# Eisenstein cohomology and CM congruences

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# This talk

Joint work with Adel Betina (University of Vienna)

1. CM congruences
2. Eisenstein cohomology
3. Eisenstein congruences

# Work of Hida

For $K/\mathbf{Q}$ imaginary quadratic let $\psi : \mathbf{A}_K^*/K^* \to \mathbf{C}^*$ be a Hecke character with $\psi_\infty(z) = z^{-1}$, Consider $p$ split in $K/\mathbf{Q}$.

**Hida (1982)**

- $p \mid L^{\mathrm{alg}}(1, \psi/\psi^c) \implies \exists f$ non-CM such that $f \equiv f_\psi \mod p$
- Base change to $K$: $\mathrm{BC}(f) \equiv \mathrm{Eis}_\psi \mod p$
- Galois repn $\rho_f|_{G_K}$ irreducible with $\overline{\rho}_f^{\mathrm{ss}} \equiv \psi + \psi^c \mod p$.
- Ribet argument implies $p \mid \#\mathrm{Cl}(K(\psi))^{\psi/\psi^c}$

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Converse proven later by Rubin:
$p \mid \#\mathrm{Cl}(K(\psi))^{\psi/\psi^c} \implies p \mid L^{\mathrm{alg}}(1, \psi/\psi^c)$

# Our results

- Prove Eisenstein congruences over imaginary quadratic field directly
- Obtain $R = T$ for residually reducible representation

# Setup

Consider

- $K \neq \mathbf{Q}(i), \mathbf{Q}(\sqrt{-3})$ with $\#\mathrm{Cl}(K) = 1$ (for simplicity)
- $p > 3$ split or inert in $K/\mathbf{Q}$
- $\mathcal{O} \subset E$ ring of integers for $E/\mathbf{Q}_p$ finite

# Bianchi modular forms

- real analytic functions on $\mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$
- automorphic representations of $\mathrm{GL}_2(\mathbf{A}_K)$
- cohomology classes in $H^i(\Gamma\backslash\mathbf{H}_3, R)$, where $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_K)$ and $R$ is an $\mathcal{O}$-algebra.

**Theorem (Harder)**

$$S_2(\Gamma, \mathbf{C}) \cong H^1_c(\Gamma\backslash\mathbf{H}_3, \mathbf{C})$$

# Cohomology of arithmetic groups

For $X_\Gamma = \Gamma\backslash \mathbf{H}_3$ have long exact sequence

$$\ldots \to H^1_c(X_\Gamma, \mathcal{O}) \to H^1(X_\Gamma, \mathcal{O}) \stackrel{\text{res}}{\to} H^1(\partial\overline{X_\Gamma}, \mathcal{O}) \stackrel{\partial}{\to} H^2_c(X_\Gamma, \mathcal{O}) \to \ldots,$$

where $\overline{X_\Gamma}$ is the Borel-Serre compactification

In terms of group cohomology:

- $H^1(X_\Gamma, \mathcal{O}) \cong H^1(\Gamma, \mathcal{O}) = \mathrm{Hom}(\Gamma, \mathcal{O})$
- $H^1(\partial\overline{X_\Gamma}, \mathcal{O}) = \mathrm{Hom}(\Gamma \cap U(\mathbf{Z}), \mathcal{O})$

# Cohomology of boundary

$$H^1(\partial\overline{X_\Gamma}, \overline{\mathbf{Q}}) = \bigoplus_{\phi} \left(\mathrm{Ind}_{B(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \overline{\mathbf{Q}}_\phi\right)^\Gamma \oplus \left(\mathrm{Ind}_{B(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \overline{\mathbf{Q}}_{w_0.\phi}\right)^\Gamma,$$

where the sum is over all $\phi = (\phi_1, \phi_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ with $\phi_\infty(z) = (z, z^{-1})$ and $w_0.\phi = (\phi_2|\cdot|, \phi_1|\cdot|^{-1})$.

- Casselman: $\left(\mathrm{Ind}_{B(\mathbf{A}_f)}^{G(\mathbf{A}_f)} \overline{\mathbf{Q}}_\phi\right)^{K_1(M)} = \Psi_\phi^{\mathrm{new}} \cdot \overline{\mathbf{Q}}$
- Use $\Psi_\phi^{\mathrm{new}}$ to define $\omega_\phi \in H^1(\partial\overline{X_\Gamma}, \mathcal{O}) \subset H^1(\partial\overline{X_\Gamma}, \overline{\mathbf{Q}}_p)$
- Hecke eigenvalue of $\omega_\phi$ for $T_v$ with $v \nmid M$ is $\phi_1(\pi_v)|\pi_v|_v^{-1} + \phi_2(\pi_v)$

# Eisenstein cohomology

G. Harder: Define $\mathrm{Eis}(\omega_\phi) \in H^1(\Gamma, \mathbf{C})$ by

$$\mathrm{Eis}(\omega_\phi)(g) = \sum_{\gamma \in (\Gamma \cap U)\backslash \Gamma} \omega_\phi(\gamma g)$$

(using meromorphic continuation)

- For $\chi = \phi_1/\phi_2$ have

$$\mathrm{res}\,\mathrm{Eis}(\omega_\phi) = \omega_\phi + * \cdot \frac{L(-1,\chi)}{L(0,\chi)} \omega_{w_0.\phi}$$

- $\mathrm{Eis}(\omega_\phi) \in H^1(\Gamma, E)$ for sufficiently large $E/\mathbf{Q}_p$.

# Denominator of $\mathrm{Eis}(\omega_\phi)$

For $c \in H^1(\Gamma, E)$ define denominator ideal

$$\delta(c) := \{a \in \mathcal{O} : a \cdot c \in \mathrm{im}\left(H^1(\Gamma, \mathcal{O}) \to H^1(\Gamma, E)\right)\}.$$

**Theorem (Berger 2005)**

$$\delta(\mathrm{Eis}(\omega_\phi)) \subseteq L^{\mathrm{alg}}(0, \chi)\mathcal{O}$$

# Anticyclotomic set-up

- $\psi : \mathbf{A}_K^*/K^* \to \mathbf{C}^*$ a Hecke character with $\psi_\infty(z) = z^{-1}$ and conductor $\mathfrak{m}$ with $\mathfrak{m} + \overline{\mathfrak{m}} = \mathcal{O}_K$.
- Put $\phi_2 = \psi$ and $\phi_1 = \phi_2^c \cdot |\cdot|$ (so $\chi = \phi_1/\phi_2 = \psi^c/\psi \cdot |\cdot|$).
- Assume $p \nmid M\varphi(M)$ for $M \in \mathbf{Z}$ given by $\mathfrak{m}\overline{\mathfrak{m}} = M\mathcal{O}_K$.

Then we have:

- Hecke eigenvalues of $\mathrm{Eis}(\omega_\phi)$ are $\psi + \psi^c$
- $\mathrm{res}\,\mathrm{Eis}(\omega_\phi) = \omega_\phi + W(\chi)\frac{L(0,\overline{\chi})}{L(0,\chi)} \cdot \omega_{\phi^c}$
- Since $\chi^c = \overline{\chi} \implies L(0,\chi) = L(0,\overline{\chi}) = L(1,\psi/\psi^c)$.
- Also root number $W(\chi) = +1$.

So, in fact:

$$\mathrm{res}\,\mathrm{Eis}(\omega_\phi) = \omega_\phi + \omega_{\phi^c}.$$

# Proof of Eisenstein congruence

We have shown:

- $\mathrm{Eis}(\omega_\phi) \in H^1(X_\Gamma, E)$ with $\mathrm{res}(\mathrm{Eis}(\omega_\phi)) \in H^1(\partial\overline{X_\Gamma}, \mathcal{O})$ and $\delta \cdot \mathrm{Eis}(\omega_\phi) \in H^1(X_\Gamma, \mathcal{O})/\mathrm{tors}$.
- By long exact sequence we know

$$\delta \cdot \partial(\mathrm{res}(\mathrm{Eis}(\omega_\phi)) = 0 \in H^2_c(X_\Gamma, \mathcal{O}).$$

- But $H^2_c(X_\Gamma, \mathcal{O})_{\mathrm{tors}} \neq 0$ possible!
- Assume there exists $c_\phi \in H^1(X_\Gamma, \mathcal{O})$ with

$$\mathrm{res}(c_\phi) = \mathrm{res}(\mathrm{Eis}(\omega_\phi)) \in H^1(\partial\overline{X_\Gamma}, \mathcal{O}).$$

- Then $d_\phi := \delta \cdot (c_\phi - \mathrm{Eis}(\omega_\phi))) \in H^1_c(X_\Gamma, \mathcal{O})$ satisfies

$$d_\phi \equiv \mathrm{Eis}(\omega_\phi) \mod \delta.$$

## Lemma

*Suppose that we have an orientation reversing involution* $\iota$ *on* $X_\Gamma$ *such that*

$$H^1(X_\Gamma, \mathcal{O}) \stackrel{\mathrm{res}}{\to} H^1(\partial \overline{X_\Gamma}, \mathcal{O})^{\varepsilon} \subset H^1(\partial \overline{X_\Gamma}, \mathcal{O}),$$

*where the superscript* $\varepsilon = \pm 1$ *indicates the* $\varepsilon$*-eigenspace for* $\iota$*.*
*Then the restriction map is surjective onto* $H^1(\partial \overline{X_\Gamma}, \mathcal{O})^{\varepsilon}$*.*

## Proof.

Use non-degeneracy of Pontryagin duality pairings

$$H^i_c(X_\Gamma, \mathcal{O})^{\pm} \times H^{3-i}(X_\Gamma, E/\mathcal{O})^{\mp} \to E/\mathcal{O}$$

$$H^1(\partial \overline{X_\Gamma}, \mathcal{O})^{\pm} \times H^1(\partial \overline{X_\Gamma}, E/\mathcal{O})^{\mp} \to E/\mathcal{O}$$

and adjointness of $\mathrm{res}$ and $\partial$ to prove for all $n$ that

$$\mathrm{im}(H^1(X_\Gamma, \mathcal{O}/\varpi^n) \stackrel{\mathrm{res}}{\to} H^1(X_\Gamma, \mathcal{O}/\varpi^n)) = \mathrm{im}(\mathrm{res})^{\perp}$$

$$\text{and } H^1(\partial \overline{X_\Gamma}, \mathcal{O}/\varpi^n)^{\varepsilon} \subset \mathrm{im}(\mathrm{res})^{\perp}.$$

## Theorem (Serre)

*For* $\Gamma = \mathrm{SL}_2(\mathcal{O}_K)$ *and* $\iota$ *induced by* $(z, r) \in H_3 \mapsto (\overline{z}, r)$ *one has*

$$H^1(X_\Gamma, \mathcal{O}) \stackrel{\text{res}}{\to} H^1(\partial \overline{X_\Gamma}, \mathcal{O})^- \subset H^1(\partial \overline{X_\Gamma}, \mathcal{O}).$$

# Generalizing Serre's result

Let $\mathbf{T}$ be the $\mathcal{O}-$algebra generated by the Hecke operators $T_v$ for $v \nmid M$ acting on $H^1(\overline{X_\Gamma}, \mathcal{O})$. Let $\mathfrak{m} \subset \mathbf{T}$ be the maximal ideal containing the ideal $J$ generated by $\{T_v - \psi(\pi_v) - \psi^c(\pi_v)\}$.

## Theorem (B-Betina)

*For* $\Gamma = \Gamma_1(M\mathcal{O}_K)$ *and* $\iota$ *induced by* $(z, r) \in H_3 \mapsto (\bar{z}, r)$ *one has*

$$H^1(X_\Gamma, \mathcal{O})_\mathfrak{m} \stackrel{\text{res}}{\to} H^1(\partial\overline{X_\Gamma}, \mathcal{O})^+_\mathfrak{m} \subset H^1(\partial\overline{X_\Gamma}, \mathcal{O})_\mathfrak{m}.$$

## Proof.

- Under our assumptions $H^1(\partial\overline{X_\Gamma}, \mathcal{O})_\mathfrak{m} \subset H^1(\partial\overline{X_\Gamma}, E)[J]$.
- By Harder and Casselman $H^1(\partial\overline{X_\Gamma}, E)[J] = \omega_\phi \cdot E \oplus \omega_{\phi^c} \cdot E$.
- $\iota(\omega_\phi) = \omega_{\phi^c}$, so $H^1(X_\Gamma, E)[J] \stackrel{\text{res}}{\to} H^1(\partial\overline{X_\Gamma}, E)^+$.

□

# Our congruence result

Putting all this together and using that $\mathrm{res}(\mathrm{Eis}(\omega_\phi)) \in H^1(\partial\overline{X_\Gamma}, \mathcal{O})^+_{\mathfrak{m}}$ we get:

**Theorem (B-Betina)**

*We have an $\mathcal{O}$-algebra surjection*

$$\mathbf{T}^0/J \twoheadrightarrow \mathcal{O}/L^{\mathrm{alg}}(1, \psi/\psi^c).$$

# R=T theorem

Generalizing previous work with Kris Klosin (where $\chi = \psi^c/\psi|\cdot|$ had to be unramified) this allows us to prove:

**Theorem (B-Betina)**

*For $k = \mathcal{O}/\varpi$ assume $\mathrm{Ext}^1_{k[G_K]}(1, \chi^{\pm}) = k$. Then we have $R_\rho = \mathbf{T}^0$ for*

$$\rho = \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix} : G_K \to \mathrm{GL}_2(k).$$