Quasi-pullback of certain Siegel modular forms and Borcherds products

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Mirror symmetry at genus one : The BCOV conjecture

X : Calabi-Yau threefold
$F_1^{\mathrm{top}}(q)$: generating function of the genus one instanton numbers of X

$$F_1^{\mathrm{top}}(q) = q^{c_2^\vee/24} \prod_{\lambda \in H_2(X,\mathbb{Z})\setminus\{0\}} \left(1 - q^{\lambda}\right)^{n_0(\lambda)/12} \prod_{k\geq 1} \left(1 - q^{k\lambda}\right)^{n_1(\lambda)}$$

where
$q^\lambda = e^{2\pi i\langle\lambda,t\rangle}$, $t \in H^2(X,\mathbb{C})$
$n_g(\lambda)$: the genus-g instanton number of X
$c_2^\vee$: the Poincaré dual of $c_2(X)$

The BCOV conjecture

Physicists Bershadsky-Cecotti-Ooguri-Vafa introduced an invariant τ_{BCOV} of Calabi-Yau threefolds, which is called the *BCOV invariant*.
(We recall its definition later.)

The BCOV conjecture

Let $\mathcal{X}^\vee \to (\Delta^*)^n$, $n = h^{1,1}(X) = h^{1,2}(X^\vee)$ be a mirror family of X. In the canonical coordinates on $(\Delta^*)^n$, the following equality holds:

$$\tau_{\mathrm{BCOV}}(X_s^\vee) = C \left\| F_1^{\mathrm{top}}(q)^2 \left(\frac{\Xi_s}{\int_{A_0} \Xi_s} \right)^{3+n-\frac{\chi}{12}} \otimes \left(q_1 \frac{\partial}{\partial q_1} \wedge \cdots \wedge q_n \frac{\partial}{\partial q_n} \right) \right\|^2$$

Here $n = h^{1,1}(X) = h^{1,2}(X_s^\vee)$, $\chi = \chi(X)$, $q = (q_1, \ldots, q_n)$ is the system of canonical coordinates on $(\Delta^*)^n$, $\Xi_s \in H^0(X_s^\vee, K_{X_s^\vee})$, $A_0 \in H_3(X_s^\vee, \mathbb{Z})$ is the cycle invariant under the monodromy, and C is a constant.

Motivation

There is a class of Calabi-Yau threefolds (Borcea-Voisin threefolds), whose BCOV invariant is given by the product of a Borcherds product, a Siegel modular form (pulled back to a domain of type IV via the Torelli map), and the Dedekind η-function.

From the BCOV conjecture for the Borcea-Voisin threefolds, we have

Conjectural Observation

The pullback of a certain Siegel modular form should be expressed as an infinite product of Borcherds type near the cusp of the domains of type IV.

In this talk, we report a result supporting this conjectural observation.

BCOV torsion

(X, g_X) : compact Kähler manifold
$\square_{p,q} = (\bar{\partial} + \bar{\partial}^*)^2$: Laplacian of (X, g_X) acting on the (p, q)-forms on X
$\zeta_{p,q}(s)$: spectral zeta function of $\square_{p,q}$

$$\zeta_{p,q}(s) := \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \dim E(\lambda; \square_{p,q})$$

Fact

$\zeta_{p,q}(s)$ converges when $\Re s > \dim X$, admits a meromorphic continuation to $\mathbb{C}$, and is holomorphic at $s = 0$.

Definition (BCOV torsion)

The BCOV torsion of (X, g_X) is defined by

$$\mathcal{T}_{\mathrm{BCOV}}(X, g_X) = \exp[-\sum_{p,q \geq 0} (-1)^{p+q} pq\, \zeta'_{p,q}(0)].$$

Calabi-Yau manifolds

Definition (Calabi-Yau manifold)

X : smooth irreducible compact Kähler n-fold is Calabi-Yau $\iff$

- $K_X = \Omega_X^n \cong \mathcal{O}_X$
- $H^q(X, \mathcal{O}_X) = 0 \quad (0 < q < n)$.

When $n = 1, 2$, Calabi-Yau manifolds are elliptic curves and $K3$ surfaces. In low dimensions, the moduli space of *polarized* Calabi-Yau manifolds is a locally symmetric variety.

Fact

- *If $n \geq 3$, there are many topological types of Calabi–Yau n-folds. (It is not known that for a fixed $n \geq 3$, the number of deformation types of Calabi-Yau n-folds is finite or not.)*
- *The global structure of the moduli space of polarized Calabi-Yau manifolds of dimension $n \geq 3$ is not understood in general.*

BCOV invariant for Calabi-Yau threefolds

Definition (Bershadsky-Cecotti-Ooguri-Vafa, Fang-Lu-Y.)

The BCOV invariant of a Calabi–Yau threefold X is defined by

$$\tau_{\mathrm{BCOV}}(X) = \mathrm{Vol}(X,\gamma)^{-3+\frac{\chi(X)}{12}} \mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma])^{-1} \mathcal{T}_{\mathrm{BCOV}}(X,\gamma)$$
$$\times \exp\left[-\frac{1}{12}\int_X \log\left(\frac{\sqrt{-1}\eta\wedge\overline{\eta}}{\gamma^3/3!}\cdot\frac{\mathrm{Vol}(X,\gamma)}{\|\eta\|^2_{L^2}}\right) c_3(X,\gamma)\right]$$

where γ is a Kähler form, $\eta \in H^0(X,\Omega^3_X)\setminus\{0\}$ is a canonical form, $\chi(X)$ is the topological Euler number, $c_3(X,\gamma)$ is the Euler form,

$$\mathrm{Vol}_{L^2}(H^2(X,\mathbb{Z}),[\gamma]) = \mathrm{Vol}(H^2(X,\mathbb{R})/H^2(X,\mathbb{Z}),\langle\cdot,\cdot\rangle_{L^2,\gamma})$$

is the covolume of the lattice $H^2(X,\mathbb{Z})$ w.r.t. the metric induced from γ.

Theorem (Fang-Lu-Y.)

$\tau_{\mathrm{BCOV}}(X)$ *is an invariant of* X*, i.e., it is independent of the choice of* γ, η. *In particular,* τ_{BCOV} *gives rise to a function on the moduli space of Calabi-Yau threefolds.*

Remark

Very recently, the BCOV invariant is extended to Calabi-Yau manifolds of higher dimensions by Eriksson-Freixas i Montplet-Mourougane and to certain pairs by Y. Zhang.

We explain the structure of the BCOV invariant for the Borcea-Voisin threefolds.

Definition

A pair (S, θ) consisting of a $K3$ surface and an involution $\theta\colon S \to S$ is called a 2-elementary $K3$ surface if $\theta^* = -1$ on $H^0(S, \Omega_S^2)$.

Calabi-Yau threefolds of Borcea-Voisin

Definition (Borcea-Voisin threefolds)

For (S, θ) : 2-elementary $K3$ surface T : elliptic curve,

$$X_{(S,\theta,T)} := \text{ crepant resolution of } \frac{S \times T}{\theta \times (-1_T)}$$

is a Calabi-Yau threefold called a Borcea-Voisin threefold. The type of $X_{(S,\theta,T)}$ is defined as the isometry class of the anti-invariant lattice of θ

$$H^2(S, \mathbb{Z})_- = \{l \in H^2(S, \mathbb{Z});\ \theta^*(l) = -l\}$$

Fact (Borcea, Voisin, Nikulin)

- *The deformation type of* $X_{(S,\theta,T)}$ *is determined by its type* $[H^2(S, \mathbb{Z})_-]$.
- $H^2(S, \mathbb{Z})_-$ *is a primitive* 2*-elementary sublattices of the* $K3$*-lattice* $\mathbb{L}_{K3} := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ *with signature* $(2, b^-)$, $0 \leq b^- \leq 19$.

Table: Primitive 2-elementary sublattices of $\mathbb{L}_{K3}$ with $b^+ = 2$ (Nikulin)

g	$\delta = 1$		$\delta = 0$
0	$(\mathbb{A}_1^+)^{\oplus 2} \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 9)$	$\mathbb{U}(2)^{\oplus 2}$
1	$\mathbb{U} \oplus \mathbb{A}_1^+ \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 9)$	$\mathbb{U} \oplus \mathbb{U}(2)$, $\mathbb{U}(2)^{\oplus 2} \oplus \mathbb{D}_4$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(2)$
2	$\mathbb{U}^{\oplus 2} \oplus \mathbb{A}_1^{\oplus t}$	$(1 \leq t \leq 9)$	$\mathbb{U}^{\oplus 2}$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4$, $\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8(2)$
3	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \leq t \leq 6)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4$, $\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4^{\oplus 2}$
4	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_6 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4^{\oplus 2}$
5	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_7 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_8$
6	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \leq t \leq 5)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8$, $\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{D}_8$
7	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(1 \leq t \leq 2)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_4 \oplus \mathbb{E}_8$
8	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_6 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 1)$	
9	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_7 \oplus \mathbb{E}_8 \oplus \mathbb{A}_1^{\oplus t}$	$(0 \leq t \leq 1)$	$\mathbb{U}^{\oplus 2} \oplus \mathbb{D}_8 \oplus \mathbb{E}_8$
10	$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2} \oplus \mathbb{A}_1$		$\mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8^{\oplus 2}$

Period domain for 2-elementary $K3$ surfaces

Let (S,θ) be a 2-elementary $K3$ surface.
Fix a lattice $\Lambda \subset \mathbb{L}_{K3} = \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8 \oplus \mathbb{E}_8$ such that

$$\Lambda \cong H^2(S,\mathbb{Z})_-$$

Λ is a 2-elementary lattice with $\mathrm{sign}(\Lambda) = (2, r-2)$, $r = \mathrm{rk}\,\Lambda$.

Definition (Domain of type IV)

For a lattice Λ with $\mathrm{sign}(\Lambda) = (2, r-2) = (2, b^-)$, define

$$\begin{aligned} \Omega_\Lambda = \Omega_\Lambda^+ \amalg \Omega_\Lambda^- &:= \{[\eta] \in \mathbb{P}(\Lambda \otimes \mathbb{C});\ \langle \eta, \eta \rangle_\Lambda = 0,\ \langle \eta, \bar{\eta} \rangle_\Lambda > 0\} \\ &\cong \frac{SO_0(2, b^-)}{SO(2) \times SO(b^-)} \end{aligned}$$

$\Omega_\Lambda^\pm$: bounded symmetric domain of type IV of $\dim \Omega_\Lambda = r - 2$

Period of 2-elementary $K3$ surfaces

(S, θ) : 2-elementary $K3$ surface $\Longrightarrow H^0(S, \Omega_S^2) \subset H^2(S, \mathbb{C})_-$
(S, θ) : type $\Lambda \Longrightarrow \exists\, \alpha\colon H^2(S, \mathbb{Z}) \cong \mathbb{L}_{K3}$ with

$$\alpha(H^2(S, \mathbb{Z})_-) = \Lambda$$

ω : nowhere vanishing holomorphic 2-form $\Longrightarrow \alpha(\omega) \in \Lambda \otimes \mathbb{C}$

Definition (Period of 2-elementary $K3$ surface)

When (S, θ) is of type Λ, its period is defined by

$$\varpi(S, \theta) := [\alpha(\omega)] = \left[\left(\cdots, \int_{\lambda_i} \omega, \cdots\right)\right] \in O^+(\Lambda)\backslash \Omega_\Lambda^+$$

where
$\{\lambda_i\}$ is a $\mathbb{Z}$-basis of $H_2(S, \mathbb{Z})_-$
$O^+(\Lambda)$ the automorphism group of Λ preserving Ω_Λ^+

Moduli space of Borcea-Voisin threefolds

Theorem

The moduli space of Borcea-Voisin threefolds of type Λ is isomorphic to

$$\mathcal{M}_\Lambda^0 \times X(1) = O^+(\Lambda)\backslash(\Omega_\Lambda^+ - \mathcal{D}_\Lambda) \times (\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H})$$

via the period mapping

$$X_{(S,\theta,T)} \mapsto (\varpi(S,\theta), \varpi(T))$$

where

$$\mathcal{D}_\Lambda = \bigcup_{d\in\Delta_\Lambda} H_d, \quad H_d = \Omega_\Lambda \cap d^\perp, \quad \Delta_\Lambda = \{d \in \Lambda;\ \langle d, d\rangle = -2\}$$

is the discriminant divisor of Ω_Λ (Heegner divisor of norm -2-vectors), $\varpi(S,\theta)$ and $\varpi(T)$ are the periods of (S,θ) and T, respectively.

Borcherds products and a formula for τ_{BCOV}

To give a formula for the BCOV invariant of the Borcea-Voisin threefolds, we recall Borcherds products.

Recall that the Dedekind η-function and the Jacobi θ-series are defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad \vartheta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} \qquad (q = e^{2\pi i\tau})$$

Definition

For a 2-elementary lattice Λ, define a modular form ϕ_Λ for $\Gamma_0(4)$ by

$$\phi_\Lambda(\tau) := \eta(\tau)^{-8}\eta(2\tau)^{8}\eta(4\tau)^{-8}\,\vartheta(\tau)^{12-r(\Lambda)}$$

To construct a Borcherds product for Λ, we lift $\phi_\Lambda(\tau)$ to a vector-valued modular form for $\mathrm{Mp}_2(\mathbb{Z})$ using the Weil representation $\rho_\Lambda\colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\Lambda^\vee/\Lambda])$.

Weil representation for 2-elementary lattices

Λ : 2-elementary lattice, i.e., $\Lambda^\vee/\Lambda = (\mathbb{Z}/2\mathbb{Z})^{\ell(\Lambda)}$
$\{\mathfrak{e}_\gamma\}_{\gamma\in\Lambda^\vee/\Lambda}$: standard basis of the group ring $\Lambda^\vee/\Lambda$
$\mathbb{C}[\Lambda^\vee/\Lambda] = \mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^{\ell(\Lambda)}] = \mathbb{C}^{2^{\ell(\Lambda)}}$: the group ring of $\Lambda^\vee/\Lambda$
$\mathrm{Mp}_2(\mathbb{Z}) := \{(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau+d});\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\}$
S, T : standard generator of $\mathrm{Mp}_2(\mathbb{Z})$

$$S := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \qquad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right)$$

Definition (Weil representation)

The Weil representation $\rho_\Lambda \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[\Lambda^\vee/\Lambda])$ is defined by

$$\rho_\Lambda(T)\,\mathfrak{e}_\gamma := e^{\pi i \gamma^2}\,\mathfrak{e}_\gamma, \qquad \rho_\Lambda(S)\,\mathfrak{e}_\gamma := \frac{i^{-\sigma(\Lambda)/2}}{\sqrt{|\Lambda^\vee/\Lambda|}} \sum_{\delta\in\Lambda^\vee/\Lambda} e^{-2\pi i \gamma\cdot\delta}\,\mathfrak{e}_\delta$$

Vector-valued modular forms for 2-elementary lattices

Definition (Vector-valued modular form for 2-elementary lattice)

$$F_\Lambda(\tau) := \sum_{\gamma \in \widetilde{\Gamma}_0(4) \backslash \mathrm{Mp}_2(\mathbb{Z})} (\phi_\Lambda|_\gamma)(\tau)\, \rho_\Lambda(\gamma^{-1})\, \mathfrak{e}_0$$

where $(f|_\gamma)(\tau) := (c\tau + d)^{-wt(f)} f(\frac{a\tau+b}{c\tau+d})$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$

Fact (Modularity of F_Λ)

$F_\Lambda(\tau)$ is a modular form of type ρ_Λ of weight $1 - b^-(\Lambda)/2$, i.e.,

$$F_\Lambda\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{1-\frac{b^-(\Lambda)}{2}} \rho_\Lambda\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \sqrt{c\tau + d}\right) \cdot F_\Lambda(\tau)$$

for all $(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \sqrt{c\tau + d}) \in \mathrm{Mp}_2(\mathbb{Z})$

Borchers products

Write

$$F_\Lambda(\tau) = \sum_{\gamma \in \Lambda^\vee/\Lambda} \mathfrak{e}_\gamma \sum_{m \in \mathbb{Z}+\gamma^2/2} c_\gamma(m) q^m$$

For simplicity, assume

$$\Lambda = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \oplus L$$

where L is a 2-elementary Lorentzian lattice
$\mathcal{C}_L = \mathcal{C}_L^+ \amalg \mathcal{C}_L^- = \{x \in L \otimes \mathbb{R};\, x^2 > 0\}$: positive cone of L

Definition (Borcherds product)

For $z \in L \otimes \mathbb{R} + i\,\mathcal{W}$ where $\mathcal{W} \subset \mathcal{C}_L^+$ is a "Weyl chamber", define

$$\Psi_\Lambda(z, F_\Lambda) := e^{2\pi i \langle \varrho, z \rangle} \prod_{\gamma \in L^\vee/L} \prod_{\lambda \in L+\gamma,\, \lambda \cdot \mathcal{W} > 0} \left(1 - e^{2\pi i \langle \lambda, z \rangle}\right)^{c_\gamma(\lambda^2/2)}$$

The Borcherds product $\Psi_\Lambda(z, F_\Lambda)$ is viewed as a formal function on Ω_Λ^+

Fact (Realization of Ω_Λ as a tube domain)

- *$L \otimes \mathbb{R} + i\,\mathcal{C}_L^+$ is isomorphic to Ω_Λ^+ via the exponential map*

$$L \otimes \mathbb{R} + i\,\mathcal{C}_L^+ \ni z \to \exp(z) := \left[\left(1, z, \frac{\langle z, z\rangle}{2}\right)\right] \in \Omega_\Lambda^+$$

- *Through this identification, $O^+(\Lambda)$ acts on $L \otimes \mathbb{R} + i\,\mathcal{C}_L^+$*

Theorem (Borcherds)

The infinite product $\Psi_\Lambda(z, F_\Lambda)$ extends to a (possibly meromorphic) automorphic form on $L \otimes \mathbb{R} + i\,\mathcal{C}_L^+ \cong \Omega_\Lambda^+$ for $O^+(\Lambda)$

The structure of the invariant: Case $(r,\delta) \neq (12,0),(20,0)$

Theorem (Ma-Y., Y.)

Let $(r,\delta) \neq (12,0),(20,0)$. $\exists$ constant C_Λ depending only on Λ s.t.

$$\tau_{\rm BCOV}(X_{(S,\theta,T)})^{2^{g-1}(2^g+1)} = C_\Lambda \, \left\|\Psi_\Lambda\left(\varpi(S,\theta), 2^{g-1}F_\Lambda\right)\right\|^2 \times \left\|\chi_g\left(\varpi(S^\theta)\right)^8\right\|^2 \left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2^g(2^g+1)}.$$

Λ : type of $X_{(S,\theta,T)}$, $\quad r = \mathrm{rk}\,\Lambda$
$\delta \in \{0,1\}$: parity of the discriminant form $q_\Lambda \colon \Lambda^\vee/\Lambda \to \mathbb{Q}/2\mathbb{Z}$
$g = g(\Lambda)$: total genus of S^θ
χ_g^8 : Siegel modular form on $\mathfrak{S}_g$ of weight $2^{g+1}(2^g+1)$

$$\chi_g(T)^8 := \prod\nolimits_{(a,b)\ \mathrm{even}} \theta_{a,b}(T)^8, \quad T \in \mathfrak{S}_g$$

$\varpi(S^\theta) \in \mathfrak{S}_g$: period of the fixed-curve $S^\theta = \{x \in S;\ \theta(x) = x\}$.

The structure of the invariant : Case $(r,\delta)=(12,0)$

Theorem (Ma-Y., Y.)

Let $(r,\delta)=(12,0)$. $\exists$ *constant* C_Λ *depending only on* Λ *s.t.*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)})^{(2^{g-1}+1)(2^g-1)} = C_\Lambda \left\|\Psi_\Lambda\left(\varpi(S,\theta),(2^{g-1}+1)F_\Lambda\right)\right\|^2 \times \left\|\Upsilon_g\left(\varpi(S^\theta)\right)\right\|^2 \times \left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2(2^{g-1}+1)(2^g-1)}$$

Υ_g : Siegel modular form of degree g and weight $2(2^g-1)(2^g+2)$

$$\Upsilon_g(T) := \chi_g(T)^8 \sum_{(a,b)\ \mathrm{even}} \theta_{a,b}(T)^{-8}$$

elementary symmetric polynomial of degree $(2^g-1)(2^{g-1}+1)$ in the even theta constants $\theta_{a,b}(T)^8$

Theorem (Ma-Y., Y.)

Let $(r,\delta)=(18,0)$. $\exists$ *constant* C_Λ *depending only on* Λ *s.t.*

$$\tau_{\mathrm{BCOV}}(X_{(S,\theta,T)})^{(2^{g-1}+1)(2^g-1)}=C_\Lambda\left\|\Psi_\Lambda\left(\varpi(S,\theta),2^{g-1}F_\Lambda+f_\Lambda\right)\right\|^2 \times\left\|\Upsilon_g\left(\varpi(S^\theta)\right)\right\|^2\left\|\eta\left(\varpi(T)\right)^{24}\right\|^{2(2^{g-1}+1)(2^g-1)}$$

f_Λ : $\mathbb{C}[\Lambda^\vee/\Lambda]$-valued elliptic modular form

$$f_\Lambda(\tau):=\frac{\theta_{\mathbb{E}_8}(\tau)}{\eta(\tau)^{24}}\qquad(\Lambda=\mathbb{U}\oplus\mathbb{U}\oplus\mathbb{E}_8\oplus\mathbb{E}_8)$$

$$f_\Lambda(\tau):=8\sum_{\gamma\in\Lambda^\vee/\Lambda}\left\{\eta\left(\frac{\tau}{2}\right)^{-8}\eta(\tau)^{-8}+(-1)^{\gamma^2}\eta\left(\frac{\tau+1}{2}\right)^{-8}\eta(\tau+1)^{-8}\right\}\mathfrak{e}_\gamma + \eta(\tau)^{-8}\eta(2\tau)^{-8}\,\mathfrak{e}_0\qquad(\Lambda=\mathbb{U}\oplus\mathbb{U}(2)\oplus\mathbb{E}_8\oplus\mathbb{E}_8)$$

The Hodge line bundle on the Siegel modular variety

Definition

The Hodge line bundle on the Siegel modular variety

$$\mathcal{A}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{S}_g$$

is the line bundle $\mathcal{F}_g = \mathrm{Sp}_{2g}(\mathbb{Z})\backslash(\mathfrak{S}_g \times \mathbb{C})$ associated to the automorphic factor

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \ni \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \det(C\tau + D) \in \mathcal{O}(\mathfrak{S}_g)$$

A section of $\mathcal{F}_g^{\otimes q}$ is identified with a Siegel modular form of weight q.

Fact (Baily-Borel)

$\mathcal{F}_g$ extends to an ample $\mathbb{Q}$-line bundle on the Satake compactification $\mathcal{A}_g^*$

The Torelli map

Recall that $\mathcal{M}_\Lambda^0 = O^+(\Lambda)\backslash(\Omega_\Lambda^+ - \mathcal{D}_\Lambda)$.

Definition (Torelli map)

The Torelli map $J_\Lambda \colon \mathcal{M}_\Lambda^0 \to \mathcal{A}_g$ is defined by

$$J_\Lambda(S,\theta) = [\mathrm{Jac}(S^\theta)] = [\varpi(S^\theta)]$$

where
$(S,\theta) \in \mathcal{M}_\Lambda^0$: 2-elementary $K3$ surface of type Λ, i.e., $H^2(S,\mathbb{Z})_- \cong \Lambda$
S^θ : fixed curve of $\theta \colon S \to S$
$[\mathrm{Jac}(S^\theta)]$: isomorphism class of the Jacobian variety of S^θ
$[\varpi(S^\theta)] \in \mathcal{A}_g$: the period of S^θ.

The Torelli map is identified with the $O^+(\Lambda)$-equivariant map

$$J_\Lambda \colon \Omega_\Lambda^+ - \mathcal{D}_\Lambda \to \mathcal{A}_g$$

Some properties of the Torelli map

Fact (Borel, Kobayashi-Ochiai)

The Torelli map extends to an $O^+(\Lambda)$-equivariant holomorphic map

$$J_\Lambda \colon \Omega_\Lambda^+ \setminus \operatorname{Sing}\mathcal{D}_\Lambda \to \mathcal{A}_g^*$$

In particular, J_Λ is a rational map from $\mathcal{M}_\Lambda$ to $\mathcal{A}_g^*$

Let $d \in \Delta_\Lambda$, i.e., $d \in \Lambda$ and $\langle d, d \rangle = -2$. Then

$$H_d = \Omega_\Lambda^+ \cap d^\perp = \{[\eta] \in \Omega_\Lambda^+;\ \langle \eta, d \rangle = 0\}$$

is identified with $\Omega^+_{\Lambda \cap d^\perp}$. Since $\Lambda \cap d^\perp$ is 2-elementary, the Torelli map $J_{\Lambda\cap d^\perp} \colon \Omega^+_{\Lambda\cap d^\perp} \setminus \operatorname{Sing}\mathcal{D}_{\Lambda\cap d^\perp} \to \mathcal{A}^*_{g'}$ is well defined, where $g' = g(\Lambda \cap d^\perp)$.

Fact (Compatibility of the Torelli maps)

$$J_\Lambda|_{H_d \setminus \operatorname{Sing}\mathcal{D}_\Lambda} = J_{\Lambda\cap d^\perp}$$

The line bundle λ_Λ^q

$J_\Lambda^* \mathcal{F}_g^{\otimes q}$ is an $O^+(\Lambda)$-equivariant holomorphic line bundle on $\Omega_\Lambda^+ \setminus \operatorname{Sing} \mathcal{D}_\Lambda$. Let

$$\iota_\Lambda \colon \Omega_\Lambda^+ \setminus \operatorname{Sing} \mathcal{D}_\Lambda \hookrightarrow \Omega_\Lambda^+$$

be the inclusion.

Definition

Define λ_Λ^q as the trivial extension of $J_\Lambda^* \mathcal{F}_g^{\otimes q}$ from $\Omega_\Lambda^+ \setminus \operatorname{Sing} \mathcal{D}_\Lambda$ to Ω_Λ^+

$$\lambda_\Lambda^q := (\iota_\Lambda)_* \mathcal{O}_{\Omega_\Lambda^+ \setminus \operatorname{Sing} \mathcal{D}_\Lambda} \left(J_\Lambda^* \mathcal{F}_g^{\otimes q} \right).$$

Since $J_\Lambda \colon \Omega_\Lambda^+ \dashrightarrow \mathcal{A}_g^*$ is a rational map, one has the following:

Fact

λ_Λ^q is an $O^+(\Lambda)$-equivariant invertible sheaf on Ω_Λ^+.

Automorphic forms on Ω_Λ^+

Fix a non-zero isotropic vector $\ell_\Lambda \in \Lambda$
Then

$$\Omega_\Lambda^+ \cong L \otimes \mathbb{R} + i\mathcal{C}_L^+,$$

where $L = \ell_\Lambda^\perp / \mathbb{Z}\ell_\Lambda$ is a Lorentzian lattice

Define the automorphic factor $j_\Lambda(\gamma, \cdot) \in \mathcal{O}^*(\Omega_\Lambda^+)$, $\gamma \in O(\Lambda)$ by

$$j_\Lambda(\gamma, [z]) := \frac{\langle \gamma(z), \ell_\Lambda \rangle}{\langle z, \ell_\Lambda \rangle} \qquad [z] \in \Omega_\Lambda^+.$$

Definition

$F \in H^0(\Omega_\Lambda^+, \lambda_\Lambda^q)$ is an automorphic form of weight (p, q) if

$$F(\gamma \cdot [z]) = j_\Lambda(\gamma, [z])^p \gamma(F([z]))$$

for all $\gamma \in O^+(\Lambda)$ and $[z] \in \Omega_\Lambda^+$

Quasi-pullback of automorphic forms

For a root $d \in \Delta_\Lambda$,

$$\frac{\langle z, d\rangle}{\langle z, \ell_\Lambda\rangle}, \qquad [z] \in \Omega_\Lambda^+$$

is a holomorphic function on Ω_Λ^+ with zero divisor $H_d = \Omega_\Lambda^+ \cap d^\perp$.

Definition (Quasi-pullback)

Let $F \in H^0(\Omega_\Lambda, \lambda_\Lambda^q)$ be a section vanishing on H_d of order k.

$$\rho^\Lambda_{\Lambda\cap d^\perp}(F) := \left\{ \left(\frac{\langle \cdot, \ell_\Lambda\rangle}{\langle \cdot, d\rangle} \right)^k \cdot F \right\}\Bigg|_{H_d}$$

is called the *quasi-pullback* of F to $H_d = \Omega^+_{\Lambda\cap d^\perp}$. Hence
$\rho^\Lambda_{\Lambda\cap d^\perp}(F) \in H^0(\Omega^+_{\Lambda\cap d^\perp}, \lambda^q_{\Lambda\cap d^\perp})$.
The case $q = 0$ is the quasi-pullback in the classical sense.

The quasi-pullback of the Siegel modular form χ_g^8

Recall that χ_g is the Siegel modular form on $\mathfrak{S}_g$ defined as

$$\chi_g = \prod\nolimits_{(a,b)\,\mathrm{even}} \theta_{a,b}.$$

Main Theorem

Let $d \in \Delta_\Lambda$. Set $\Lambda' := \Lambda \cap d^\perp$ and $g' := g(\Lambda')$. Assume $g' = g - 1$ and $(r,\delta), (r(\Lambda'), \delta(\Lambda')) \neq (12,0), (20,0)$. Then $\exists$ constant $C_{\Lambda,d}$ s.t.

$$\frac{\rho^\Lambda_{\Lambda'}(J^*_\Lambda \chi_g^{8(2^{g'}+1)})}{J^*_{\Lambda'}\chi_{g'}^{16(2^g+1)}} = C_{\Lambda,d}\left(\frac{\rho^\Lambda_{\Lambda'}\left(\Psi_\Lambda(\cdot,(2^{g'}+1)2^{g-1}F_\Lambda)\right)}{\Psi_{\Lambda'}(\cdot,2(2^g+1)2^{g'-1}F_{\Lambda'})}\right)^{-1}$$

In particular, $\rho^\Lambda_{\Lambda'}(J^*_\Lambda \chi_g^{8(2^{g'}+1)})/J^*_{\Lambda'}\chi_{g'}^{16(2^g+1)}$ is a Borcherds product, since $\rho^\Lambda_{\Lambda'}\left(\Psi_\Lambda(\cdot,(2^{g'}+1)2^{g-1}F_\Lambda)\right)$ is a Borcherds product by S. Ma.

Idea of the proof

Set

$$\Phi_\Lambda := \Psi_\Lambda(\cdot, 2^{g-1}F_\Lambda) \otimes J_\Lambda^* \chi_g^8.$$

Then τ_{BCOV} is given as follows:

$$\tau_{\mathrm{BCOV}} = C_\Lambda \left\| \Phi_\Lambda^{\frac{1}{2^{g-1}(2^g+1)}} \right\|^2 \|\eta^{24}\|^2.$$

The Main Theorem is a consequence of the following formula, which determines how the BCOV invariants behave under "transition".

Theorem

$$\rho_{\Lambda'}^{\Lambda}\left(\Phi_\Lambda^{2^{g'}+1}\right) = C_{\Lambda,d}\,\Phi_{\Lambda'}^{2(2^g+1)}$$

We first prove that the L.H.S. is an automorphic form. Then we prove the above equality by comparing the weights and the divisors of Φ_Λ and $\Phi_{\Lambda'}$.