The Zelevinsky–Aubert duality for classical groups

Hiraku Atobe

Hokkaido University

(Joint work with Alberto Mínguez)

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Setting

Let

- F be a p-adic field;
- $G \in \{\mathrm{GL}_n(F), \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)\}$ be split;
- $P = MN$ be a standard parabolic subgroup of G;
- $\mathrm{Ind}_P^G(\pi_M)$ be the (normalized) parabolically induction of $\pi_M \in \mathrm{Rep}(G)$
- $\mathrm{Jac}_P(\pi)$ be the Jacquet module of $\pi \in \mathrm{Rep}(G)$;
- $\pi \mapsto [\pi]$ be the semisimplification.

Zelevinsky–Aubert duality

For $\pi \in \mathrm{Rep}(G)$, define

$$D_G(\pi) \coloneqq \sum_{P=MN} (-1)^{\dim A_M} [\mathrm{Ind}_P^G(\mathrm{Jac}_P(\pi))],$$

where P runs over all standard parabolic subgroups of G.

Theorem (Aubert 1996)

For any irreducible π, there exists $\epsilon \in \{\pm 1\}$ such that

$$\hat{\pi} \coloneqq \epsilon \cdot D_G(\pi)$$

is also irreducible. Moreover, $\hat{\hat{\pi}} \cong \pi$.

Call $\hat{\pi}$ the *Zelevinsky–Aubert dual* of π.

History (partial)

- Alvis ('79, '82) and Curtis ('80) defined/studied D_G for reductive group G over a finite field $\mathbb{F}_q$.
- Deligne–Lusztig ('83) showed that $D_G(R_T^G(\theta)) = \pm R_T^G(\theta)$, where $R_T^G(\theta)$ is the Deligne–Lusztig induction of $\theta \in \mathrm{Irr}(T(\mathbb{F}_q))$. As an application, one can determine $\dim R_T^G(\theta)$.
- Zelevinsky ('80) defined a similar involution for $G = \mathrm{GL}_n(F)$ generated by Steinberg $\leftrightarrow$ trivial. He predicted that it preserves the irreducibility.
- After a work of S.-I. Kato ('93), Aubert ('96) defined D_G for general reductive group G over F, and showed that it preserves the irreducibility.

Motivation and Main Result

Conjecture

If π is irreducible and unitary, then so is $\hat{\pi}$.

In particular, this duality would produce many unitary representations.

- Mœglin–Waldspurger ('86) gave an algorithm to compute $\hat{\pi}$ for $G = \mathrm{GL}_n(F)$. Together with Tadić's result on the unitary dual, it shows the conjecture in this case.

Main Result (A.-Mínguez)

We give an explicit algorithm for $\hat{\pi}$ for $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$.

Known results for $G = \mathrm{Sp}_{2n}(F), \mathrm{SO}_{2n+1}(F)$

- It is trivial that if π is supercuspidal, then $\hat{\pi} = \pi$ (Aubert ('96)).
- Hanzer ('09) showed that if π is a strongly positive discrete series, then $\hat{\pi}$ is unitary.
- Arthur ('13) established an endoscopic classification. As a consequence, he proved that if π is tempered, then $\hat{\pi}$ is unitary.
- Matić ('17, '19) gave an explicit formula for $\hat{\pi}$ when π is discrete series which is strongly positive or is in the first induction step.
- C. Jantzen ('18) gave an algorithm for $\hat{\pi}$ of good parity when π is in the "half-integral case".

Notation

We explain Jantzen's algorithm for $\hat{\tau}$ for $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$.

- Let ρ be a supercuspidal representation of $\mathrm{GL}_d(F)$.
- A *segment* is a set

$$[x, y]_\rho = \{\rho|\cdot|^x, \rho|\cdot|^{x-1}, \dots, \rho|\cdot|^y\},$$

  where $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Z}_{\geq 0}$.
- Denote by $\Delta_\rho[x, y]$ (resp. $Z_\rho[y, x]$) the unique irreducible subrepresentation (resp. quotient) of induced representation

$$\rho|\cdot|^x \times \cdots \times \rho|\cdot|^y := \mathrm{Ind}_P^{\mathrm{GL}_{d(x-y+1)}(F)}(\rho|\cdot|^x \boxtimes \cdots \boxtimes \rho|\cdot|^y).$$

Jacquet modules of $\Delta_\rho[x,y]$ and $Z_\rho[y,x]$

- $\Delta_\rho[x,y]$ is an essentially discrete series representation. ($\Delta_{\mathbf{1}_{\mathrm{GL}_1(F)}}[\frac{n-1}{2}, -\frac{n-1}{2}]$ is called a *Steinberg* representation.)
- If $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, then $Z_\rho[y,x]$ is a character of $\mathrm{GL}_{x-y+1}(F)$.

Proposition (Zelevinsky 1980)

When P_d is a standard parabolic with Levi $\mathrm{GL}_d(F) \times \mathrm{GL}_{d(x-y)}(F)$,

$$\mathrm{Jac}_{P_d}(\Delta_\rho[x,y]) = \rho|\cdot|^x \boxtimes \Delta_\rho[x-1,y],$$
$$\mathrm{Jac}_{P_d}(Z_\rho[y,x]) = \rho|\cdot|^y \boxtimes Z_\rho[y+1,x].$$

Here, we set $\Delta_\rho[y-1,y] \coloneqq \mathbf{1}_{\mathrm{GL}_0(F)}$ and $Z_\rho[x+1,x] \coloneqq \mathbf{1}_{\mathrm{GL}_0(F)}$.

Derivatives

We introduce the following notion.

Definition (essentially due to Jantzen and Mínguez)

The k-th *left ρ-derivative* of $\tau \in \mathrm{Rep}(\mathrm{GL}_n(F))$ is a semisimple representation $L_\rho^{(k)}(\tau)$ satisfying

$$[\mathrm{Jac}_{P_{dk}}(\tau)] = \rho^k \boxtimes L_\rho^{(k)}(\tau) + \sum_i \tau_i' \boxtimes \tau_i'',$$

where $\tau_i' \boxtimes \tau_i'' \in \mathrm{Irr}(\mathrm{GL}_{dk}(F) \times \mathrm{GL}_{n-dk}(F))$ such that $\tau_i' \not\cong \rho^k$.

When $L_\rho^{(k)}(\tau) \neq 0$ but $L_\rho^{(k+1)}(\tau) = 0$, call $L_\rho^{(k)}(\tau)$ the *highest ρ-derivative*. Similarly, one can define the *right ρ-derivative* $R_\rho^{(k)}(\tau)$.

Properties of derivatives

Let $\mathrm{soc}(\Pi)$ denote the *socle* of Π, i.e., the maximal semisimple subrepresentation of Π.

Theorem (Jantzen 2007, Mínguez 2009)

Let τ be an irreducible representation of $\mathrm{GL}_n(F)$.

- The highest ρ-derivative $L_\rho^{(k)}(\tau)$ is irreducible.
- τ can be recovered from $L_\rho^{(k)}(\tau)$ by

$$\tau = \mathrm{soc}\left(\rho^k \times L_\rho^{(k)}(\tau)\right).$$

- There is an explicit formula for the highest ρ-derivative $L_\rho^{(k)}(\tau)$.
- There is an explicit formula for $\mathrm{soc}(\rho^r \times \tau)$, which is irreducible.

Algorithm for computing $\hat{\tau}$

The analogous results for $R_\rho^{(k)}$ are also known.
The derivatives are related with the Zelevinsky dual as follows.

Proposition

For $\tau \in \mathrm{Irr}(\mathrm{GL}_n(F))$,

$$L_\rho^{(k)}(\hat{\tau}) = R_\rho^{(k)}(\tau)\hat{}\,.$$

Using this proposition together with explicit formulas for derivatives and socles, one can compute $\hat{\tau}$ by induction on n.

Example

We have $Z_\rho[y,x]\hat{} = \Delta_\rho[x,y]$ since

$$L_{\rho|\cdot|^x}^{(1)}(\Delta_\rho[x,y]) = \Delta_\rho[x-1,y], \quad R_{\rho|\cdot|^x}^{(1)}(Z_\rho[y,x]) = Z_\rho[y,x-1].$$

Remark

- Mœglin–Waldspurger (1986) gave an algorithm to compute the Zelevinsky dual $\hat{\tau}$ by a different form.
- The idea to use derivatives for an algorithm to compute $\hat{\tau}$ is due to Jantzen (2007).
- There are three explicit formulas for the highest derivatives and socles given by Janzten (2007), Mínguez (2009), and Lapid–Mínguez (2016).
- We will use the explicit formula of Lapid–Mínguez, which uses the notion of *best matching functions*.

Derivatives

From now, let $G = \mathrm{Sp}_{2n}(F)$ or $G = \mathrm{SO}_{2n+1}(F)$.
Fix an irreducible supercuspidal representation ρ of $\mathrm{GL}_d(F)$.

Definition

The k-th *ρ-derivative* of π is a semisimple representation $D_\rho^{(k)}(\pi)$ satisfying

$$[\mathrm{Jac}_{P_{dk}}(\pi)] = \rho^k \boxtimes D_\rho^{(k)}(\pi) + \sum_i \tau_i \boxtimes \pi_i,$$

where P_{dk} is a parabolic of G with Levi $\mathrm{GL}_{dk}(F) \times G_0$, and $\tau_i \boxtimes \pi_i \in \mathrm{Irr}(\mathrm{GL}_{dk}(F) \times G_0)$ such that $\tau_i \not\cong \rho^k$.

If $D_\rho^{(k)}(\pi) \neq 0$ but $D_\rho^{(k+1)}(\pi) = 0$, call $D_\rho^{(k)}(\pi)$ the *highest ρ-derivative*.

Compatibility with the Aubert duality

As in the GL_n-case, derivatives are compatible with the Aubert duality.

Proposition

We have

$$D_{\rho}^{(k)}(\hat{\pi}) = D_{\rho^\vee}^{(k)}(\pi)\widehat{\ }.$$

Here, $\rho^\vee$ denotes the contragredient of ρ.

To obtain an algorithm for $\hat{\pi}$, we want to solve the following questions:

- Is the highest derivative $D_{\rho}^{(k)}(\pi)$ irreducible?
- Can we establish an explicit formula for $D_{\rho}^{(k)}(\pi)$?
- Can π be recovered from $D_{\rho}^{(k)}(\pi)$?

Properties of derivatives

Proposition (Jantzen 2018, Lapid–Tadić 2020, A.–Mínguez)

Suppose that ρ is **not self-dual**. Let π be irreducible representation of G.

- The highest derivative $D_\rho^{(k)}(\pi)$ is irreducible.
- The socle $\mathrm{soc}(\rho^r \rtimes \pi)$ is irreducible.
- They are related as

$$\pi = \mathrm{soc}\left(\rho^k \rtimes D_\rho^{(k)}(\pi)\right).$$

Hence, we want explicit formulas for $D_\rho^{(k)}(\pi)$ and for $\mathrm{soc}(\rho^r \rtimes \pi)$.

From now, fix an irreducible **self-dual** supercuspidal representation ρ of $\mathrm{GL}_d(F)$, and consider $\rho|\cdot|^x$-derivative $D_{\rho|\cdot|^x}^{(k)}(\pi)$ for $x \in \mathbb{R}$.

Langlands classification

As a parametrization of $\mathrm{Irr}(G)$, we will use the *Langlands classification*.

Theorem (Langlands, Arthur)

Any $\pi \in \mathrm{Irr}(G)$ can be written as

$$\begin{aligned}\pi &= L(\Delta_{\rho_1}[x_1, y_1], \dots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\mathrm{temp}}) \\ &\coloneqq \mathrm{soc}\left(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi(\phi, \eta)\right),\end{aligned}$$

where

- ρ_i is **unitary** supercuspidal, and $x_1 + y_1 \leq \cdots \leq x_r + y_r < 0$;
- $\pi(\phi, \eta)$ is an irreducible tempered representation.
- (ϕ, η) is the tempered L-parameter for $\pi(\phi, \eta)$.

Jantzen's algorithm

Jantzen (2018) suggested an algorithm to compute the highest $\rho|\cdot|^x$-derivative of π for $x \neq 0$.
The first step is to rewrite

$$\pi = \mathrm{soc}\left(\tau \rtimes \pi_A\right),$$

where

- $\tau = \mathrm{soc}(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r])$ with $[x_i, y_i]_{\rho_i} \neq [x-1, -x]_\rho$ for any i;
- $\pi_A = L(\Delta_\rho[x-1, -x]^t; \pi(\phi, \eta))$ with $t \geq 0$.

Explicit formula for the highest derivatives

- $\pi = \mathrm{soc}(\tau \rtimes \pi_A)$ with $\pi_A = L(\Delta_\rho[x-1,-x]^t; \pi(\phi,\eta))$.
- Use the best matching functions for $R^{(a)}_{\rho|\cdot|^{-x}}(\tau)$ and $L^{(k)}_{\rho|\cdot|^x}(\tau)$.
- Use an A-*parameter* for $D^{(k_A)}_{\rho|\cdot|^x}(\pi_A)$.

By combining them with Jantzen's algorithm, we have:

Theorem (A.–Mínguez)

Assume that $\rho|\cdot|^x$ is **not self-dual**. Let $\pi \in \mathrm{Irr}(G)$. Then we have explicit formulas for the highest $\rho|\cdot|^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ and for the socle $\mathrm{soc}((\rho|\cdot|^x)^r \rtimes \pi)$ in terms of the *best matching functions* and A-*parameters*.

The first step of the algorithm

We can now state the first step of our algorithm.

Algorithm: Step 1

Let π be an irreducible representation of G. If there exist ρ and $x \in \mathbb{R}$ such that $\rho|\cdot|^x$ is **not self-dual** and $\pi_0 := D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ for $k > 0$, then use

$$\hat{\pi} = \mathrm{soc}\left((\rho|\cdot|^{-x})^k \rtimes \hat{\pi}_0\right)$$

to reduce the computation of $\hat{\pi}$ to the one of $\hat{\pi}_0$.

Problem

Step 1 reduces the problem to the case where π is irreducible such that

$$D_\rho^{(1)}(\pi) \neq 0 \implies \rho \text{ is self-dual.} \qquad (*)$$

From now, assume that ρ is **self-dual**. In this case, the ρ-derivatives $D_\rho^{(k)}(\pi)$ is difficult, and π cannot be recovered from $D_\rho^{(k)}(\pi)$ in general.

Example

If σ is supercuspidal such that $\rho \rtimes \sigma = \pi_+ \oplus \pi_-$. Then

$$D_\rho^{(1)}(\pi_+) = D_\rho^{(1)}(\pi_-) = \sigma.$$

But it is easy to see that $\hat{\pi}_+ = \pi_-$ by definition (Aubert).

Tempered case

As in this example, we can compute $\hat{\pi}$ if π is tempered.

Proposition (A.–Mínguez)

Suppose that $\pi = \pi(\phi, \eta)$ is tempered and satisfies $(*)$. Set $\{\rho \subset \phi \mid \text{even mult.}\} = \{\rho_1, \dots, \rho_r\}$ and $y_i = \max\{\frac{d_i - 1}{2} \mid \rho_i \boxtimes S_{d_i} \subset \phi\}$. Assume that $y_1 \geq \cdots \geq y_t > 0 = y_{t+1} = \cdots = y_r$. Then

$$\hat{\pi} = L(\Delta_{\rho_1}[0, -y_1], \dots, \Delta_{\rho_t}[0, -y_t]; \pi(\phi', \eta')),$$

where

$$\phi' = \phi - \bigoplus_{i=1}^{t} \rho_i \boxtimes (S_1 \oplus S_{2y_i+1})$$

and $\eta'(\rho \boxtimes S_d) \neq \eta(\rho \boxtimes S_d) \iff \rho \in \{\rho_1, \dots, \rho_r\}$.

New derivatives

The remaining case is that π is non-tempered and satisfies $(*)$.
The key idea to deal with this case is *to define new derivatives*.

Definition (A.-Mínguez)

Define the k-th $\Delta_\rho[0,-1]$-*derivative* $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ and the k-th $Z_\rho[0,1]$-*derivative* $D^{(k)}_{Z_\rho[0,1]}(\pi)$ as semisimple representations satisfying

$$[\mathrm{Jac}_{P_{2dk}}(\pi)] = \Delta_\rho[0,-1]^k \boxtimes D^{(k)}_{\Delta_\rho[0,-1]}(\pi) + Z_\rho[0,1]^k \boxtimes D^{(k)}_{Z_\rho[0,1]}(\pi) + (\text{others}).$$

One can define the notions of the *highest* $\Delta_\rho[0,-1]$-*derivatives* and the *highest* $Z_\rho[0,1]$-*derivatives*.

Properties

These derivatives are substitute for the ρ-derivatives.

Proposition (A.–Mínguez)

Suppose that π is irreducible and satisfies $(*)$.

1. The highest $\Delta_\rho[0,-1]$-derivative $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$ (resp. the highest $Z_\rho[0,1]$-derivative $D^{(k)}_{Z_\rho[0,1]}(\pi)$) is irreducible.
2. The socles $\mathrm{soc}(\Delta_\rho[0,-1]^r \rtimes \pi)$ (resp. $\mathrm{soc}(Z_\rho[0,1]^r \rtimes \pi)$) is irreducible.
3. $\pi \cong \mathrm{soc}(Z_\rho[0,1]^k \rtimes D^{(k)}_{Z_\rho[0,1]}(\pi))$.
4. $D^{(k)}_{Z_\rho[0,1]}(\hat{\pi}) = D^{(k)}_{\Delta_\rho[0,-1]}(\pi)\hat{}$.

- Actually, we only assume a weaker condition than $(*)$.

The explicit formulas

Moreover:

Theorem (A.–Mínguez)

Suppose that π satisfies $(*)$. Then we have explicit formulas for

- the highest $\Delta_\rho[0,-1]$-derivatives $D^{(k)}_{\Delta_\rho[0,-1]}(\pi)$;
- the highest $Z_\rho[0,1]$-derivatives $D^{(k)}_{Z_\rho[0,1]}(\pi)$;
- the socle $\mathrm{soc}(Z_\rho[0,1]^r \rtimes \pi)$

in terms of matching functions and A-parameters.

- Actually, we only assume a weaker condition than $(*)$.

Completion of the algorithm

Algorithm: Step 2 and Step 3

Let π be an irreducible representation satisfying that

$$D_\rho^{(1)}(\pi) \neq 0 \implies \rho \text{ is self-dual.} \qquad (*)$$

- If π is non-tempered, can find ρ such that $\pi_0 := D^{(k)}_{\Delta_\rho[0,-1]}(\pi) \neq 0$ for $k > 0$, and use
$$\hat{\pi} = \mathrm{soc}\left(Z_\rho[0,1]^k \rtimes \hat{\pi}_0\right)$$
to reduce the computation of $\hat{\pi}$ to the one of $\hat{\pi}_0$.
- If π is tempered, we have an explicit formula for $\hat{\pi}$.

Example

Now we set $\rho = \mathbf{1}_{\mathrm{GL}_1(F)}$, and drop ρ in the notation.
Let $\pi(x_1^{\eta_1}, \dots, x_r^{\eta_r}) = \pi(\phi, \eta)$ with $\phi = \oplus_{i=1}^r S_{2x_i+1}$ and $\eta(S_{2x_i+1}) = \eta_i$.

Example

Let us consider $L(\Delta[0,-2]; \pi(0^-, 0^-, 1^+)) \in \mathrm{Sp}_{10}(F)$, which satisfies $(\ast)$. Hence

$$\begin{array}{c} L(\Delta[0,-2]; \pi(0^-, 0^-, 1^+)) \\ \Big\downarrow {\scriptstyle D^{(1)}_{\Delta[0,-1]}} \\ L(|\cdot|^{-2}; \pi(0^-, 0^-, 1^+)) \\ \Big\downarrow {\scriptstyle D^{(1)}_{|\cdot|^{-2}}} \\ \pi(0^-, 0^-, 1^+). \end{array}$$

By the explicit formula, $\hat{\pi}(0^-, 0^-, 1^+) = L(\Delta[0,-1]; \pi(0^+))$.

Example

Example (continued)

Now take socles

$$\begin{array}{c} L(\Delta[0,-1];\pi(0^+)) \\ \Big\downarrow{\scriptstyle \mathrm{soc}(|\cdot|^2 \rtimes -)} \\ L(\Delta[0,-2];\pi(0^+)) \\ \Big\downarrow{\scriptstyle \mathrm{soc}(Z[0,1] \rtimes -)} \\ L(\Delta[0,-2];\pi(0^-,0^-,1^+)). \end{array}$$

We conclude that $L(\Delta[0,-2];\pi(0^-,0^-,1^+))$ is fixed by the Zelevinsky–Aubert duality.

Remarks

Remark

- Even if we start at a tempered representation, we need to compute $\Delta_\rho[0,-1]$-derivatives and the $Z_\rho[0,1]$-socles in general.
- As an application of $Z_\rho[0,1]$-derivatives, one can refine Mœglin's explicit construction of local A-packets (A.).
- In that work, a conjectural "formula" for $\hat{\pi}$ for π of *Arthur type* was formulated.
- Although the explicit formulas for derivatives and for socles are complicated, it would be easy to write a computer program.