

# THE BRAUER-MANIN PAIRING, CLASS FIELD THEORY AND MOTIVIC HOMOLOGY

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ABSTRACT. For a smooth proper variety over a  $p$ -adic field, its Brauer group and abelian fundamental group are related to higher Chow groups by the Brauer-Manin pairing and class field theory. We generalize this relation to smooth (possibly non-proper) varieties, using motivic homology and a variant of Wiesend's ideal class group. Several examples are discussed.

## 1. INTRODUCTION

**1.1. The Brauer-Manin pairing and class field theory.** Let  $p$  be a prime number, and  $k$  a finite extension of  $\mathbb{Q}_p$ . Let  $X$  be a smooth variety over  $k$ . For  $i \in \mathbb{Z}_{\geq 0}$ , we write  $X_{(i)}$  for the set of all points of  $X$  of dimension  $i$ . Let  $\mathrm{Br}(X)$  be the cohomological Brauer group of  $X$ , and let  $\pi_1^{ab}(X)$  be the abelian étale fundamental group of  $X$ . For any  $x \in X_{(0)}$ , local class field theory yields canonical maps

$$\psi_x^* : \mathrm{Br}(x) \cong \mathbb{Q}/\mathbb{Z}, \quad \rho_x : k(x)^* \rightarrow \pi_1^{ab}(x).$$

As for the first map, it is convenient for our purpose to consider its dual. Putting  $A^* := \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$  for an abelian group  $A$ , we define  $\psi_x$  to be the composition of the dual of  $\psi_x^*$  and the canonical inclusion

$$\psi_x : \mathbb{Z} \hookrightarrow \hat{\mathbb{Z}} \cong \mathrm{Br}(x)^*.$$

Since both of  $\mathrm{Br}(-)^*$  and  $\pi_1^{ab}(-)$  are covariant functorial, we get homomorphisms

$$(1.1.1) \quad \tilde{\psi}_X : Z_0(X) := \bigoplus_{x \in X_{(0)}} \mathbb{Z} \rightarrow \mathrm{Br}(X)^*, \quad \tilde{\rho}_X : Z_0^1(X) := \bigoplus_{x \in X_{(0)}} k(x)^* \rightarrow \pi_1^{ab}(X)$$

by taking the direct sum of the  $\psi_x$ 's and  $\rho_x$ 's.

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When  $X$  is proper over  $k$ , Manin [24] and Bloch/Saito [2, 30] observed that  $\tilde{\psi}_X$  and  $\tilde{\rho}_X$  factor respectively through

$$CH_0(X) := \operatorname{coker}\left[\bigoplus_{y \in X_{(1)}} k(y)^* \rightarrow Z_0(X)\right],$$

$$SK_1(X) := \operatorname{coker}\left[\bigoplus_{y \in X_{(1)}} K_2 k(y) \rightarrow Z_0^1(X)\right];$$

the induced pairing  $CH_0(X) \times \operatorname{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$  and the induced map  $SK_1(X) \rightarrow \pi_1^{ab}(X)$  are called the *Brauer-Manin pairing* and *reciprocity map of class field theory* respectively. Both are studied intensively by several authors; for the Brauer-Manin pairing, see [29, 4, 5, 6, 32, 47]; for the reciprocity map, see [41, 42, 17, 18, 33, 48].

If  $X$  is not proper over  $k$ , however,  $\tilde{\psi}_X$  and  $\tilde{\rho}_X$  do not factor through  $CH_0(X)$  or  $SK_1(X)$ . To this end, we shall introduce good quotients of  $Z_0(X)$  and  $Z_0^1(X)$ .

**1.2. Wiesend's tame ideal class group.** Let  $V$  be a variety over a field  $F$ . Take  $y \in V_{(1)}$ . Let  $C(y)$  be the closure of  $\{y\}$  in  $V$ ,  $\tilde{C}(y) \twoheadrightarrow C(y)$  the normalization,  $\tilde{C}(y) \hookrightarrow \bar{C}(y)$  the smooth completion, and  $C_\infty(y) = \bar{C}(y) \setminus \tilde{C}(y)$ . For  $x \in C_\infty(y)$ , we take a uniformizer  $\pi_x \in F(y)^*$  at  $x$ . We define for  $r \in \mathbb{Z}_{>0}$

$$UK_r^M F(y) := \ker\left[K_r^M F(y) \rightarrow \bigoplus_{x \in C_\infty(y)} (K_{r-1}^M F(x) \oplus K_r^M F(x))\right].$$

Here the  $x$ -component of the map is defined by  $a \mapsto (\partial_x(a), \partial_x(\{\pi_x\} \cup a))$  for  $a \in K_r^M F(y)$ , where  $\partial_x$  is the tame symbol at  $x$ . This group does not depend on the choice of  $\pi_x$ . If  $V$  is proper over  $F$ , then  $UK_r^M F(y) = K_r^M F(y)$ . For example,  $UK_1^M F(y)$  is the group of rational functions on  $\bar{C}(y)$  which takes value 1 at all points of  $C_\infty(y)$ . The following definition is a natural outcome of an idea of Wiesend [46].

**Definition 1.1.** Let  $V$  be a variety over a field  $F$ , and let  $r \in \mathbb{Z}_{\geq 0}$ . We define *Wiesend's tame ideal class group* of degree  $r$  to be

$$C_r(V) := \operatorname{coker}\left[\bigoplus_{y \in V_{(1)}} UK_{r+1}^M F(y) \hookrightarrow \bigoplus_{y \in V_{(1)}} K_{r+1}^M F(y) \xrightarrow{(*)} \bigoplus_{x \in V_{(0)}} K_r^M F(x)\right],$$

where  $(*)$  is the boundary map of Gersten complex of the Milnor  $K$ -sheaf. (In particular, its image is in the direct sum.)

By definition,  $C_0(V)$  and  $C_1(V)$  are quotients of  $Z_0(V)$  and  $Z_0^1(V)$  respectively. If  $V$  is proper over  $F$ , then we have  $C_0(V) = CH_0(V)$  and  $C_1(V) = SK_1(V)$ .

**Remark 1.2.** Suppose that  $V = \bar{C} \setminus C_\infty$ , where  $\bar{C}$  is a smooth projective geometrically irreducible curve over  $F$  and  $C_\infty$  is a closed reduced subvariety of  $\bar{C}$ . Then our definition of  $C_0(V)$  coincides with that of the *group of classes of divisors on  $\bar{C}$  prime to  $C_\infty$  modulo  $C_\infty$ -equivalence* defined in [37, Chap. V, §2]. It is proved [37, Chap. V, Thm. 1] that  $\ker(\deg : C_0(V) \rightarrow \mathbb{Z})$  is represented by a semi-abelian variety  $J$ , which is called the *generalized*

*Jacobian variety* of  $\bar{C}$  with modulus  $C_\infty$ . (Here the degree map  $\deg : C_0(V) \rightarrow \mathbb{Z}$  is induced by the usual degree map  $Z_0(V) \rightarrow \mathbb{Z}$ .) For simplicity, we call  $J$  “the generalized Jacobian of  $V$ ”. One can also interpret  $C_0(V)$  as the Picard group of the singular curve obtained from  $\bar{C}$  by contracting all points of  $C_\infty$  to one point [37, Chap. V, §2 and Chap. IV. §4]. On the other hand,  $C_0(V)$  is also isomorphic to the relative Picard group  $\text{Pic}(\bar{C}, C_\infty)$  (see [39, Theorem 3.1]).

**1.3. Motivic homology.** Let  $V$  be a smooth variety over a perfect field  $F$ , and let  $i, j \in \mathbb{Z}$ . Its motivic homology  $H_i^M(V, \mathbb{Z}(j))$  is defined to be the group of homomorphisms  $\text{Hom}_{\text{DM}_{\text{Nis}}^{\text{eff},-}(F)}(\mathbb{Z}(j)[i], M(V))$  in Voevodsky’s category (cf. [11], [28, (14.17)]). When  $j = 0$ , the group  $H_i^M(V, \mathbb{Z}(0))$  agrees with *Suslin’s algebraic singular homology*  $h_i(V)$  [39], and admits a (relatively simple) description in terms of algebraic cycles (cf. [43, (3.2.7)], [28, (14.18)]). There is a similar but complicated description also for  $j \neq 0$  (cf. [11, (9.4)]). The following theorem, which plays a crucial role in our paper, provides a simpler description in a special case.

**Theorem 1.3.** *Let  $V$  be a smooth variety over a perfect field  $F$ , and let  $r \in \mathbb{Z}_{\geq 0}$ . Then there is a canonical isomorphism*

$$(1.3.1) \quad C_r(V) \cong H_{-r}^M(V, \mathbb{Z}(-r)).$$

When  $r = 0$ , using the comparison of  $H_0^M(V, \mathbb{Z}(0))$  with  $h_0(V)$  recalled above, this follows from a result of Schmidt ([35] Theorem 5.1). We shall prove this theorem by reducing to the case  $r = 0$  in §2. In what follows, we often identify  $C_r(V)$  and  $H_{-r}^M(V, \mathbb{Z}(-r))$ .

**Remark 1.4.** For a smooth projective variety  $V$  over a *finite* field, unramified class field theory [22] relates  $CH_0(V)$  with the abelian fundamental group of  $V$ . This has been generalized by Schmidt and Spiess [36] to smooth (possibly non-proper) variety  $V$ , in which  $CH_0(V)$  is replaced by Suslin’s algebraic singular homology  $h_0(V)$ . (Note that  $h_0(V) \cong C_0(V)$  by Schmidt’s theorem mentioned above. See also recent works of Geisser [12, 13] for a further generalization.) The basic strategy of our paper is to follow their argument over a  $p$ -adic base field.

**1.4. Open varieties over a  $p$ -adic field.** We go back to the situation of §1.1. Schmidt and Spiess constructed a map connecting motivic homology and étale cohomology with compact support, which we will recall in §3. As an application, we deduce the following proposition in §4.

**Proposition 1.5.** *Let  $X$  be a smooth variety over a finite extension  $k$  of  $\mathbb{Q}_p$ . The homomorphisms (1.1.1) induce well-defined homomorphisms*

$$(1.4.1) \quad \psi_X : C_0(X) \rightarrow \text{Br}(X)^*, \quad \rho_X : C_1(X) \rightarrow \pi_1^{ab}(X).$$

In §5, we consider the case where  $X$  is a curve. In this case, the map  $\psi_X$  was already studied by Scheiderer-van Hamel [34] (see Theorem 5.1 below), and the map  $\rho_X$  is closely related to work of Hiranouchi [16] (see Theorem 5.4 and Remark 5.5 below). When  $X$  is of dimension two or higher, the maps  $\psi_X$  and  $\rho_X$  are not very close to an isomorphism even if  $X$  is projective over  $k$ . We study several examples of surfaces in §6. As a sample, here we mention the following result. (We write  $\bar{V} = V \times_k \bar{k}$  for a variety  $V$  over a field  $k$  with an algebraic closure  $\bar{k}$ .)

**Theorem 1.6.** *Let  $X$  be a smooth projective geometrically irreducible surface over a finite extension  $k$  of  $\mathbb{Q}_p$ . Suppose that  $\bar{X}$  is rational. Let  $U$  be an open subvariety of  $X$ .*

- (1) *Suppose that the irreducible components of  $\bar{X} \setminus \bar{U}$  generate the Néron-Severi group  $NS(\bar{X})$  of  $\bar{X}$ . Then, the kernel of  $\psi_U$  is the maximal divisible subgroup of  $C_0(U)$ . However, there is an example of  $U$  such that  $\ker(\psi_U \otimes \mathbb{Z}/n\mathbb{Z}) \neq 0$  for all  $n \in \mathbb{Z}_{>0}$  divisible by some (fixed) integer  $N$ .*
- (2) *The kernel of  $\rho_U$  is the maximal divisible subgroup of  $C_1(U)$ , and  $\rho_U \otimes \mathbb{Z}/n\mathbb{Z}$  is bijective for all  $n \in \mathbb{Z}_{>0}$ .*

Note that, if  $X$  is a (projective smooth) rational surface, then  $\ker(\psi_X) = 0$  and  $\ker(\psi_X \otimes \mathbb{Z}/n\mathbb{Z}) = 0$  for all sufficiently divisible  $n$  (cf. Theorem 6.2). Note also that there are examples of (non-rational) projective smooth surfaces  $X$  and  $X'$  for which  $\ker(\psi_X)$  and  $\ker(\rho_{X'})$  are not divisible (cf. [29, 33]).

**1.5. Conventions.** Let  $A$  be an abelian group. For a non-zero integer  $n$ , we write  $A[n]$  and  $A/n$  for the kernel and cokernel of the map  $n : A \rightarrow A$ . This notation is sometimes used when  $n = \infty$ , in which case we mean  $A[n] = A_{\text{Tor}}$  is the subgroup of torsion elements in  $A$ , and  $A/n = A \otimes \mathbb{Q}/\mathbb{Z}$ . We write  $A_{\mathbb{Q}}$  for  $A \otimes_{\mathbb{Z}} \mathbb{Q}$ . We define  $A_{\text{Div}} := \text{Im}[\text{Hom}(\mathbb{Q}, A) \rightarrow \text{Hom}(\mathbb{Z}, A) = A]$  to be the maximal divisible subgroup in  $A$ , and  $A_{\text{div}} := \bigcap_{n \in \mathbb{Z}_{>0}} nA$  the subgroup of divisible elements in  $A$ . Note that we always have  $A_{\text{Div}} \subset A_{\text{div}}$ , and that  $A_{\text{Div}} = A_{\text{div}}$  holds if  $A[n]$  is finite for all  $n \in \mathbb{Z}_{>0}$ .

Let  $f : A \rightarrow B$  be a homomorphism of abelian groups. We write  $f/n$  for  $f \otimes \mathbb{Z}/n$  when  $n \in \mathbb{Z}_{>0}$ , and for  $f \otimes \mathbb{Q}/\mathbb{Z}$  when  $n = \infty$ . Let  $m, n \in \mathbb{Z}_{>0}$ . The map  $\ker(f/n) \rightarrow \ker(f/nm)$  induced by the map  $m : A \rightarrow A$  is called the canonical map, so that  $\{\ker(f/n)\}_n$  becomes an inductive system whose limit is  $\ker(f \otimes \mathbb{Q}/\mathbb{Z})$ . The map  $\ker(f/nm) \rightarrow \ker(f/n)$  induced by the identity map on  $A$  is called the canonical map, so that  $\{\ker(f/n)\}_n$  becomes an inverse system.

Let  $F$  be a field. A separated scheme of finite type over  $F$  is called a variety over  $F$ . Let  $X$  be a variety over  $F$ . For  $i \in \mathbb{Z}_{\geq 0}$ , we write  $X_{(i)}$  and  $X^{(i)}$  for the set of all points on  $X$  of dimension  $i$  and of codimension  $i$  respectively. We write  $\bar{F}$  for an algebraic closure of  $F$  and  $\bar{X}$  for the base change  $X \times_{\text{Spec } F} \text{Spec } \bar{F}$ . A closed subset of  $X$  is always regarded as a *reduced* subvariety of  $X$ .

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## 2. WIESEND'S IDEAL CLASS GROUP AND MOTIVIC HOMOLOGY

Let  $F$  be a perfect field. Except in the beginning of §2.1 and in §2.3, we will assume the characteristic of  $F$  is zero.

**2.1. Motivic homology and cohomology.** We recall some facts from [43] and [28]. Let  $\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)$  denote the rigid triangulated tensor category of effective motivic complexes ([43] 3.1, [28] 14.1). There is a functor  $M$  from the category of varieties over  $F$  to  $\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)$ . For a variety  $X$  and  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ , we write  $M(X, \mathbb{Z}/n\mathbb{Z}) = M(X) \otimes^{\mathbb{L}} \mathbb{Z}/n\mathbb{Z}$  (cf. §1.5). Let  $i, j \in \mathbb{Z}$ . Motivic cohomology and motivic homology of a variety  $X$  are defined by

$$\begin{aligned} H_M^i(X, \mathbb{Z}(j)) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)}(M(X), \mathbb{Z}(j)[i]), \\ H_i^M(X, \mathbb{Z}(j)) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(F)}(\mathbb{Z}(j)[i], M(X)). \end{aligned}$$

For  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ , their coefficient version are defined by

$$\begin{aligned} H_M^i(X, \mathbb{Z}/n\mathbb{Z}(j)) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(k)}(M(X), \mathbb{Z}/n\mathbb{Z}(j)[i]), \\ H_i^M(X, \mathbb{Z}/n\mathbb{Z}(j)) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(k)}(\mathbb{Z}(j)[i], M(X, \mathbb{Z}/n\mathbb{Z})). \end{aligned}$$

They fit into (obvious) exact sequences

$$(2.1.1) \quad 0 \rightarrow H_M^i(X, \mathbb{Z}(j))/n \rightarrow H_M^i(X, \mathbb{Z}/n\mathbb{Z}(j)) \rightarrow H_M^{i+1}(X, \mathbb{Z}(j))[n] \rightarrow 0,$$

$$(2.1.2) \quad 0 \rightarrow H_i^M(X, \mathbb{Z}(j))/n \rightarrow H_i^M(X, \mathbb{Z}/n\mathbb{Z}(j)) \rightarrow H_{i-1}^M(X, \mathbb{Z}(j))[n] \rightarrow 0.$$

Motivic cohomology  $H_M^i(X, \mathbb{Z}(j))$  is contravariantly functorial in  $X$ . It also has covariant functionality: if  $f : X \rightarrow Y$  is a proper flat equidimensional morphism of relative dimension  $d$ , then there is an induced map  $H_M^i(X, \mathbb{Z}(j)) \rightarrow H_M^{i-2d}(Y, \mathbb{Z}(j-d))$ . Motivic homology  $H_i^M(X, \mathbb{Z}(j))$  is covariantly functorial in  $X$ . It also has contravariant functionality: if  $f : X \rightarrow Y$  is a proper flat equidimensional morphism of relative dimension  $d$ , then there is an induced map  $H_i^M(Y, \mathbb{Z}(j)) \rightarrow H_{i+2d}^M(X, \mathbb{Z}(j+d))$ .

The following fact will play an important role in the proof of Theorem 1.3: for any variety  $X$ , there is a decomposition

$$(2.1.3) \quad H_i^M(X \times \mathbb{G}_m, \mathbb{Z}(j)) \cong H_{i-1}^M(X, \mathbb{Z}(j-1)) \oplus H_i^M(X, \mathbb{Z}(j)).$$

This is deduced from Mayer-Vietoris sequence, projective bundle formula and  $\mathbb{A}^1$ -homotopy invariance.

From now through the end of §2.2, we assume the characteristic of  $F$  is zero. If  $X$  is a smooth variety, we have a canonical isomorphism

$$(2.1.4) \quad H_M^i(X, \mathbb{Z}(j)) \cong CH^j(X, 2j - i),$$

where the right hand side is Bloch's higher Chow group. If  $X$  is a smooth projective variety of pure dimension  $d$ , we also have

$$(2.1.5) \quad H_i^M(X, \mathbb{Z}(j)) \cong H_M^{2d-i}(X, \mathbb{Z}(d-j)) \cong CH^{d-j}(X, i-2j).$$

In particular, if  $X = \text{Spec } F$ , then we have for any  $r \in \mathbb{Z}_{\geq 0}$

$$(2.1.6) \quad H_{-r}^M(\text{Spec } F, \mathbb{Z}(-r)) \cong H_M^r(\text{Spec } F, \mathbb{Z}(r)) \cong CH^r(\text{Spec } F, r) \cong K_r^M F.$$

Let  $Z$  be a closed subvariety of a smooth variety  $X$ . Suppose  $Z$  is smooth and of pure codimension  $c$ . Then we have a long exact sequence

$$(2.1.7) \quad \cdots \rightarrow H_{i+1}^M(X, \mathbb{Z}(j)) \rightarrow H_{i+1-2c}^M(Z, \mathbb{Z}(j-c)) \rightarrow H_i^M(X \setminus Z, \mathbb{Z}(j)) \rightarrow H_i^M(X, \mathbb{Z}(j)) \rightarrow \cdots.$$

**Lemma 2.1.** *Let  $X$  be a smooth variety of pure dimension  $d$ , and let  $i, j \in \mathbb{Z}$ .*

- (1) *Suppose either  $j < 0$ ,  $i > j + d$  or  $i > 2j$ . Then, we have  $H_M^i(X, \mathbb{Z}(j)) = 0$ .*
- (2) *Suppose  $j > i$ . Then, we have  $H_i^M(X, \mathbb{Z}(j)) = 0$ .*

*The same holds for the  $\mathbb{Z}/n$ -coefficient version for any  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ .*

*Proof.* For (1), see [28, (3.6), (19.3)]. If  $X$  is projective, (2) follows from (1) and (2.1.5). The general case follows by induction on  $\dim X$  and (2.1.7).  $\square$

By Lemma 2.1 and (2.1.2), we can identify  $H_i^M(X, \mathbb{Z}(i))/n = H_i^M(X, \mathbb{Z}/n(i))$  for any  $i \in \mathbb{Z}$  and  $n \in \mathbb{Z}_{>0}$ , which will be frequently used without further notice.

**Remark 2.2.** If  $X$  is a smooth variety of pure dimension  $d$ , then motivic homology  $H_i^M(X, \mathbb{Z}(j))$  is isomorphic to *motivic cohomology with compact support*  $H_{M,c}^{2d-i}(X, \mathbb{Z}(d-j))$ . However, the work of Geisser (cf. Remark 1.4) suggests that motivic homology would be better for a further generalization, so we opt to use homology theory.

**2.2. Motivic complex.** We recall some results on motivic complex. Let  $X$  be a smooth variety over  $F$ , and let  $j \in \mathbb{Z}_{\geq 0}$ . There is a complex  $\mathbb{Z}(j)_X$  of Zariski sheaves on  $X$  [28, (3.1)], which is concentrated in degrees  $\leq j$ . The hypercohomology of  $\mathbb{Z}(j)_X$  agrees with motivic cohomology [28, (14.16)]:

$$H_{\text{Zar}}^i(X, \mathbb{Z}(j)_X) \cong H_M^i(X, \mathbb{Z}(j)).$$

Thanks to the recent resolution of Bloch-Kato conjecture by Rost and Voevodsky (see [38, 44, 45]), the following important result of Suslin-Voevodsky [40] (see also Geisser-Levine [14]) holds unconditionally.

**Theorem 2.3** ([40, 14]). *Let  $X$  be a smooth variety over  $F$ , and let  $\pi : X_{\text{et}} \rightarrow X_{\text{Zar}}$  be the natural map of sites. Let  $j \in \mathbb{Z}_{\geq 0}$  and  $n \in \mathbb{Z}_{> 0}$ .*

(1) *There is a canonical isomorphism*

$$(2.2.1) \quad \pi^* \mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \cong \mu_n^{\otimes j}.$$

*Consequently, we have a canonical map for any  $i \in \mathbb{Z}_{\geq 0}$*

$$(2.2.2) \quad H_M^i(X, \mathbb{Z}/n(j)) \rightarrow H_{\text{et}}^i(X, \mu_n^{\otimes j}).$$

(2) *The map (2.2.1) induces an isomorphism*

$$\mathbb{Z}(j)_X \otimes^{\mathbb{L}} \mathbb{Z}/n \rightarrow \tau_{\leq j} R\pi_* \mu_n^{\otimes j}.$$

*Consequently, the map (2.2.2) is an isomorphism if either  $i \leq j$  or  $j \geq \dim X + \text{cd } F$ . (Here  $\text{cd}$  means the cohomological dimension.) It is an injection if  $i = j + 1$ .*

In what follows, we frequently write  $H_{\text{et}}^i(X, \mathbb{Z}/n(j))$  for  $H_{\text{et}}^i(X, \mu_n^{\otimes j})$ .

**2.3. Wiesend's ideal class group.** In this subsection, we do not assume  $\text{Char } F = 0$ . We shall prove Theorem 1.3, after introducing a simple lemma.

**Lemma 2.4.** *Let  $r \in \mathbb{Z}_{\geq 0}$ . Let  $f : X \rightarrow X'$  be a morphism of varieties over  $F$ . There is a unique homomorphism  $f_* : C_r(X) \rightarrow C_r(X')$  characterized by the following property: for any  $x \in X_{(0)}$ , the diagram*

$$\begin{array}{ccc} K_r^M F(x) & \rightarrow & C_r(X) \\ \downarrow^{N_{F(x)/F(f(x))}} & & \downarrow_{f_*} \\ K_r^M F(f(x)) & \rightarrow & C_r(X') \end{array}$$

*commutes. (Here the upper and lower horizontal maps are the natural map to the  $x$ -component and  $f(x)$ -component respectively.) This makes  $C_r$  a covariant functor on the category of varieties over  $F$ .*

*Proof.* Uniqueness is clear by the definition of  $C_r(X)$ . Functoriality follows from that of the norm map of Milnor  $K$ -groups. We prove well-definedness. Take  $y \in X_{(1)}$  and put  $y' = f(y)$ . We need to show that the image of  $UK_{r+1}^M F(y)$  in  $C_r(X')$  is trivial. We consider two cases separately. First we assume  $y' \in X'_{(0)}$ . With the notation of §1.2, we regard  $\bar{C}(y)$  as a curve over  $F(y')$ . Weil reciprocity [15, Proposition 7.4.4] shows that the composition map

$$K_{r+1}^M F(y) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in \bar{C}(y)_{(0)}} K_r^M F(x) \xrightarrow{\oplus N_{F(x)/F(y')}} K_r^M F(y')$$

is the zero-map. In view of the definition of  $UK_{r+1}^M F(y)$ , this proves our assertion in this case. Next, we assume  $y' \in X'_{(1)}$  so that  $f$  induces a morphism  $f_y : \bar{C}(y) \rightarrow \bar{C}(y')$  such that  $f_y^{-1}(C_{\infty}(y')) \subset C_{\infty}(y)$ . It suffices to show that the image of  $UK_{r+1}^M F(y)$  by the norm

map  $K_{r+1}^M F(y) \rightarrow K_{r+1}^M F(y')$  is contained in  $UK_{r+1}^M F(y')$ . This follows from the following basic fact on the tame symbol. (We omit its proof.)  $\square$

**Lemma 2.5.** *Let  $L/K$  be a finite extension of fields. Let  $v$  a discrete valuation on  $K$ , and let  $\{v_1, \dots, v_f\}$  be the set of all extensions of  $v$  to  $L$ . We write  $C$  and  $D_i$  ( $1 \leq i \leq f$ ) for the residue field of  $v$  and  $v_i$  respectively. Let  $r \in \mathbb{Z}_{>0}$ . For any  $\lambda \in K_r^M L$ , we have*

$$\partial_v N_{L/K} \lambda = \sum_{i=1}^f N_{D_i/C} \partial_{v_i} \lambda.$$

*Proof of Theorem 1.3.* The case  $r = 0$  is a conjunction of the comparison theorem of motivic homology with Suslin's homology ([43, Corollary 3.2.7], [28, Proposition 14.18]) and the comparison theorem of Suslin's homology and  $C_0(V)$  [35, Theorem 5.1]. We proceed by induction on  $r$ .

We construct a commutative diagram

$$(2.3.1) \quad \begin{array}{ccc} \bigoplus_{y \in (V \times \mathbb{G}_m)_{(1)}} UK_{r+1}^M F(y) & \xrightarrow{\partial} & \bigoplus_{x \in (V \times \mathbb{G}_m)_{(0)}} K_r^M F(x) \\ \downarrow f & & \downarrow g \\ \bigoplus_{w \in V_{(1)}} UK_{r+2}^M F(w) & \xrightarrow{\partial} & \bigoplus_{z \in V_{(0)}} K_{r+1}^M F(z). \end{array}$$

The two horizontal maps  $\partial$  are given by the tame symbol. For  $x \in (V \times \mathbb{G}_m)_{(0)}$  and  $z \in V_{(0)}$ , the  $(x, z)$ -component of  $g$  is given as follows. It is the zero-map if  $z \neq p_0(x)$ , where  $p_0 : V \times \mathbb{G}_m \rightarrow V$  is the projection. Suppose  $z = p_0(x)$ , so that  $F(x)$  is a finite extension of  $F(z)$ . The composition  $x \rightarrow V \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  defines an element  $\xi(x) \in \mathbb{G}_m(F(x)) = F(x)^*$ . Now the map in question is given by

$$K_r^M F(x) \xrightarrow{\text{mult. by } \xi(x)} K_{r+1}^M F(x) \xrightarrow{N_{F(x)/F(z)}} K_{r+1}^M F(z).$$

Next, let  $y \in (V \times \mathbb{G}_m)_{(1)}$  and  $w \in V_{(1)}$ . If  $w \neq p_0(y)$ , then the  $(y, w)$ -component of  $f$  is the zero-map. Suppose  $w = p_0(y)$ , so that  $F(y)$  is a finite extension of  $F(w)$ . The composition  $y \rightarrow V \times \mathbb{G}_m \rightarrow \mathbb{G}_m$  defines an element  $\xi(y) \in \mathbb{G}_m(F(y)) = F(y)^*$ . Now the  $(y, w)$ -component of  $f$  is given by

$$UK_{r+1}^M F(y) \xrightarrow{\text{mult. by } \xi(y)} UK_{r+2}^M F(y) \xrightarrow{N_{F(y)/F(w)}} UK_{r+2}^M F(w).$$

We check the commutativity. Let  $y \in (V \times \mathbb{G}_m)_{(1)}$  and  $\lambda \in UK_{r+1}^M F(y)$ . Put  $w = p_0(y)$ . When  $w \in V_{(0)}$ , by definition we have  $\partial(f(\lambda)) = 0$ , and setting  $z = w$  we have

$$\begin{aligned}
g(\partial(\lambda)) &= g\left(\sum_{x \in C(y)} \partial_x \lambda\right) = \sum_{x \in \tilde{C}(y)} N_{F(x)/F(z)} \{\partial_x \lambda, \xi(x)\} \\
&\stackrel{(1)}{=} \sum_{x \in \tilde{C}(y)} N_{F(x)/F(z)} \partial_x \{\lambda, \xi(y)\} \\
&\stackrel{(2)}{=} \left[ \sum_{x \in \tilde{C}(y)} + \sum_{x \in C_\infty(y)} \right] (N_{F(x)/F(z)} \partial_x \{\lambda, \xi(y)\}) \\
&\stackrel{(3)}{=} 0.
\end{aligned}$$

Here, at (1) we used the fact that for all  $x \in \tilde{C}(y)$  we have  $\text{ord}_x \xi(y) = 0$  and  $\partial_x \{\xi(y), \pi_x\} = \xi(x)$ , where  $\pi_x \in k(y)^*$  is a uniformizer at  $x$ . At (2), we used the definition of  $UK_{r+1}^M F(y)$  (that is,  $\partial_x(\lambda) = 0$  and  $\partial_x \{\lambda, \pi_x\} = 0$  for all  $x \in C_\infty(y)$ ). The equality (3) is Weil reciprocity [15, Proposition 7.4.4]. Now we suppose  $w \in V_{(1)}$ . We take  $z \in V_{(0)}$ , and we write  $\{x_1, \dots, x_n\}$  for the set of all points on  $\tilde{C}(y)$  above  $z$ . We have

$$\begin{aligned}
z\text{-component of } \partial(f(\lambda)) &= \partial_z N_{F(y)/F(w)} \{\lambda, \xi(y)\} \\
&\stackrel{(1)}{=} \sum_{i=1}^n N_{F(x_i)/F(z)} \partial_{x_i} \{\lambda, \xi(y)\} \\
&\stackrel{(2)}{=} \sum_{i=1}^n N_{F(x_i)/F(z)} \{\partial_{x_i}(\lambda), \xi(x_i)\} \\
&= z\text{-component of } g(\partial(\lambda)).
\end{aligned}$$

At (1), we used Lemma 2.5. The equality (2) follows from the fact that  $\text{ord}_x \xi(y) = 0$  and  $\partial_x \{\xi(y), \pi_x\} = \xi(x)$  for all  $x \in \tilde{C}(y)$ . This proves the commutativity.

It can be seen that both  $f$  and  $g$  are surjective without difficulty. As a consequence, we get a surjection

$$\alpha : C_r(V \times \mathbb{G}_m) \rightarrow C_{r+1}(V).$$

It follows from the definition that the composition  $C_r(V) = C_r(V \times \{1\}) \rightarrow C_r(V \times \mathbb{G}_m) \xrightarrow{\alpha} C_{r+1}(V)$  is the zero-map.

Next, we consider a diagram (cf. Kahn [20, (4.3)])

$$\begin{array}{ccc}
\bigoplus_{x \in (V \times \mathbb{G}_m)_{(0)}} K_r^M F(x) & \rightarrow & H_{-r}^M(V \times \mathbb{G}_m, \mathbb{Z}(-r)) \\
\downarrow g & & \downarrow \\
\bigoplus_{z \in V_{(0)}} K_{r+1}^M F(z) & \rightarrow & H_{-r-1}^M(V, \mathbb{Z}(-r-1)).
\end{array}
\tag{2.3.2}$$

Here the horizontal maps are induced by the functoriality of motivic homology and (2.1.6). The right vertical map is the projection with respect to the decomposition (2.1.3). The left vertical map  $g$  was defined above.

We denote by  $P_r(V)$  the assertion that the diagram (2.3.2) is commutative. We also denote by  $Q_r(V)$  the assertion that  $\bigoplus_{x \in V(0)} K_r^M F(x) \rightarrow H_{-r}^M(V, \mathbb{Z}(-r))$  induces an isomorphism  $C_r(V) \cong H_{-r}^M(V, \mathbb{Z}(-r))$  (which is what the theorem claims). We claim:

- (1) We have  $Q_0(V)$  for all  $V$ .
- (2) Fix  $r \in \mathbb{Z}_{\geq 0}$  and a finite extension  $F'/F$ . If  $Q_r(\mathbb{G}_m \times \text{Spec } F')$  holds, then  $P_r(\text{Spec } F')$  holds.
- (3) Fix  $r \in \mathbb{Z}_{\geq 0}$ . If  $P_r(\text{Spec } F')$  holds for any finite extension  $F'/F$ , then  $P_r(V)$  holds for all  $V$ .
- (4) Fix  $r \in \mathbb{Z}_{\geq 0}$  and  $V$ . If  $P_r(V), Q_r(V)$  and  $Q_r(V \times \mathbb{G}_m)$  hold, then  $Q_{r+1}(V)$  holds.

Indeed, (1) was already remarked at the beginning of the proof. (2) holds because we have  $Q_{r+1}(\text{Spec } F')$  by (2.1.6). (By the definition of  $C_r(V)$ , if one has  $Q_r(V \times \mathbb{G}_m)$  and  $Q_{r+1}(V)$ , then  $P_r(V)$  becomes trivial.) (3) is clear from the definition of  $P_r(V)$ . We prove (4). By  $P_r(V), Q_r(V \times \mathbb{G}_m)$  and the surjectivity of  $f$ , we get a well-defined map (!) in the commutative diagram

$$\begin{array}{ccc} C_r(V \times \mathbb{G}_m)/C_r(V) & \xrightarrow{\alpha} & C_{r+1}(V) \\ \downarrow & & \downarrow^{(!)} \\ H_{-r}^M(V \times \mathbb{G}_m, \mathbb{Z}(-r))/H_{-r}^M(V, \mathbb{Z}(-r)) & \cong & H_{-r-1}^M(V, \mathbb{Z}(-r-1)). \end{array}$$

By  $Q_r(V)$  and  $Q_r(V \times \mathbb{G}_m)$ , the left vertical map is an isomorphism. It follows the right vertical map is also an isomorphism, which shows (4). Now the theorem follows by induction.  $\square$

### 3. ETALE COHOMOLOGY WITH COMPACT SUPPORT

In this section,  $F$  is a field of characteristic zero.

**3.1. The map  $c_{X,n}^{i,j}$ .** Let  $V$  be a smooth irreducible variety over  $F$  of dimension  $d$ . Schmidt and Spiess [36, Proposition 3.1 and p. 26 Remarks] constructed a canonical homomorphism

$$(3.1.1) \quad c_{V,n}^{i,j} : H_{2d-i}^M(V, \mathbb{Z}/n(d-j)) \rightarrow H_{et,c}^i(V, \mathbb{Z}/n(j))$$

for any  $i, j \in \mathbb{Z}_{\geq 0}$  and  $n \in \mathbb{Z}_{>0}$ , which is functorial in  $V$ . If  $V$  is projective,  $c_{V,n}^{i,j}$  coincides with the map (2.2.2) under the identification (2.1.5) and  $H_{et,c}^i(V, \mathbb{Z}/n(j)) = H_{et}^i(V, \mathbb{Z}/n(j))$ . The maps  $c_{X,n}^{i,j}$  are functorial with respect to the sequences (2.1.7) and the Gysin sequence in etale cohomology. By taking the inductive limit, we also have

$$c_{V,\infty}^{i,j} : H_{2d-i}^M(V, \mathbb{Q}/\mathbb{Z}(d-j)) \rightarrow H_{et,c}^i(V, \mathbb{Q}/\mathbb{Z}(j)).$$

The source of the map  $c_{V,n}^{2d+r,d+r}$  can be identified with  $C_r(V)/n$  by Theorem 1.3.

**Remark 3.1.** In [36, Proposition 3.1 and p. 26 Remarks (b)], this homomorphism is constructed when  $k$  is a finite field. In [36, p. 26 Remarks (a)], it is pointed out that the same construction works over any perfect field, by using relative Poincaré duality instead

of its absolute version. When  $k$  is a  $p$ -adic field, one can also use the absolute version (based on  $H_{\text{Gal}}^2(k, \mathbb{Z}/n\mathbb{Z}(1)) \cong \mathbb{Z}/n$ ).

**Proposition 3.2.** *Let  $V$  be a smooth variety over  $F$  of dimension  $d$ , and let  $i, j \in \mathbb{Z}_{\geq 0}$ ,  $n \in \mathbb{Z}_{>0}$ . If  $i \leq j$  or  $j \geq d + \text{cd } F$  then  $c_{V,n}^{i,j}$  is an isomorphism. If  $i = j + 1$ , then  $c_{V,n}^{i,j}$  is an injection.*

*Proof.* By induction on  $\dim V$  and (2.1.7), this can be reduced to the case where  $V$  is projective, which is proved in Theorem 2.3.  $\square$

**3.2. Curves.** Let  $X$  be a smooth projective irreducible curve over  $F$ , and let  $U$  be an open dense subscheme of  $X$ . Put  $Z = X \setminus U$ .

**Lemma 3.3.** *Let  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ .*

- (1) *We have  $\ker(c_{U,n}^{2,1}) = 0$ . The map  $\text{coker}(c_{U,n}^{2,1}) \rightarrow \text{coker}(c_{X,n}^{2,1})$  is injective.*
- (2) *We have  $\ker(c_{U,n}^{3,2}) = 0$ . If  $\text{cd } F \leq 2$ , then we have  $\text{coker}(c_{U,n}^{3,2}) \cong \text{coker}(c_{X,n}^{3,2})$ .*

*Proof.* We consider a commutative diagram with exact rows

$$\begin{array}{ccccccc} H_1^M(X, \mathbb{Z}/n(0)) & \rightarrow & C_1(Z)/n & \rightarrow & C_0(U)/n & \rightarrow & C_0(X)/n \rightarrow 0 \\ \downarrow c_{X,n}^{1,1} & & \downarrow c_{Z,n}^{1,1} & & \downarrow c_{U,n}^{2,1} & & \downarrow c_{X,n}^{2,1} \\ H_{\text{et}}^1(X, \mathbb{Z}/n(1)) & \rightarrow & H_{\text{et}}^1(Z, \mathbb{Z}/n(1)) & \xrightarrow{(*)} & H_{\text{et},c}^2(U, \mathbb{Z}/n(1)) & \rightarrow & H_{\text{et}}^2(X, \mathbb{Z}/n(1)) \end{array}$$

The two left vertical maps are bijective and two right vertical maps are injective by Proposition 3.2, and (1) follows. The proof of (2) is similar.  $\square$

**3.3. Surfaces.** Let  $X$  be a smooth projective irreducible surface over  $F$ . Let  $V \subset X$  be an open subvariety such that  $\dim(X \setminus V) = 0$ . Let  $U \subset V$  be an open subvariety such that  $V \setminus U$  is a (not necessary connected) smooth curve. Given a smooth surface  $U$ , one can always find such  $V$  and  $X$ . Let  $C$  be the smooth compactification of  $V \setminus U$  (that is, the normalization of  $X \setminus U$ ).

**Proposition 3.4.** *Let  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ .*

- (1) *There is a canonical homomorphism*

$$\eta_n : \text{coker}(c_{V,n}^{3,2}) \rightarrow \text{coker}(c_{V \setminus U,n}^{3,2}),$$

*for which there are exact sequences*

$$\begin{aligned} \ker(\eta_n) &\xrightarrow{(*)} \ker(c_{U,n}^{4,2}) \rightarrow \ker(c_{X,n}^{4,2}), \\ \text{coker}(\eta_n) &\xrightarrow{(**)} \text{coker}(c_{U,n}^{4,2}) \rightarrow \text{coker}(c_{V,n}^{4,2}). \end{aligned}$$

*If  $H_{\text{et},c}^3(V, \mathbb{Z}/n(2)) \rightarrow H_{\text{et},c}^3(V \setminus U, \mathbb{Z}/n(2))$  is injective, then  $(*)$  is injective too. If  $c_{X,n}^{4,2}$  is injective, then  $(**)$  is injective too. If  $\text{cd } F \leq 2$ , then we have  $\text{coker}(c_{V \setminus U,n}^{3,2}) \cong \text{coker}(c_{C,n}^{3,2})$ ,  $\text{coker}(c_{V,n}^{3,2}) \cong \text{coker}(c_{X,n}^{3,2})$  and  $\text{coker}(c_{V,n}^{4,2}) \cong \text{coker}(c_{X,n}^{4,2})$  (i.e., one can replace  $V \setminus U$  and  $V$  by  $C$  and  $X$  in the above sequences).*

(2) Suppose  $F$  is a finite extension of  $\mathbb{Q}_p$ . Then there is an exact sequence

$$\text{coker}(c_{X,n}^{4,3}) \rightarrow \ker(c_{U,n}^{5,3}) \rightarrow \ker(c_{X,n}^{5,3}),$$

and we have  $\text{coker}(c_{U,n}^{5,3}) \cong \text{coker}(c_{X,n}^{5,3})$ .

Before we give a proof of this proposition, we record a well-known lemma which describes the kernel and cokernel of  $c_{X,n}^{i,j}$ . We set  $\mathcal{H}^i(\mathbb{Z}/n(j)) := R^i \pi_* \mu_n^{\otimes j}$  for  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$  and  $i, j \in \mathbb{Z}$ . (See Theorem 2.3 for the definition of  $\pi$ .)

**Lemma 3.5.** (1) Let  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ . There is an exact sequences

$$\begin{aligned} 0 \rightarrow H_1^M(X, \mathbb{Z}/n(0)) &\xrightarrow{c_{X,n}^{3,2}} H_{\text{et}}^3(X, \mathbb{Z}/n\mathbb{Z}(2)) \\ &\rightarrow H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Z}/n(2))) \rightarrow C_0(X)/n \xrightarrow{c_{X,n}^{4,2}} H_{\text{et}}^4(X, \mathbb{Z}/n(2)). \end{aligned}$$

(2) Let  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ . There is an exact sequence

$$\begin{aligned} 0 \rightarrow H_0^M(X, \mathbb{Z}/n(-1)) &\xrightarrow{c_{X,n}^{4,3}} H_{\text{et},c}^4(X, \mathbb{Z}/n(3)) \rightarrow H_{\text{Zar}}^0(X, \mathcal{H}^4(\mathbb{Z}/n(3))) \\ &\rightarrow C_1(X)/n \xrightarrow{c_{X,n}^{5,3}} H_{\text{et},c}^5(X, \mathbb{Z}/n(3)) \rightarrow H_{\text{Zar}}^1(X, \mathcal{H}^4(\mathbb{Z}/n(3))) \rightarrow 0. \end{aligned}$$

(3) Let  $n \in \mathbb{Z}_{>0}$ . The canonical map  $H_{\text{Zar}}^0(X, \mathcal{H}^i(\mathbb{Z}/n(i-1))) \rightarrow H_{\text{Zar}}^0(X, \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}(i-1)))[n]$  is bijective for  $i = 3, 4$ .

*Proof.* Since  $X$  is projective, (1) and (2) follow from (2.1.5) and Bloch-Ogus theory. (3) is a consequence of Bloch-Kato conjecture.  $\square$

*Proof of Proposition 3.4 (1).* We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} H_1^M(V, \mathbb{Z}/n\mathbb{Z}(0)) & \rightarrow & C_1(V \setminus U)/n & \rightarrow & C_0(U)/n & \rightarrow & C_0(V)/n \rightarrow 0 \\ \downarrow c_{V,n}^{3,2} & & \downarrow c_{V \setminus U,n}^{3,2} & & \downarrow c_{U,n}^{4,2} & & \downarrow c_{V,n}^{4,2} \\ H_{\text{et},c}^3(V, \mathbb{Z}/n\mathbb{Z}(2)) & \rightarrow & H_{\text{et},c}^3(V \setminus U, \mathbb{Z}/n\mathbb{Z}(2)) & \rightarrow & H_{\text{et},c}^4(U, \mathbb{Z}/n\mathbb{Z}(2)) & \rightarrow & H_{\text{et},c}^4(V, \mathbb{Z}/n\mathbb{Z}(2)). \end{array}$$

Proposition 3.2 shows the injectivity of  $c_{V,n}^{3,2}$  and  $c_{V \setminus U,n}^{3,2}$ . If  $\text{cd } F \leq 2$ , then we have  $\text{coker}(c_{V \setminus U,n}^{3,2}) \cong \text{coker}(c_{C,n}^{3,2})$  by Lemma 3.3. Using the following lemma, a diagram chase completes the proof.  $\square$

**Lemma 3.6.** The map  $\ker(c_{V,n}^{4,2}) \rightarrow \ker(c_{X,n}^{4,2})$  is injective. If  $\text{cd } F \leq 2$ ,  $\text{coker}(c_{V,n}^{i,2}) \rightarrow \text{coker}(c_{X,n}^{i,2})$  is bijective for  $i = 3, 4$ .

*Proof.* This follows from Proposition 2.1, 3.2 and (2.1.7).  $\square$

*Proof of Proposition 3.4 (2).* We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} H_0^M(V, \mathbb{Z}/n\mathbb{Z}(1)) & \rightarrow & C_2(V \setminus U)/n & \rightarrow & C_1(U)/n & \rightarrow & C_1(V)/n \rightarrow 0 \\ \downarrow c_{V,n}^{4,3} & & \downarrow c_{V \setminus U,n}^{4,3} & & \downarrow c_{U,n}^{5,3} & & \downarrow c_{V,n}^{5,3} \\ H_{\text{et},c}^4(V, \mathbb{Z}/n\mathbb{Z}(3)) & \rightarrow & H_{\text{et},c}^4(V \setminus U, \mathbb{Z}/n\mathbb{Z}(3)) & \rightarrow & H_{\text{et},c}^5(U, \mathbb{Z}/n\mathbb{Z}(3)) & \rightarrow & H_{\text{et},c}^5(V, \mathbb{Z}/n\mathbb{Z}(3)) \rightarrow 0. \end{array}$$

Proposition 3.2 shows the injectivity of the left two vertical maps. In Proposition 4.3 below, we will show that  $c_{V \setminus U, n}^{4,3}$  is bijective. Using the following lemma, a diagram chase completes the proof.  $\square$

**Lemma 3.7.** *The map  $\ker(c_{V, n}^{5,3}) \rightarrow \ker(c_{X, n}^{5,3})$  is injective. If  $\text{cd } F \leq 2$ , then  $\text{coker}(c_{V, n}^{i,3}) \rightarrow \text{coker}(c_{X, n}^{i,3})$  is bijective for  $i = 3, 4$ .*

*Proof.* This follows from Proposition 2.1, 3.2 and (2.1.7).  $\square$

#### 4. VARIETIES OVER A $p$ -ADIC FIELD

In this section,  $k$  is a finite extension of  $\mathbb{Q}_p$ . Let  $X$  be a smooth geometrically irreducible variety over  $k$  of dimension  $d$ .

**4.1. The Brauer-Manin pairing.** By Poincaré duality, we have a canonical isomorphism

$$H_{et, c}^{2d}(X, \hat{\mathbb{Z}}(d)) \cong H_{et}^2(X, \mathbb{Q}/\mathbb{Z}(1))^*.$$

Since  $\text{Br}(X)$  is a torsion group, the Kummer sequence implies an exact sequence

$$0 \rightarrow \text{Pic}(X) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H_{et}^2(X, \mathbb{Q}/\mathbb{Z}(1)) \rightarrow \text{Br}(X) \rightarrow 0,$$

hence we have an injective homomorphism  $\text{Br}(X)^* \rightarrow H_{et, c}^{2d}(X, \hat{\mathbb{Z}}(d))$ . We consider a diagram

$$\begin{array}{ccc} Z_0(X) & \xrightarrow{\tilde{\psi}_X} & \text{Br}(X)^* \\ \downarrow \text{surj.} & & \downarrow \text{inj.} \\ C_0(X) & \rightarrow & H_{et, c}^{2d}(X, \hat{\mathbb{Z}}(d)), \end{array}$$

where the lower horizontal map is the composition of

$$C_0(X) \rightarrow \lim_{\leftarrow n} C_0(X)/n \xrightarrow{(c_{X, n}^{2d, d})^n} H_{et, c}^{2d}(X, \hat{\mathbb{Z}}(d)).$$

This diagram is seen to be commutative by reducing to the case  $\dim X = 0$ . This shows the first part of Proposition 1.5 (the well-definedness of  $\psi_X$ ). The same argument shows the following simple lemma.

**Lemma 4.1.** *For any  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ , we have an isomorphism  $\ker(\psi_X/n) \cong \ker(c_{X, n}^{2d, d})$  and an exact sequence*

$$0 \rightarrow \text{coker}(\psi_X/n) \rightarrow \text{coker}(c_{X, n}^{2d, d}) \rightarrow (\text{Pic}(X)/n)^* \rightarrow 0.$$

4.2. **Class field theory.** By Poincaré duality, we have a canonical isomorphism

$$H_{et,c}^{2d+1}(X, \hat{\mathbb{Z}}(d+1)) \cong H_{et}^1(X, \mathbb{Q}/\mathbb{Z})^* \cong \pi_1^{ab}(X).$$

We consider a diagram

$$\begin{array}{ccc} Z_0^1(X) & \xrightarrow{\tilde{\rho}_X} & \pi_1^{ab}(X) \\ \downarrow \text{surj.} & & \downarrow \cong \\ C_1(X) & \rightarrow & H_{et,c}^{2d+1}(X, \hat{\mathbb{Z}}(d+1)). \end{array}$$

where the lower horizontal map is the composition of

$$C_1(X) \rightarrow \lim_{\leftarrow n} C_1(X)/n \xrightarrow{(c_{X,n}^{2d+1,d+1})_n} H_{et,c}^{2d+1}(X, \hat{\mathbb{Z}}(d+1)).$$

This diagram is seen to be commutative by reducing to the case  $\dim X = 0$ . This shows the second part of Proposition 1.5 (the well-definedness of  $\rho_X$ ). The same argument shows the following simple lemma.

**Lemma 4.2.** *For any  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ , we have isomorphisms  $\ker(\rho_X/n) \cong \ker(c_{X,n}^{2d+1,d+1})$  and  $\text{coker}(\rho_X/n) \cong \text{coker}(c_{X,n}^{2d+1,d+1})$ .*

4.3. **Higher degree.** We consider the groups  $C_r(X)$  when  $r \geq 2$ . Proposition 4.3 below shows that  $C_r(X)$  is uniquely divisible if  $r \geq 3$ , and that  $C_2(X) \rightarrow C_2(\text{Spec } k) = K_2(k)$  is a surjective map with uniquely divisible kernel. Note that  $K_2(k)$  is the direct sum of a uniquely divisible group and a finite group isomorphic to  $\mu(k)$  (see [27]).

**Proposition 4.3.** *Let  $X$  be a smooth geometrically irreducible variety over  $k$ . Suppose that there exists a smooth projective variety  $Y$  that contains  $X$  as an open dense subvariety. Let  $i, j \in \mathbb{Z}$  and suppose  $j \leq -2$ .*

- (1) *If  $i < -2$ , then  $H_i^M(X, \mathbb{Z}(j))$  is uniquely divisible.*
- (2) *The map  $H_{-2}^M(X, \mathbb{Z}(j)) \rightarrow H_{-2}^M(\text{Spec } k, \mathbb{Z}(j))$  induced by the structure morphism is surjective with uniquely divisible kernel. Consequently,*

$$c_{X,n}^{2d+2,d+2} : C_2(X)/n \rightarrow H_{et,c}^{2d+2}(X, \mathbb{Z}/n(d+2))$$

*is bijective for any  $n \in \mathbb{Z}_{>0}$ .*

*Proof.* Since  $j \leq -2$ , Proposition 3.2 shows that

$$(4.3.1) \quad c_{X,n}^{2d-i,d-j} : H_i^M(X, \mathbb{Z}/n(j)) \rightarrow H_{et,c}^{2d-i}(X, \mathbb{Z}/n(d-j))$$

is an isomorphism for any  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ . If  $i < -2$ , then we have  $H_{et,c}^{2d-i}(X, \mathbb{Z}/n\mathbb{Z}(d-j)) = 0$  since  $\text{cd } k = 2$ . We also have  $H_{et,c}^{2d+2}(X, \mathbb{Q}/\mathbb{Z}(d-j)) = 0$ , because this group is dual to  $H_{\text{Gal}}^0(k, \hat{\mathbb{Z}}(j+1))^* = 0$  under Poincaré duality. Now (1) follows from (2.1.2).

(2) follows from the four claims (a)-(d) below. In what follows we write  $H_*^M(k, -)$  for  $H_*^M(\text{Spec } k, -)$ . Put  $H_{-2}^M(X, \mathbb{Z}(j))_0 := \ker[H_{-2}^M(X, \mathbb{Z}(j)) \rightarrow H_{-2}^M(k, \mathbb{Z}(j))]$ .

- (a)  $H_{-2}^M(X, \mathbb{Z}(j)) \rightarrow H_{-2}^M(k, \mathbb{Z}(j))$  is surjective.
- (b)  $H_{-2}^M(X, \mathbb{Z}(j))/n \rightarrow H_{-2}^M(k, \mathbb{Z}(j))/n$  is bijective for any  $n \in \mathbb{Z}_{>0}$ .



*Proof.* By Poincaré duality, it suffices to show  $H_{\text{Gal}}^0(k, H_{\text{et}}^1(\bar{X}, \mathbb{Z}_l(j+1))) = 0$  for all prime number  $l$ . There is an exact sequence

$$0 \rightarrow H_{\text{et}}^1(\bar{Y}, \mathbb{Z}_l(j+1)) \rightarrow H_{\text{et}}^1(\bar{X}, \mathbb{Z}_l(j+1)) \rightarrow H_{\text{et}, \bar{Y} \setminus X}^2(\bar{Y}, \mathbb{Z}_l(j+1)).$$

By excision and purity, we have  $H_{\text{Gal}}^0(k, H_{\text{et}, \bar{Y} \setminus X}^2(\bar{Y}, \mathbb{Z}_l(j+1))) = 0$ . Thus we are reduced to showing  $H_{\text{Gal}}^0(k, H_{\text{et}}^1(\bar{Y}, \mathbb{Z}_l(j+1))) = 0$ , but this is a result of Jannsen [19, Theorem 4.2 and 5.3 for the case  $l \neq p$  and  $l = p$  respectively].  $\square$

**4.4. Auxiliary lemmas.** For future use, we record a few simple lemmas. The following lemma is often used concurrently with the fact that the groups  $\pi_1^{ab}(X)$  and  $\text{Br}(X)^*$  have no non-trivial divisible elements for any smooth variety  $X$ . (Cf. §1.5.)

**Lemma 4.5.** *Let  $f : A \rightarrow B$  be a homomorphism of abelian groups.*

(1) *Suppose  $B_{\text{div}} = 0$ . Then there is an exact sequence*

$$0 \rightarrow A_{\text{div}} \rightarrow \ker(f) \rightarrow \varprojlim \ker(f/n).$$

(2) *Suppose that there is an  $N \in \mathbb{Z}_{>0}$  such that the canonical map  $\ker(f/nN) \rightarrow \ker(f \otimes \mathbb{Q}/\mathbb{Z})$  is bijective for all  $n \in \mathbb{Z}_{>0}$ . Then we have  $\varprojlim \ker(f/n) = 0$ . (Hence  $\ker(f) = A_{\text{div}}$  if  $B_{\text{div}} = 0$ .)*

*Proof.* (1) is a direct consequence of the definition. We prove (2). By assumption, the canonical map  $\ker(f/mN) \rightarrow \ker(f/mnN)$  is bijective for all  $n, m \in \mathbb{Z}_{>0}$ , which implies that the canonical map  $\ker(f/mnN) \rightarrow \ker(f/nN)$  is the zero-map. Thus we have  $\varprojlim \ker(f/n) = 0$ .  $\square$

**Lemma 4.6.** *Let  $X$  be a smooth irreducible variety over  $k$  of dimension  $d$ , and let  $i, j \in \mathbb{Z}$ . If  $i+2 \geq j+d$  or  $j \geq d+2$ , then we have  $H_i^M(X, \mathbb{Z}(j))_{\text{div}} = H_i^M(X, \mathbb{Z}(j))_{\text{Div}}$ . In particular, we have  $C_r(X)_{\text{div}} = C_r(X)_{\text{Div}}$  for any  $r \in \mathbb{Z}_{\geq 0}$  when  $d \leq 2$ . If  $d = 1$ , then  $C_0(X)_{\text{Div}} = 0$ .*

*Proof.* It suffices to show that  $H_i^M(X, \mathbb{Z}(j))[n]$  is finite for any  $n \in \mathbb{Z}_{>0}$ . Proposition 3.2 and (2.1.2) reduce this to the finiteness of  $H_{\text{et}, c}^{2d-i-1}(X, \mathbb{Z}/n(d-j))$ , which is well-known. For the last statement, we recall that  $C_0(X)$  is isomorphic to the relative Picard group (cf. Remark 1.2), which has no non-trivial divisible subgroup by a theorem of Mattuck [26].  $\square$

## 5. CURVES OVER A LOCAL FIELD

In this section,  $k$  is a finite extension of  $\mathbb{Q}_p$ . Let  $X$  be a smooth projective irreducible curve over  $k$  and let  $U$  be an open dense subscheme of  $X$ . Put  $Z = X \setminus U$ .

**5.1. The Brauer-Manin pairing.** The following theorem was proved by Scheiderer and van Hamel [34, (3.5)]. When  $U = X$ , this theorem is due to Lichtenbaum [23].

**Theorem 5.1** (Scheiderer/van Hamel [34]). *The homomorphism  $\psi_U : C_0(U) \rightarrow \text{Br}(U)^*$  is an injection with dense image. The induced map  $\psi_U/n : C_0(U)/n \rightarrow \text{Br}(U)^*/n$  is an isomorphism for all  $n \in \mathbb{Z}_{>0}$ .*

We deduce Theorem 5.1 assuming Lichtenbaum's result. We may assume  $X$  is geometrically irreducible over  $k$ . Let  $n \in \mathbb{Z}_{>0}$ . Lemmas 3.3 and 4.1 show the injectivity of  $\psi_U/n$ . By Lemma 4.1, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{coker}(\psi_U/n) & \rightarrow & \text{coker}(c_{U,n}^{2,1}) & \rightarrow & (\text{Pic}(U)/n)^* \rightarrow 0 \\ & & & & \downarrow (*) & & \downarrow \\ & & & & \text{coker}(c_{X,n}^{2,1}) & \cong & (\text{Pic}(X)/n)^*. \end{array}$$

The bottom horizontal map is bijective because  $\text{coker}(\psi_X/n) = 0$  by Lichtenbaum's result. By Lemma 3.3,  $(*)$  is injective, thus  $\text{coker}(\psi_U/n) = 0$ . The rest of assertion follows from Lemmas 4.5 and 4.6.  $\square$

**5.2. Class field theory.** We recall a result for projective curves due to Bloch and Saito.

**Theorem 5.2** ([2] [30]). (1) *The kernel of the homomorphism  $\rho_X : C_1(X)(= SK_1(X)) \rightarrow \pi_1^{ab}(X)$  coincides with  $C_1(X)_{\text{Div}}$ . Moreover,  $\rho_X/n$  is injective for all  $n \in \mathbb{Z}_{>0}$ .*  
(2) *The group  $D(X) := \pi_1^{ab}(X)/\overline{\text{Im}(\rho_X)}$  is isomorphic to  $\hat{\mathbb{Z}}^{\oplus r(X)}$  for some  $r = r(X) \in \mathbb{Z}_{\geq 0}$ . We have  $\text{coker}(\rho_X/n) \cong D(X)/n$  for any  $n \in \mathbb{Z}_{>0}$ .*

**Remark 5.3.** If  $X$  has potentially good reduction, then  $r(X) = 0$ . In general, we have an inequality  $r(X) \leq g(X)$ , where  $g(X)$  the genus of  $X$  [30, (6.2)]. In particular, We have  $D(X) = 0$  when  $g(X) = 0$ .

The following generalization follows from Lemmas 3.3, 4.2, 4.5 and 4.6.

**Theorem 5.4.** (1) *The kernel of the homomorphism  $\rho_U : C_1(U) \rightarrow \pi_1^{ab}(U)$  coincides with  $C_1(U)_{\text{Div}}$ . Moreover,  $\rho_U/n$  is injective for all  $n \in \mathbb{Z}_{>0}$ .*  
(2) *Set  $D(U) := \pi_1^{ab}(U)/\overline{\text{Im}(\rho_U)}$ . The canonical maps  $D(U) \rightarrow D(X)$  and  $\text{coker}(\rho_U/n) \rightarrow \text{coker}(\rho_X/n) (\cong D(X)/n)$  are bijective for any  $n \in \mathbb{Z}_{>0}$ .*

**Remark 5.5.** Following Hiranouchi [16], we define

$$C_1^w(U) := \text{coker}[K_2(k(X)) \rightarrow (\bigoplus_{x \in U_{(0)}} k(x)^*) \oplus (\bigoplus_{z \in Z} K_2(k(X)_z))],$$

where  $k(X)_z$  is the completion of  $k(X)$  at  $z$ . He constructed the reciprocity map  $\rho'_U : C_1^w(U) \rightarrow \pi_1^{ab}(U)$  and showed that the kernel of  $\rho'_U$  coincides with  $C_1^w(U)_{\text{Div}}$ , and that  $\pi_1^{ab}(U)/\overline{\text{Im}(\rho'_U)} \cong D(X)$ . It is easy to see the following:

- The natural projection  $\pi : C_1^w(U) \rightarrow C_1(U)$  fits into an exact sequence

$$\bigoplus_{z \in Z} UK_2k(X)_z \rightarrow C_1^w(U) \xrightarrow{\pi} C_1(U) \rightarrow 0.$$

Here we put  $UK_2k(X)_z = \ker[K_2k(X)_z \rightarrow k(z)^* \oplus K_2k(z)]$  where the map is defined by  $a \mapsto (\partial_z(a), \partial_z(\{\pi_z\} \cup a))$  for some uniformizer  $\pi_z \in k(X)_z$  at  $z$  (and  $\partial_z$  is the tame symbol). Note that  $UK_2k(X)_z$  is uniquely divisible.

- We have  $\rho'_U = \rho_U \circ \pi$ .

Hence Hiranouchi's result can be recovered by Theorem 5.4.

## 6. SURFACES OVER A LOCAL FIELD

We keep assuming  $k$  is a finite extension of  $\mathbb{Q}_p$ . Let  $X$  be a smooth projective geometrically connected surface over  $k$ . Let  $U \subset V \subset X$  be open dense subsets such that  $\dim(X \setminus V) = 0$  and such that  $V \setminus U$  is a smooth curve. Let  $C$  be the normalization of  $X \setminus U$ , and set  $D(C) = \bigoplus_i D(C_i)$ , where  $C_i$  are the irreducible components of  $C$ , and  $D(C_i)$  is the group defined in Theorem 5.2.

6.1.  $\mathbb{P}^2$  **minus curves.** As the first example, we show the following.

**Proposition 6.1.** *Suppose  $X \cong \mathbb{P}^2$ . Let  $d$  be the greatest common divisor of the degrees (as a subvariety of  $\mathbb{P}^2$ ) of the irreducible components of  $X \setminus U$ . Then, there is an exact sequence*

$$0 \rightarrow C_0(U)/nd \xrightarrow{\psi_U/nd} \text{Br}(U)^*/nd \rightarrow D(C)/nd \rightarrow 0$$

for any  $n \in \mathbb{Z}_{>0}$ . We have  $\ker(\psi_U) = C_0(U)_{\text{Div}}$  and  $\text{Br}(U)^*/\overline{\text{Im}(\psi_U)} \cong D(C)$ .

For the map  $\rho_U : C_1(U) \rightarrow \pi_1^{ab}(U)$ , see Theorem 6.5 below.

*Proof.* Let  $n \in \mathbb{Z}_{>0}$ . Note that  $\psi_X/n : C_0(X)/n \rightarrow \text{Br}(X)^*/n$  is bijective, and that  $H_{\text{Zar}}^0(X, \mathcal{H}^i(\mathbb{Q}/\mathbb{Z}(i-1))) = 0$  for  $i = 3, 4$ . By Lemmas 3.5, 4.1, and Proposition 3.4, we have  $\ker(\psi_U/n) = 0$ . By Lemmas 4.5 and 4.6, we get  $\ker(\psi_U) = C_0(U)_{\text{Div}}$ .

Note that  $\mathbb{Z} = \text{Pic}(X) \rightarrow \text{Pic}(U)$  is a surjection with kernel  $d\mathbb{Z}$ . By Lemma 4.1, we get a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & \text{coker}(\psi_U/nd) & \rightarrow & \text{coker}(c_{U,nd}^{4,2}) & \rightarrow & \mathbb{Z}/d\mathbb{Z} & \rightarrow 0 \\ & & & \downarrow & & \downarrow \text{inj.} & \\ 0 = & \text{coker}(\psi_X/nd) & \rightarrow & \text{coker}(c_{X,nd}^{4,2}) & \cong & \mathbb{Z}/nd. & \end{array}$$

By Proposition 3.4 and Lemma 3.5, we get an exact sequence

$$0 \rightarrow \text{coker}(c_{C,nd}^{3,2}) \rightarrow \text{coker}(c_{U,nd}^{4,2}) \rightarrow \text{coker}(c_{X,nd}^{4,2}).$$

It follows that  $\text{coker}(\psi_U/nd) \cong \text{coker}(c_{C,nd}^{3,2})$ , which is isomorphic to  $D(C)/nd$  by Lemma 4.2 and Theorem 5.2. We are done.  $\square$

**6.2. Rational surface.** Suppose that  $\bar{X} := X \times_{\text{Spec } k} \text{Spec } \bar{k}$  is a *rational surface* over  $\bar{k}$  (that is,  $\bar{X}$  is birational to  $\mathbb{P}_{\bar{k}}^2$ ), and that  $X(k) \neq \emptyset$ . In this situation, the Chow group  $C_0(X) = CH_0(X)$  and related maps such as  $\psi_X$  and  $c_{X,n}^{4,2}$  have been studied by several authors (see, for example, [25, 1, 8, 3, 4, 5]). We will briefly recall a few results among them.

We set  $A_0(X) := \ker[C_0(X) = CH_0(X) \xrightarrow{\deg} \mathbb{Z}]$ . Let  $S = \text{Hom}(NS(\bar{X}), \bar{k}^*)$  be the Néron-Severi torus of  $X$ . Recall that we have the Bloch map [1], [8]

$$\phi_X : A_0(X) \rightarrow H_{\text{Gal}}^1(k, S).$$

It is known [3] that  $\phi_X$  is injective. Note that  $H_{\text{Gal}}^1(k, S)$  is finite (hence so is  $A_0(X)$ ). We will also need the following result.

**Theorem 6.2** (Saito [31], see also [4]). *There is an  $N = N_X \in \mathbb{Z}_{>0}$  such that  $c_{X,nN}^{4,2}$  is injective for all  $n \in \mathbb{Z}_{>0}$ . Moreover, we have  $\ker(\psi_X) = 0$ .*

Now we state our main result.

**Theorem 6.3.** *We suppose the following the condition:*

$$(*) \quad \text{the irreducible components of } \bar{X} \setminus \bar{U} \text{ generate } NS(\bar{X}).$$

*Then we have the following.*

- (1)  $\ker(\psi_U) = C_0(U)_{\text{Div}}$ .
- (2)  $F_0(U) := \ker(\psi_U \otimes \mathbb{Q}/\mathbb{Z})$  is finite. There is an  $N \in \mathbb{Z}_{>0}$  such that for all  $n \in \mathbb{Z}_{>0}$  the canonical map  $\ker(\psi_U/nN) \rightarrow F_0(U)$  is bijective.
- (3) There is an injection  $F_0(U) \hookrightarrow \text{coker}(\phi_X)$ , which is bijective if  $D(C) = 0$ .

In Example 6.8, 6.10 below, we shall provide concrete examples for which  $F_0(U) \neq 0$ . Theorem 1.6 (1) is a consequence of this theorem and examples. Note that Parimala and Suresh [29] have constructed smooth projective surfaces  $X$  and  $X'$  such that  $\ker(\psi_X) \neq C_0(X)_{\text{Div}}$  and  $\ker(\psi_{X'}/2) \neq 0$ . It seems difficult to control the cokernel of  $\psi_U$ , because, as Proposition 6.1 shows, it can easily become bigger and bigger. If we fix  $X$  and vary  $U$ , the above theorem immediately implies the following.

**Proposition 6.4.** *There exists an open dense subvariety  $U_0 \subset X$  such that, for any open dense subvariety  $U \subset U_0$ , the natural map  $F_0(U) \rightarrow F_0(U_0)$  is an isomorphism.*

*Proof.* We take  $U_* \subset X$  that satisfies (\*). By Theorem 6.3 (3), the map  $F_0(U) \rightarrow F_0(U_*)$  is injective for any  $U \subset U_*$ . Since  $F_0(U_*)$  is finite, there exists  $U_0 \subset U_*$  such that

$$F_0(U_0) = \bigcap_{U \subset U_*} F_0(U).$$

The proposition holds with this  $U_0$ . □

As for the map  $\rho_U$ , we prove the following result. Theorem 1.6 (2) follows from this.

**Theorem 6.5.** *The map  $\rho_U/n : C_1(U)/n \rightarrow \pi_1^{ab}(U)/n$  is an isomorphism for any  $n \in \mathbb{Z}_{>0}$ . We have  $\ker(\rho_U) = C_1(U)_{\text{Div}}$  and  $\pi_1^{ab}(U) = \overline{\text{Im}(\rho_U)}$ .*

Note that Sato [33] has constructed smooth projective K3 surfaces  $X, X'$  such that  $\ker(\rho_X) \neq C_1(X)_{\text{Div}}$  and  $\ker(\rho_{X'}/2n) \neq 0$  for all  $n \in \mathbb{Z}_{>0}$ .

Before we start the proof of Theorems 6.3 and 6.5, we introduce a few lemmas.

**Lemma 6.6.** *We have a canonical isomorphism*

$$H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \cong \text{coker}[\phi_X : A_0(X) \rightarrow H_{\text{Gal}}^1(k, S)].$$

*In particular,  $H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$  is finite.*

*Proof.* By Kummer sequence, we get an exact sequence for any  $n \in \mathbb{Z}_{>0}$

$$0 \rightarrow S(k)/n \rightarrow H_{\text{Gal}}^1(k, S[n]) \rightarrow H_{\text{Gal}}^1(k, S)[n] \rightarrow 0.$$

Using the facts that  $H_{\text{Gal}}^1(k, S)$  is a torsion group, and that Hochschild-Serre spectral sequence induces an isomorphism  $H_{\text{et}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) \cong H_{\text{Gal}}^1(k, S_{\text{Tor}})$ , we get an exact sequence which fits into the lower row in the commutative diagram (note that  $CH_0(X)_{\text{Tor}} = A_0(X)$ )

$$\begin{array}{ccccccc} 0 \rightarrow & H_M^3(X, \mathbb{Z}(2)) \otimes \mathbb{Q}/\mathbb{Z} & \rightarrow & H_M^3(X, \mathbb{Q}/\mathbb{Z}(2)) & \rightarrow & CH_0(X)_{\text{Tor}} & \rightarrow 0 \\ & \downarrow \cong & & \downarrow c_{X, \infty}^{3,2} & & \downarrow \phi_X & \\ 0 \rightarrow & S(k) \otimes \mathbb{Q}/\mathbb{Z} & \rightarrow & H_{\text{et}}^3(X, \mathbb{Q}/\mathbb{Z}(2)) & \rightarrow & H_{\text{Gal}}^1(k, S) & \rightarrow 0. \end{array}$$

Here the upper row is exact by (2.1.2). The left vertical isomorphism is given by [3, Theorem C]. (Here we used the assumption  $X(k) \neq \emptyset$ ). Now the assertion follows from Theorem 6.2 and Lemma 3.5.  $\square$

**Lemma 6.7.** *If (\*) is satisfied, then  $H_{\text{et},c}^3(V, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H_{\text{et},c}^3(V \setminus U, \mathbb{Q}/\mathbb{Z}(2))$  is injective.*

*Proof.* It suffices to show that  $H_{\text{et},c}^3(U, \mathbb{Q}/\mathbb{Z}(2)) \rightarrow H_{\text{et},c}^3(V, \mathbb{Q}/\mathbb{Z}(2))$  is the zero map. By Poincaré duality, this amounts to showing that  $H_{\text{et}}^3(V, \hat{\mathbb{Z}}(1)) \rightarrow H_{\text{et}}^3(U, \hat{\mathbb{Z}}(1))$  is the zero map. By Gysin sequence  $H_{\text{et}}^3(X, \hat{\mathbb{Z}}(1)) \rightarrow H_{\text{et}}^3(V, \hat{\mathbb{Z}}(1)) \rightarrow H_{\text{et}}^0(X \setminus V, \hat{\mathbb{Z}}(-1)) = 0$ , it suffices to show that  $H_{\text{et}}^3(X, \hat{\mathbb{Z}}(1)) \rightarrow H_{\text{et}}^3(U, \hat{\mathbb{Z}}(1))$  is the zero-map. By Hochschild-Serre spectral sequence, we have  $H_{\text{et}}^3(X, \hat{\mathbb{Z}}(1)) = H_{\text{Gal}}^1(k, H_{\text{et}}^2(\bar{X}, \hat{\mathbb{Z}}(1)))$ , and we are reduced to showing that  $NS(\bar{X}) \otimes \hat{\mathbb{Z}} = H_{\text{et}}^2(\bar{X}, \hat{\mathbb{Z}}(1)) \rightarrow H_{\text{et}}^2(\bar{U}, \hat{\mathbb{Z}}(1))$  is the zero map, but this follows from the assumption (\*).  $\square$

*Proof of Theorem 6.3.* Let  $N_1 = N_X \in \mathbb{Z}_{>0}$  be the natural number appearing in Theorem 6.2, and  $N_2$  the order of  $H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ . Put  $N = N_1 N_2$  and take any  $n \in \mathbb{Z}_{>0}$ . By Theorem 6.2 and Lemma 3.5, we have  $\text{coker}(c_{X, nN}^{3,2}) \cong H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Z}/nN(2)))$ . Hence we have a commutative diagram which contains the map  $\eta_n$  appearing in Proposition 3.4:

$$\begin{array}{ccc} H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Z}/nN(2))) & \xrightarrow{\eta_n} & \text{coker}(c_{C, nN}^{3,2}) \cong (\mathbb{Z}/nN)^{\oplus r(C)} \\ \parallel & & \downarrow \cap \\ H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) & \xrightarrow{\eta_\infty} & \text{coker}(c_{C, \infty}^{3,2}) \cong (\mathbb{Q}/\mathbb{Z})^{\oplus r(C)}, \end{array}$$

where the two right horizontal isomorphisms are given by Theorem 5.2. We get  $\ker(\eta_{nN}) \cong \ker(\eta_\infty)$ . Then by Theorem 6.2 and Proposition 3.4, we have a commutative diagram,

$$\begin{array}{ccc} \ker(\eta_{nN}) & \rightarrow & \ker(\psi_U/nN) \\ \downarrow \cong & & \downarrow \\ \ker(\eta_\infty) & \cong & \ker(\psi_U \otimes \mathbb{Q}/\mathbb{Z}) = F_0(U), \end{array}$$

in which the top horizontal map is surjective. Hence all the maps in this diagram are bijective. This proves (2). If  $D(C) = 0$ , then we have  $\ker(\eta_\infty) = H_{\text{Zar}}^0(X, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))$ , and Lemma 6.6 proves (3). (1) follows from Lemma 4.5.  $\square$

*Proof of Theorem 6.5.* By Proposition 3.4 and Lemma 4.1, it suffices to show the bijectivity of  $c_{X,n}^{i,3}$  for  $i = 4, 5$  and  $n \in \mathbb{Z}_{>0}$ . Probably this is well-known to the specialists, but the author could not find a suitable reference. For the completeness sake, we include a proof here.

Let  $n \in \mathbb{Z}_{>0} \cup \{\infty\}$ . The injectivity of  $c_{X,n}^{4,3}$  follows from Proposition 3.2. We consider the commutative diagram with exact rows

$$(6.2.1) \quad \begin{array}{ccccccc} 0 \rightarrow H_0^M(X, \mathbb{Z}(-1))/n & \rightarrow & H_0^M(X, \mathbb{Z}/n(-1)) & \rightarrow & C_1(X)[n] & \rightarrow & 0 \\ & & \downarrow c_{X,n}^{4,3} & & \downarrow & & \\ 0 \rightarrow H_{\text{Gal}}^2(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2)) & \rightarrow & H_{\text{et}}^4(X, \mathbb{Z}/n(3)) & \rightarrow & H_{\text{Gal}}^0(k, \mathbb{Z}/n(1)) & \rightarrow & 0. \end{array}$$

Here the upper and lower rows are given by (2.1.2) and by Hochschild-Serre spectral sequence respectively. The right vertical map is given by

$$C_1(X)[n] \rightarrow C_1(\text{Spec } k)[n] = k^*[n] = H_{\text{Gal}}^0(k, \mathbb{Z}/n(1)).$$

The left vertical map is defined by the commutativity of this diagram. The existence of a  $k$ -rational point of  $X$  shows that both rows are split exact. It also shows the surjectivity of the right vertical map. We show the surjectivity of the left vertical map. Since the corestriction map

$$H_{\text{Gal}}^2(k', NS(\bar{X}) \otimes \mathbb{Z}/n(2)) \rightarrow H_{\text{Gal}}^2(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2))$$

is surjective for any finite extension  $k'/k$ , it suffices to show this surjectivity when  $X$  is a *split* rational surface (that is,  $k(X)$  is purely transcendental over  $k$ ). This follows from Lemma 3.5 because  $H_{\text{Zar}}^0(X, \mathcal{H}^4(\mathbb{Z}/n(3))) = 0$  when  $X$  is a split rational surface. (More explicitly, one can directly show the surjectivity of the composite map

$$\text{Pic}(X) \otimes K_2 k \xrightarrow{\text{prod.}} H_M^4(X, \mathbb{Z}(3)) \cong H_0^M(X, \mathbb{Z}(-1)) \rightarrow H_{\text{Gal}}^2(k, NS(\bar{X}) \otimes \mathbb{Z}/n(2)),$$

where the first map is given by the product structure.)

Next, we consider  $c_{X,n}^{5,3}$ . By Hochschild-Serre spectral sequence, we have an isomorphism

$$H_{\text{et}}^5(X, \mathbb{Z}/n(3)) \cong H_{\text{Gal}}^1(k, \mathbb{Z}/n(1)) \cong k^*/n.$$

It suffices to show that the structure morphism induces an isomorphism  $C_1(X) \rightarrow C_1(\text{Spec } k) = k^*$ . By the existence of a  $k$ -rational point of  $X$ , one knows that this map is split surjective. We set  $V(X) := \ker[C_1(X) \rightarrow C_1(\text{Spec } k) = k^*]$ . Since  $V(X') = 0$  if  $X'$  is a split rational surface, the norm argument shows that  $V(X)$  is a torsion group. It suffices to show  $V(X)_{\text{Tor}} = 0$ . This follows from a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \rightarrow V(X)_{\text{Tor}} \rightarrow & C_1(X)_{\text{Tor}} & \xrightarrow{(!)} & \mu(k) \rightarrow 0 \\
& \uparrow \text{surj.} & & \parallel \\
& H_0^M(X, \mathbb{Q}/\mathbb{Z}(-1)) & \xrightarrow{(!)} & H_0^M(\text{Spec } k, \mathbb{Q}/\mathbb{Z}(1)) \\
& \downarrow c_{X,\infty}^{4,3} & & \parallel \\
& H_{\text{ét}}^4(X, \mathbb{Q}/\mathbb{Z}(3)) & \cong & H_{\text{Gal}}^0(k, \mathbb{Q}/\mathbb{Z}(1)).
\end{array}$$

Here the maps (!) are split surjective, and  $c_{X,\infty}^{4,3}$  is injective by Proposition 3.2. The lower horizontal map is bijective because the first term in the bottom row of (6.2.1) vanishes when  $n = \infty$ . The middle upper vertical map is given by (2.1.2).  $\square$

**Example 6.8.** Let  $a, b \in k^*$ . Suppose  $p \neq 2$ ,  $a \neq b$  and  $\text{ord}_k(a) = \text{ord}_k(b) = r$ . We also take  $d \in k^*$  such that  $k(\sqrt{d})/k$  is a non-trivial unramified extension. Let  $X$  be a smooth projective surface with function field  $K := k(x, y)[z]/(x^2 - dy^2 - z(z - a)(z - b))$  (a so-called *Châtelet surface*). Let  $C$  be a divisor on  $X$ , and put  $U = X \setminus C$ . We assume that the irreducible components of  $\bar{C}$  generate  $NS(\bar{X})$ , and that  $D(C) = 0$ . (When  $X$  is minimal with respect to the fibration corresponding to the field extension  $K/k(z)$ , we can take  $C$  to be the union of the four singular fibers and the ‘ $\infty$ -section’, so that each component has genus zero. (Cf. Remark 5.3.) See [7] for details.) We claim the following.

- (1) If  $r$  is even and  $\text{ord}_k(a - b) = r$ , then  $F_0(U) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ .
- (2) If  $r$  is even and  $\text{ord}_k(a - b) > r$ , then  $F_0(U) \cong \mathbb{Z}/2$ .
- (3) If  $r$  is odd, then  $F_0(U) = 0$ .

It is shown in loc. cit. that  $H_{\text{Gal}}^1(k, S) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ . Hence the claim follows from Theorem 6.3 and the following theorem.

**Theorem 6.9** (Colliot-Thélène, [9](4.7)). (1) *If  $r$  is even and  $\text{ord}_k(a - b) = r$ , then*

$$A_0(X) = 0.$$

- (2) *If  $r$  is even and  $\text{ord}_k(a - b) > r$ , then  $A_0(X) \cong \mathbb{Z}/2$ .*
- (3) *If  $r$  is odd, then  $A_0(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ .*

One can extend this example to the case where  $k(\sqrt{d})/k$  is a ramified extension, by using a result of Dalawat [10].

**Example 6.10.** Let  $a \in k^*$  and let  $X$  be a cubic surface defined by  $T_0^3 + T_1^3 + T_2^3 + aT_3^3 = 0$  in  $\mathbb{P}_k^3$ . Let  $\zeta \in \bar{k}$  be a primitive cubic root of unity. Set  $r = \text{ord}_k(a)$ . Let  $C$  be a divisor on  $X$ , and put  $U = X \setminus C$ . We assume that the irreducible components of  $\bar{C}$  generate

$NS(\bar{X})$ , and that  $D(C) = 0$ . (For example, we can take  $C$  to be the union of the 27 lines on  $X$ . Cf. Remark 5.3) We claim the following.

- (1) If  $p \neq 3$  and  $r \equiv 0 \pmod{3}$ , then  $F_0(U) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$ .
- (2) If  $p \neq 3, r \not\equiv 0 \pmod{3}$  and  $\zeta \notin k$ , then  $F_0(U) \cong \mathbb{Z}/3$ .
- (3) Suppose  $\zeta \in k$ . If  $p \neq 3$ , assume  $r \not\equiv 0 \pmod{3}$ . If  $p = 3$ , assume  $r \equiv 1 \pmod{3}$ . Then  $F_0(X) = 0$ .

Manin [25] has observed that  $H_{\text{Gal}}^1(k, S) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$ . Hence the claim follows from Theorem 6.3 and the following theorem.

**Theorem 6.11** (Saito/Sato [32] (5.1.1)). (1) *If  $p \neq 3$  and  $r \equiv 0 \pmod{3}$ , then  $A_0(X) = 0$ .*  
 (2) *If  $p \neq 3, r \not\equiv 0 \pmod{3}$  and  $\zeta \notin k$ , then  $A_0(X) \cong \mathbb{Z}/3$ .*  
 (3) *Suppose  $\zeta \in k$ . If  $p \neq 3$  assume  $r \not\equiv 0 \pmod{3}$ . If  $p = 3$ , assume  $r \equiv 1 \pmod{3}$ . Then  $A_0(X) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$ .*

#### REFERENCES

- [1] S. Bloch, On the Chow groups of certain rational surfaces. Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 1, 41–59.
- [2] S. Bloch, Algebraic  $K$ -theory and classfield theory for arithmetic surfaces. Ann. of Math. (2) 114 (1981), no. 2, 229–265.
- [3] J-L. Colliot-Thélène, Hilbert’s Theorem 90 for  $K_2$ , with application to the Chow groups of rational surfaces. Invent. Math. 71 (1983), no. 1, 1–20.
- [4] J-L. Colliot-Thélène, Cycles algébriques de torsion et  $K$ -théorie algébrique. Arithmetic algebraic geometry (Trento, 1991), 1–49, Lecture Notes in Math., 1553, Springer, Berlin, 1993.
- [5] J-L. Colliot-Thélène, L’arithmétique du groupe de Chow des zéro-cycles. J. Théor. Nombres Bordeaux 7 (1995), no. 1, 51–73.
- [6] J-L. Colliot-Thélène, S. Saito, Zéro-cycles sur les variétés  $p$ -adiques et groupe de Brauer. Internat. Math. Res. Notices 1996, no. 4, 151–160.
- [7] J-L. Colliot-Thélène, J-J. Sansuc, La descente sur les variétés rationnelles. Journées de Géométrie Algébrique d’Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, pp. 223–237, Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.
- [8] J-L. Colliot-Thélène, J-J. Sansuc, On the Chow groups of certain rational surfaces: a sequel to a paper of S. Bloch. Duke Math. J. 48 (1981), no. 2, 421–447.
- [9] D. Coray, M. Tsfasman, Arithmetic on singular Del Pezzo surfaces. Proc. London Math. Soc. (3) 57 (1988), no. 1, 25–87.
- [10] C. Dalawat, Le groupe de Chow d’une surface de Châtelet sur un corps local. Indag. Math. (N.S.) 11 (2000), no. 2, 173–185.
- [11] E. Friedlander, V. Voevodsky, Bivariant cycle cohomology. Cycles, transfers, and motivic homology theories, 138–187, Ann. of Math. Stud., 143, Princeton Univ. Press, Princeton, NJ, 2000.
- [12] T. Geisser, Arithmetic homology and an integral version of Kato’s conjecture J. Reine Angew. Math. 644 (2010), 1–22.
- [13] T. Geisser, On Suslin’s singular homology and cohomology. Doc. Math. 2010, Extra volume: Andrei A. Suslin sixtieth birthday, 223–249.

- [14] T. Geisser, M. Levine, The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. *J. Reine Angew. Math.* 530 (2001), 55–103.
- [15] P. Gille, T. Szamuely, Central simple algebras and Galois cohomology. Cambridge Studies in Advanced Mathematics, 101. Cambridge University Press, Cambridge, 2006.
- [16] T. Hiranouchi, Class field theory for open curves over  $p$ -adic fields *Math. Z.* 266 (2010), no. 1, 107–113.
- [17] U. Jannsen, S. Saito, Kato homology of arithmetic schemes and higher class field theory over local fields. Kazuya Kato’s fiftieth birthday. *Doc. Math.* 2003, Extra Vol., 479–538.
- [18] U. Jannsen, S. Saito, Bertini theorems and Lefschetz pencils over discrete valuation rings, with applications to higher class field theory, to appear in *J. of Algebraic Geometry*.
- [19] U. Jannsen, Weights in arithmetic geometry. *Jpn. J. Math.* 5 (2010), no. 1, 73–102.
- [20] B. Kahn, Relatively unramified elements in cycle modules. *J. K-theory* 7 (2011), 409–427.
- [21] K. Kato, A generalization of local class field theory by using  $K$ -groups I. *J. Fac. Sci. Univ. Tokyo*, no. 26, (1979), 303–376.
- [22] K. Kato, S. Saito, Unramified class field theory of arithmetical surfaces. *Ann. of Math. (2)* 118 (1983), no. 2, 241–275.
- [23] S. Lichtenbaum, Duality theorems for curves over  $p$ -adic fields. *Invent. Math.* 7 1969 120–136.
- [24] Y. Manin, Le groupe de Brauer-Grothendieck en géométrie diophantienne. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 401–411.
- [25] Y. Manin, Cubic forms: algebra, geometry, arithmetic. North-Holland Mathematical Library, Vol. 4. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, 1974.
- [26] A. Mattuck, Abelian varieties over  $p$ -adic ground fields. *Ann. Math.* 62(2), 92–119 (1955).
- [27] A. Merkurjev, On the torsion in  $K_2$  of local fields. *Ann. of Math. (2)* 118 (1983), no. 2, 375–381.
- [28] C. Mazza, V. Voevodsky, C. Weibel, Lecture notes on motivic cohomology. Clay Mathematics Monographs, 2. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.
- [29] R. Parimala, V. Suresh, Zero-cycles on quadric fibrations: finiteness theorems and the cycle map, *Invent. Math.* 122 (1995), no. 1, 83–117.
- [30] S. Saito, Class field theory for curves over local fields. *J. Number Theory* 21 (1985), no. 1, 44–80.
- [31] S. Saito, On the cycle map for torsion algebraic cycles of codimension two. *Invent. Math.* 106 (1991), no. 3, 443–460.
- [32] S. Saito, K. Sato, Zero-cycles on varieties over  $p$ -adic fields and Brauer groups. arXiv:0906.2273.
- [33] K. Sato, Non-divisible cycles on surfaces over local fields. *J. Number Theory* 114 (2005), no. 2, 272–297.
- [34] C. Scheiderer, J. van Hamel, Cohomology of tori over  $p$ -adic curves. *Math. Ann.* 326 (2003), no. 1, 155–183.
- [35] A. Schmidt, Singular homology of arithmetic schemes. *Algebra Number Theory* 1 (2007), no. 2, 183–222.
- [36] A. Schmidt, M. Spieß, Singular homology and class field theory of varieties over finite fields. *J. Reine Angew. Math.* 527 (2000), 13–36.
- [37] J.-P. Serre, Groupes algébriques et corps de classes. Hermann, 1959.
- [38] A. Suslin, S. Joukhovitski, Norm varieties. *J. Pure Appl. Algebra* 206 (2006), no. 1-2, 245–276.
- [39] A. Suslin, V. Voevodsky, Singular homology of abstract algebraic varieties. *Invent. Math.* 123 (1996), no. 1, 61–94.
- [40] A. Suslin, V. Voevodsky, Bloch-Kato conjecture and motivic cohomology with finite coefficients. The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

- [41] T. Szamuely, Sur l'application de réciprocité pour une surface fibrée en coniques définie sur un corps local. *K-Theory* 18 (1999), no. 2, 173–179.
- [42] T. Szamuely, Sur la théorie des corps de classes pour les variétés sur les corps  $p$ -adiques. *J. Reine Angew. Math.* 525 (2000), 183–212.
- [43] V. Voevodsky, Triangulated categories of motives over a field. In: *Cycles, transfers, and motivic homology theories*, 188–238, *Ann. of Math. Stud.*, 143, Princeton Univ. Press, Princeton, NJ, 2000.
- [44] V. Voevodsky, On motivic cohomology with  $\mathbf{Z}/l$ -coefficients. *Ann. of Math. (2)* 174 (2011), no. 1, 401–438.
- [45] C. Weibel, The norm residue isomorphism theorem. *J. Topology* (2009) 2 (2): 346–372.
- [46] G. Wiesend, Class field theory for arithmetic schemes. *Math. Z.* 256 (2007), no. 4, 717–729. (With an appendix by A. Schmidt.)
- [47] T. Yamazaki, On Chow and Brauer groups of a product of Mumford curves. *Math. Ann.* 333 (2005), no. 3, 549–567.
- [48] T. Yamazaki, Class field theory for a product of curves over a local field. *Math. Z.* 261 (2009), no. 1, 109–121.

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