6.14, the left hand side is equal to

$$\prod_{v \mid \infty} \frac{|H^0(K_v, C \otimes \mu_n)|}{||n||_v}.$$

By the assumption that both T_1 and T_2 are split by a totally real field, $C \otimes \mu_n$ is isomorphic to a direct sum of copies of $\mathbb{Z}/n\mathbb{Z}$ as G_{K_v} -modules for any $v|\infty$. This completes the proof. \square

4.5. Main result. We now finish the proof of our main result Theorem 1.4. We recall the statement, including the function field case.

Theorem 4.8. *Let K be a global field, and let T be a torus over T . Assume that T admits a motivic interpretation.*

- (1) *Suppose that K is a totally real number field, and that T is split by a totally real field L over K . Then the equality*

$$|L_K(X(T), -1)| = \frac{|K^T(O_K)|}{|W^T(K)|}$$

holds up to a power of 2.

- (2) *Suppose $\text{Char}(K) = p > 0$. Then the equality*

$$|L_K(X(T), -1)| = \frac{|K^T(C)|}{|W^T(K)|}$$

holds. (Both sides are rational numbers prime to p .)

Proof. Since $L_K(X(T), s)$ is real analytic (as $X(T)$ being an integral representation), it is enough to show the equality after taking the m -th power for some $m \in \mathbb{Z}_{>0}$. Both sides of the equation is stable under isogeny (by Proposition 4.7). By [13], there exist tori P, Q over K such that

- Both P and Q are quasi-trivial and split by L/K .
- $T^{\oplus m} \oplus P$ is isogenous to Q for some $m \in \mathbb{Z}_{>0}$.

Hence we are reduced to the case $T = \text{Res}_K^M \mathbb{G}_m$ for a subextension M/K of L/K . In this case, we have $L_K(X(T), s) = L_K(\text{Ind}_K^M X(\mathbb{G}_m), s) = \zeta_M(s)$, $K^T(C) = K_2(C_M)$ and $W^T(K) = W_2(M)$. Thus we are reduced to the case $T = \mathbb{G}_m$, which is a theorem of Wiles [21] in the number field case, or of Tate [18] in the function field case. \square

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REFERENCES

- [1] A. Bak, U. Rehmann, K_2 -analogs of Hasse's norm theorems. *Comment. Math. Helv.* 59 (1984), no. 1, 1–11.
- [2] J.-L. Colliot-Thélène, Hilbert's Theorem 90 for K_2 , with application to the Chow groups of rational surfaces. *Invent. Math.* 71 (1983), no. 1, 1–20.

- [3] J.-L. Colliot-Thélène, J.-J. Sansuc, La R -équivalence sur les tores. *Ann. Sci. École Norm. Sup.* (4) 10 (1977), no. 2, 175–229.
- [4] S. Endo, T. Miyata, On a classification of the function fields of algebraic tori, *Nagoya Math. J.* 56 (1975), 85–104.
- [5] H. Garland, A finiteness theorem for K_2 of a number field. *Ann. of Math.* (2) 94 (1971), 534–548.
- [6] S. Lichtenbaum, The construction of weight-two arithmetic cohomology. *Invent. Math.* 88 (1987), no. 1, 183–215.
- [7] S. Lichtenbaum, New results on weight-two motivic cohomology. *The Grothendieck Festschrift*, Vol. III, 35–55, *Progr. Math.*, 88, Birkhäuser, 1990.
- [8] A. Merkurjev, On the torsion in K_2 of local fields. *Ann. of Math.* (2) 118 (1983), no. 2, 375–381.
- [9] A. Merkurjev, A. Suslin, K -cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Math. USSR Izvestiya* 21 (1983), 307–340.
- [10] S. Mochizuki, Motivic interpretation of Milnor K -groups attached to Jacobian varieties. Preprint, Arxiv: math.KT/0603241.
- [11] C. Moore, Group extensions of p -adic and adelic linear groups. *Inst. Hautes Études Sci. Publ. Math.* No. 35 1968 157–222.
- [12] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*. *Grundlehren der Mathematischen Wissenschaften*, 323. Springer-Verlag, Berlin, 2000.
- [13] T. Ono, Arithmetic of algebraic tori. *Ann. of Math.* (2) 74 1961 101–139.
- [14] J.-P. Serre, *Cohomologie galoisienne*. Fifth edition. *Lecture Notes in Mathematics*, 5. Springer-Verlag, Berlin, 1994.
- [15] *Algebraic groups and class fields*. *Graduate Texts in Mathematics*, 117. Springer-Verlag, New York, 1988.
- [16] M. Spiess, T. Yamazaki. A counterexample to generalizations of the Milnor-Bloch-Kato conjecture. Preprint, arXiv:0706.4354.
- [17] M. Somekawa, On Milnor K -groups attached to semi-abelian varieties. *K-Theory* 4 (1990), no. 2, 105–119.
- [18] J. Tate, Symbols in arithmetic. *Actes du Congrès International des Mathématiciens (Nice, 1970)*, Tome 1, pp. 201–211. Gauthier-Villars, Paris, 1971.
- [19] J. Tate, Relations between K_2 and Galois cohomology. *Invent. Math.* 36 (1976), 257–274.
- [20] J. Tong, Application d’Albanese pour les courbes et contractions. *Math. Ann.* 338 (2007), no. 2, 405–420.
- [21] A. Wiles, The Iwasawa conjecture for totally real fields. *Ann. of Math.* (2) 131 (1990), no. 3, 493–540.
- [22] V. Voevodsky, *Triangulated categories of motives over a field*. *Cycles, transfers, and motivic homology theories*, 188–238, *Ann. of Math. Stud.*, 143, Princeton Univ. Press, Princeton, NJ, 2000
- [23] T. Yamazaki, Class field theory for a product of curves over a local field. To appear in *Math. Z.*

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