On Supercuspidal Representations of Sp_{2n} and Langlands Parameters

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My Project

To speculate the Langlands parameter of an explicitly constructed supercuspidal representation by means of checking

- the formal degree conjecture
  (due to Hiraga, Ichino, Ikeda:J.Amer.Math.Soc.(2008)), and

- the root number conjecture
  (due to Gross, Reeder:Duke Math.J. (2010))

In this talk, on $Sp_{2n}(F)$ ($F/\mathbb{Q}_p$: finite ext., $p \neq 2$)

1) To construct a supercuspidal representation π of $Sp_{2n}(F)$,

2) To find a candidate of the Langlands parameter of π:

$$\varphi : W_F \to SO_{2n+1}(\mathbb{C}),$$

3) To check the formal degree conjecture :

$$\begin{array}{l}\text{formal degree of } \pi \text{ w.r. to} \\ \text{Euler-Poincaré measure(abs.val.)}\end{array} = \frac{\dim \chi}{|\mathcal{A}_\varphi|} \cdot \left| \frac{\gamma(\mathrm{Ad} \circ \varphi, 0)}{\gamma(\varphi_0, \mathrm{Ad}, 0)} \right|,$$

($\varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}_{2n}} SO_{2n+1}(\mathbb{C})$
: the principal parameter,
$\mathcal{A}_\varphi = Z_{SO_{2n+1}(\mathbb{C})}(\mathrm{Im}\, \varphi)$ and $\widehat{\mathcal{A}_\varphi} \ni \chi \leftrightarrow \pi \in \mathbf{\Pi}_\varphi$: L-packet),

4) To check the root number cojecture : $w(\mathrm{Ad} \circ \varphi) = \pi(\epsilon)$ ($\epsilon = -1_{2n}$)

$F/\mathbb{Q}_p$: a finite extension ($p \neq 2$)

$O \subset F$: the integer ring, $\mathfrak{p} = \varpi O \subset O$: the maximal ideal

$\mathbb{F} = O/\mathfrak{p}$: the residue class field, $|\mathbb{F}| = q$

$G = Sp_{2n}$: an O-group scheme

$\mathfrak{g} = \mathrm{Lie}(G)$: the Lie algebra O-scheme

$\psi : F \to \mathbb{C}^\times$:unitary character s.t. $\{x \in F \mid \psi(xO) = 1\} = O$

• Construction of supercuspidal representation

Th (Stevens (Invent. math. (2008)).

$$\pi : \text{supercuspidal representation of } G(F)$$
$$\Rightarrow \pi = \text{cmp-ind}_J^{G(F)}\delta \ \text{ with } J \subset G(F)\text{:open compact, } \delta \in J^{\frown}$$

$$J = G(O), \quad \delta : G(O) \xrightarrow{\text{can.}} G(O/\mathfrak{p}^r) \xrightarrow{\delta} GL_{\mathbb{C}}(V_\delta) \ (1 \leq r\text{:minimal})$$

$$r = 1 \ \Rightarrow \begin{cases} \pi = \text{cmp-ind}_{G(O)}^{G(F)}\delta : \text{depth-zero supercuspidal representation,} \\ G(\mathbb{F})^{\frown}\text{: representation theory of finite reductive group} \end{cases}$$

$\underline{r > 1}$ Need a good parametrization of $G(O/\mathfrak{p}^r)^{\frown}$

$\underline{r > 1}$ $r = l + l'$, $l \geq l' > 0$, l:minimal, i.e. $l' = \begin{cases} l & : r = 2l, \\ l-1 & : r = 2l-1 \end{cases}$

$$K_l(O/\mathfrak{p}^r) \overset{\text{def.}}{=} \text{Ker}\left[G(O/\mathfrak{p}^r) \xrightarrow{\text{can.}} G(O/\mathfrak{p}^l)\right] \tilde{\to} \mathfrak{g}(O/\mathfrak{p}^{l'})$$

$$\delta \in G(O/\mathfrak{p}^r)^{\widehat{}} \Rightarrow \delta|_{K_l(O/\mathfrak{p}^r)} = \left(\bigoplus_{\beta \in \Omega} \psi(\varpi^{-l'} B(\beta, *))\right)^m \text{ by Clifford's theorem}$$

with $\Omega \subset \mathfrak{g}(O/\mathfrak{p}^{l'})$ adjoint $G(O/\mathfrak{p}^{l'})$-orbit

$$B : \mathfrak{g}(O/\mathfrak{p}^{l'}) \times \mathfrak{g}(O/\mathfrak{p}^{l'}) \to O/\mathfrak{p}^{l'} : \text{trace form}$$

Fix an adjoint $G(O/\mathfrak{p}^{l'})$-orbit $\Omega \subset \mathfrak{g}(O/\mathfrak{p}^{l'})$, and put

$$\Omega^{\widehat{}} = \left\{\delta \in G(O/\mathfrak{p}^r)^{\widehat{}} : \text{corresponds to } \Omega\right\}$$

Th (Takase (to appear on Pacific J.Math.)).

$$\left\{\theta \in G_\beta(O/\mathfrak{p}^r)^\wedge \,\middle|\, \begin{array}{l} \theta = \psi(\varpi^{-l'}B(\beta,*)) \\ \text{on } G_\beta(O/\mathfrak{p}^r) \cap K_l(O/\mathfrak{p}^r) \end{array} \right\} \ni \theta \xleftrightarrow{1:1} \delta_{\beta,\theta} \in \Omega^\wedge$$

if $\Omega \pmod{\mathfrak{p}} \subset \mathfrak{g}(\mathbb{F})$ *is regular and non-singular, where* $\beta \in \mathfrak{g}(O)$ *s.t.* $\beta \pmod{\mathfrak{p}^{l'}} \in \Omega$ *and* $G_\beta = Z_G(\beta)$.

Rem. 1) *For* $G = Sp_{2n}$,

$$\begin{aligned} &\overline{\Omega} = \Omega \pmod{\mathfrak{p}} \subset \mathfrak{g}(\mathbb{F}) : \text{regular} \\ \iff &\text{with respect to } \beta \pmod{\mathfrak{p}} \in \mathfrak{g}(\mathbb{F}) \subset M_{2n}(\mathbb{F}) \\ &\text{the characteristic polynomial} = \text{the minimal polynomial.} \end{aligned}$$

In this case $G_\beta(O) = G(O) \cap O[\beta]^\times$ *is commutative.*

2) $\dim \delta_{\beta,\theta} = \sharp\overline{\Omega} \cdot q^{(r-2)(\dim G - \mathrm{rank}\, G)}$ *(Shechter:J. Pure Appl. Alg. (2019))*

Starting from a tamely ramified extension

K_+/F : tamely ramified field extension, $(K_+ : F) = n$,
K/K_+ : quadratic field extension with $\mathrm{Gal}(K/K_+) = \langle \tau \rangle$ s.t.

$$K/K_+ : \text{unramified} \quad \text{or} \quad K/F : \text{totally ramified},$$

then $\exists \beta \in O_K$ s.t. $\beta + \beta^\tau = 0$ and $O_K = O[\beta]$, and

$$D(x,y) = \frac{1}{2} T_{K/F}\left(\omega^{-1} \varpi_{K_+}^{1-e_+} x^{\tau} y\right) \quad \left(\begin{array}{l} O_K = O_{K_+}[\omega] \text{ s.t. } \omega^\tau = -\omega \\ e_+ = e(K_+/F) \end{array} \right)$$

is a symplectic F-form on K s.t.

$$D(x\beta, y) + D(x, y\beta) = 0 \text{ for } \forall x, y \in K, \qquad \therefore \beta \in \mathfrak{sp}(K, D)$$

$\beta \in \mathfrak{sp}_{2n}(O)$ by the regular representation
w.r.to a suitable symplectic base on K

Prop (Shintani (J.Math.Soc.Japan (1968))).

The characteristic polynomial of $\beta \pmod{\mathfrak{p}} \in \mathfrak{sp}_{2n}(\mathbb{F}) \subset M_{2n}(\mathbb{F})$ *is equal to its minimal polynomial.*

$$\delta_{\beta,\theta} \in G(O/\mathfrak{p}^r)^\wedge \text{ corresponding to } \theta : G_\beta(O/\mathfrak{p}^r) \to \mathbb{C}^\times$$

$$G_\beta(O_F) = \left\{\varepsilon \in O_K^\times \mid N_{K/K_+}(\varepsilon) = 1\right\} \stackrel{\text{put}}{=} U_{K/K_+},$$

$$G_\beta(O) \xrightarrow{\text{can.}} G_\beta(O/\mathfrak{p}^r) \subset (O_K/\mathfrak{p}_K^{er})^\times \quad (e = e(K/F))$$

$$\theta : U_{K/K_+} \xrightarrow{\text{can.}} (O_K/\mathfrak{p}_K^{er})^\times \to \mathbb{C}^\times$$

$$\delta_{\beta,\theta} : G(O) \xrightarrow{\text{can.}} G(O/\mathfrak{p}^r) \xrightarrow{\delta_{\beta,\theta}} GL_{\mathbb{C}}(V_{\delta_{\beta,\theta}}))$$

Th.A *If* $\left[\frac{r}{2}\right] = l' > \mathrm{Max}\{2, 2(e-1)\}$ *then*

1) $\pi_{\beta,\theta} = \text{cmp-ind}_{G(O)}^{G(F)} \delta_{\beta,\theta}$ *is an irreducible supercuspidal rep. of* $G(F)$,

2) *w.r. to the Haar measure* $d_{G(F)}(x)$ *s.t.* $\mathrm{vol}(G(O)) = 1$,

$$\text{the formal degree of } \pi_{\beta,\theta} = \dim \delta_{\beta,\theta}.$$

Rem. *the Euler-Poincaré measure* μ_G *on* $G(F)$ *is*

$$d\mu_G(x) = (-1)^n q^{n^2} \prod_{k=1}^{n} \left(1 - q^{-(2k-1)}\right) \times d_{G(F)}(x).$$

(See J.-P. Serre:Ann.Math.Stud. (1971)).

<u>Rem</u>. 1) *the multiplicity of* $\delta_{\beta,\theta}$ *in* $\pi_{\beta,\theta}|_{G(O)}$ *is one,*

2) $\delta_{\beta,\theta}$ *is the unique irreducible constituent of* $\pi_{\beta,\theta}|_{G(O)}$ *which factors through* $G(O) \xrightarrow{\text{can.}} G(O/\mathfrak{p}^r)$.

3) $\pi_{\beta,\theta}$ *is generic if* K/F *is unramified.*

• Candidate of Langlands parameter

K/F: tamely ramified Galois extension of s.t. $O_K = O[\beta]$

$$\theta : U_{K/K_+} \xrightarrow{\text{can.}} (O_K/\mathfrak{p}_K^{er})^\times \to \mathbb{C}^\times$$

Put $\vartheta = \theta \cdot c$ with

$$c : U_{K/K_+} \to \mathbb{C}^\times \ \text{ s.t. } \ c(x) = 1 \ \forall x \in U_{K/K_+} \cap (1 + \mathfrak{p}_K^{el}) \ \text{ and}$$

	$e =$ odd	$e =$ even	
		$f = f(K/F) = 1$	$f =$ even
$c(-1)$	1	$(-1)^{\frac{(q-1)(n+1)}{4}}$	$-(-1)^{\frac{q-1}{2}\frac{n}{2}}$

$$\widetilde{\vartheta} : K^\times \xrightarrow{x \mapsto x^{1-\tau}} U_{K/K_+} \xrightarrow{\vartheta = \theta \cdot c} \mathbb{C}^\times \quad (\mathrm{Gal}(K/K_+) = \langle \tau \rangle)$$

$$W_{K/F} = W_F/\overline{[W_K, W_K]} \supset W_K/\overline{[W_K, W_K]} \xleftarrow[\text{l.c.f.t.}]{\sim} K^\times,$$

$$1 \to K^\times \to W_{K/F} \to \mathrm{Gal}(K/F) \to 1 : \text{exact}$$

$$\Updownarrow$$

$$[\alpha_{K/F}] \in H^2(\mathrm{Gal}(K/F), K^\times) : \text{fund. class}$$

$\boldsymbol{\Theta} = \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}$:irreducible rep. of $W_{K/F}$ on

$V_{\boldsymbol{\Theta}} = \{\mathrm{Gal}(K/F) \to \mathbb{C} : \text{function}\}$

$$S(\varphi, \psi) = \sum_{\gamma \in \mathrm{Gal}(K/F)} \widetilde{\vartheta}(\alpha_{K/F}(\gamma, \tau))^{-1} \varphi(\gamma)\psi(\gamma\tau) \quad (\varphi, \psi \in V_{\boldsymbol{\Theta}})$$

: $W_{K/F}$-invariant quad. form on $V_{\boldsymbol{\Theta}}$

$$\mathbf{\Theta} = \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta} : W_{K/F} \to O(V_{\mathbf{\Theta}}, S) = O_{2n}(\mathbb{C})$$

$$\begin{array}{l}
\varphi : W_K \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\mathbf{\Theta} \oplus \det \mathbf{\Theta}} SO_{2n+1}(\mathbb{C}) \\
\quad \Uparrow \\
\theta : U_{K/K_+} \xrightarrow{\text{can.}} (O_K/\mathfrak{p}_K^{er})^\times \to \mathbb{C}^\times \\
\quad \Downarrow \\
\pi_{\beta,\theta} = \text{cmp-ind}_{G(O)}^{G(F)} \delta_{\beta,\theta} : \text{supercuspidal rep of } Sp_{2n}(F)
\end{array}$$

Th.B *The formal degree conjecture and the root number conjecture are valid w.r. to* $\pi_{\beta\,\theta}$ *and* φ*.*

• **Local factors of a representation** $\Phi : W_F \to GL_{\mathbb{C}}(V_{\Phi})$

$W_F = \langle \mathrm{Fr} \rangle \ltimes I_F$ with $I_F = \mathrm{Gal}(F^{\mathrm{alg}}/F^{\mathrm{ur}})$, $\mathrm{Fr}|_{F^{\mathrm{ur}}} = \mathrm{Frobenius}^{-1}$

$$L(\Phi, s) = \det\left(1 - q^{-s}\Phi(\mathrm{Fr})\Big|_{V_{\Phi}^{I_F}}\right)^{-1}$$

$$= \begin{cases} 1 & : f = f(K/F) = 1, \\ \left(1 + q^{-\frac{f}{2}s}\right)^{-1} & : f > 1 \end{cases} \qquad \text{if } \Phi = \mathrm{Ad} \circ \varphi$$

$$\varepsilon(\Phi, \psi, d_F(x), s) = w(\Phi) \cdot q^{a(\Phi)(\frac{1}{2} - s)} \qquad (a(\mathrm{Ad} \circ \varphi) = 2n^2 r)$$

$$\{x \in F \mid \psi(xO_F) = 1\} = O_F, \qquad \int_{O_F} d_F(x) = 1$$

$$\gamma(\Phi, s) = \varepsilon(\Phi, s) \cdot \frac{L(\Phi^{\vee}, 1 - s)}{L(\Phi, s)}$$

$$\varphi_0 : W_F \times SL_2(\mathbb{C}) \xrightarrow{\text{proj.}} SL_2(\mathbb{C}) \xrightarrow{\text{Sym}_{2n}} SO_{2n+1}(\mathbb{C})$$

$$L(\varphi_0, \mathrm{Ad}, s) = \prod_{k=1}^{n} \left(1 - q^{-(2k-1)} q^{-s}\right)^{-1}$$

$$\varepsilon(\varphi_0, \mathrm{Ad}, \psi, d_F(x), s) = w(\varphi_0, \mathrm{Ad}) \cdot q^{a(\varphi_0, \mathrm{Ad})(\frac{1}{2} - s)} \quad \begin{pmatrix} w(\varphi_0, \mathrm{Ad}) = 1, \\ a(\varphi_0, \mathrm{Ad}) = 2n^2 \end{pmatrix}$$

$$|\mathcal{A}_\varphi| = 2 \qquad \therefore \dim \chi = 1 \text{ for } \forall \chi \in \widehat{\mathcal{A}_\varphi}$$

<u>**Th**</u> (formal degree conjecture).

the formal degree of $\pi_{\beta,\theta}$ w.r. to abs. val. of Euler-Poincaré measure

$$= \frac{1}{|\mathcal{A}_\varphi|} \left| \frac{\gamma(\mathrm{Ad} \circ \varphi, 0)}{\gamma(\varphi_0, \mathrm{Ad}, 0)} \right|.$$

• **Structure of** $\mathrm{Ad}\circ(\boldsymbol{\Theta}\oplus\det\boldsymbol{\Theta})$ **($\boldsymbol{\Theta}=\mathrm{Ind}_{K^\times}^{W_{K/F}}\widetilde{\vartheta}$)**

$$\mathfrak{so}_{2n+1}(\mathbb{C}) = \{X\in M_{2n+1}(\mathbb{C}) \mid X + {}^tX = 0\}$$
$$\mathrm{Ad}(g)X = gX\,{}^tg \text{ for } g\in SO_{2n+1}(\mathbb{C})$$

and $(\det\boldsymbol{\Theta})|_{K^\times}=1$, hence

$$\begin{aligned}\mathrm{Ad}\circ(\boldsymbol{\Theta}\oplus\det\boldsymbol{\Theta}) &= \bigwedge^2(\boldsymbol{\Theta}\oplus\det\boldsymbol{\Theta})\\ &= \left(\bigwedge^2\boldsymbol{\Theta}\right)\oplus(\det\boldsymbol{\Theta}\otimes\boldsymbol{\Theta}) = \left(\bigwedge^2\boldsymbol{\Theta}\right)\oplus\boldsymbol{\Theta}\end{aligned}$$

$$\chi_{\boldsymbol{\Theta}}(g) = \begin{cases}0 & :\sigma\neq 1,\\ \displaystyle\sum_{\gamma\in\mathrm{Gal}(K/F)}\widetilde{\vartheta}(x^\gamma) & :\sigma=1\end{cases}, \quad \chi_{\wedge^2\boldsymbol{\Theta}}(g) = \frac{1}{2}\left\{\chi_{\boldsymbol{\Theta}}(g)^2 - \chi_{\boldsymbol{\Theta}}(g^2)\right\}$$

for $g=(\sigma,x)\in W_{K/F} = \mathrm{Gal}(K/F)\underset{\alpha_{K/F}}{\ltimes} K^\times$

The structure of the order-two elements of Gal(K/F)

K_0/F : maximal unramified subext. of K/F

$$\mathrm{Gal}(K/F) = \langle \delta, \rho \rangle \text{ with } \begin{cases} \mathrm{Gal}(K/K_0) = \langle \delta \rangle, \\ \rho|_{K_0} \in \mathrm{Gal}(K_0/F) : \text{Frobenius}^{-1} \end{cases}$$

- $\rho^{-1}\delta\rho = \delta^q$ (Iwasawa:Transaction A.M.S. (1955)),
- $\rho^f = \delta^m$ ($0 \leq m < e = e(K/F)$, $f = f(K/F)$)

Prop.

$$H = \{\gamma \in \mathsf{Gal}(K/F) \mid \gamma^2 = 1\} \subset Z(\mathsf{Gal}(K/F))$$

and

$$H = \begin{cases} \{1, \delta^{\frac{e}{2}}\} & : f = \text{odd or } \begin{cases} e = \text{even}, \\ m = \text{odd} \end{cases} \\ \{1, \rho^{\frac{f}{2}}\delta^{-\frac{m}{2}}\} & : e = \text{odd}\,, m = \text{even} \\ \{1, \rho^{\frac{f}{2}}\delta^{\frac{e-m}{2}}\} & : e = \text{odd}\,, m = \text{odd} \\ \{1, \delta^{\frac{e}{2}}, \rho^{\frac{f}{2}}\delta^{-\frac{m}{2}}, \rho^{\frac{f}{2}}\delta^{\frac{e-m}{2}}\} & : f = \text{even}\,, e = \text{even}\,, m = \text{even}. \end{cases}$$

Th.

$$\mathsf{Ad}\circ(\boldsymbol{\Theta}\oplus\det\boldsymbol{\Theta})$$
$$=\bigoplus_{\substack{\pi\in\mathsf{Gal}(K/F)\widehat{\ }\\ \pi(\tau)\neq 1}}\pi^{\dim\pi}\oplus\mathsf{Ind}_{K^\times}^{W_{K/F}}\widetilde{\vartheta}\oplus\bigoplus_{\{\gamma\neq\gamma^{-1}\}}\mathsf{Ind}_{K^\times}^{W_{K/F}}\widetilde{\vartheta}_\gamma\oplus\bigoplus_{\substack{\gamma^2=1\\ \gamma\neq 1,\tau}}\mathsf{Ind}_{W_{K/K_\gamma}}^{W_{K/F}}\chi_\gamma$$

where, for $1\neq\gamma\in\mathsf{Gal}(K/F)$,

$$\widetilde{\vartheta}_\gamma(x)=\widetilde{\vartheta}(x^{1+\gamma})\quad(x\in K^\times)$$

$$\chi_\gamma: W_{K/K_\gamma}=W_{K_\gamma}/\overline{[W_K,W_K]}\xrightarrow[\text{can.}]{}W_{K_\gamma}/\overline{[W_{K_\gamma},W_{K_\gamma}]}\xrightarrow[\text{l.c.f.t.}]{\sim}K_\gamma^\times\xrightarrow[(*,K/K_\gamma)\cdot\widetilde{\vartheta}]{}\mathbb{C}^\times$$

with

$$\mathsf{Gal}(K/K_\gamma)=\langle\gamma\rangle,\quad (x,K/K_\gamma)=\begin{cases}1 & : x\in N_{K/K_\gamma}(K^\times),\\ -1 & : x\notin N_{K/K_\gamma}(K^\times)\end{cases}$$

via $W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\text{proj.}} \mathrm{Gal}(K/F)$

$$\bigoplus_{\substack{\pi \in \mathrm{Gal}(K/F)\widehat{} \\ \pi(\tau)\neq 1}} \pi^{\dim \pi} = \mathrm{Ind}_{W_K}^{W_F} \mathbf{1}_K - \mathrm{Ind}_{W_{K_+}}^{W_F} \mathbf{1}_{K_+}$$

$\therefore \varepsilon(\mathrm{Ad} \circ \varphi, \psi, d_F(x), s)$ is a product of powers of q and Gauss sums:

$$G(\chi, \psi_K) = q^{-fn/2} \sum_{t \in (O_K/\mathfrak{p}_K^n)^\times} \chi\left(\varpi_K^{-(n+d)} t\right) \cdot \psi_K\left(\varpi_K^{-(n+d)} t\right)$$

for $\chi : K^\times \to \mathbb{C}^\times$ s.t. $\chi|_{O_K^\times} \neq 1$ with $f = f(K/F)$,

$$n = \mathrm{Min}\{0 < l \in \mathbb{Z} \mid \chi(1 + \mathfrak{p}_K^l) = 1\}, \qquad \mathcal{D}(K/\mathbb{Q}_p) = \mathfrak{p}_K^d$$

and

$$\psi_K : K \xrightarrow{T_{K/\mathbb{Q}_p}} \mathbb{Q}_p \xrightarrow{\text{can.}} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z} \xrightarrow{\exp(2\pi\sqrt{-1}*)} \mathbb{C}^\times$$

We have a trivial cancellation

$$G(\chi,\psi_K)\cdot G(\chi^{-1},\psi_K)=\chi(-1)$$

and

Th (Fröhlich-Queyrut;Inv.Math. (1973)).

$$K/E : \text{quad. ext. s.t. } K=E(\varepsilon),\varepsilon^2\in E^\times, \quad \chi|_{E^\times}=1$$
$$\Rightarrow \ G(\chi,\psi_K)=\chi(\varepsilon)$$

Then we have

Th. $w(\mathrm{Ad}\circ\varphi)=\theta(-1)$.

$$\pi_{\beta,\theta}=\text{cmp-ind}_{G(O)}^{G(F)}\delta_{\beta,\theta} \quad \therefore \pi_{\beta,\theta}(-1_{2n})=\delta_{\beta,\theta}(-1_{2n})=\theta(-1)$$

Cor (root number conjecture). $w(\mathrm{Ad}\circ\varphi)=\pi_{\beta,\theta}(-1_{2n})$

• **quasi-invariant symplectic form on $\mathbf{\Theta} = \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}$**

$\nu : \mathrm{Gal}(K/F) \to \mathbb{C}^\times$ s.t. $\nu(\tau) = -1$

$$D_\nu(\varphi, \psi) = \sum_{\gamma \in \mathrm{Gal}(K/F)} \nu(\gamma) \cdot \widetilde{\vartheta}(\alpha_{K/F}(\gamma, \tau))^{-1} \cdot \varphi(\gamma)\psi(\gamma\tau) \quad (\varphi, \psi \in V_{\mathbf{\Theta}})$$

:quasi-$W_{K/F}$-invariant symplectic form on $V_{\mathbf{\Theta}}$

$$\mathbf{\Theta}_\nu : W_{K/F} \xrightarrow{\mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}} GSp_{\mathbb{C}}(V_{\mathbf{\Theta}}, D_\nu) = GSp_{2n}(\mathbb{C})$$

<u>Sp_4 case</u>

$$\varphi_\nu : W_F \xrightarrow{\text{can.}} W_{K/F} \xrightarrow{\mathbf{\Theta}_\nu} GSp_4(\mathbb{C}) \xrightarrow{\text{can.}} GSp_4(\mathbb{C})/\mathbb{C}^\times 1_4 \xrightarrow{\simeq} SO_5(\mathbb{C})$$

<u>**Rem**</u>**.** *Two φ_ν's are conjugate by an inner automorphism of $SO_5(\mathbb{C})$.*

Sp_4 case

$$\mathrm{Gal}(K/F) : \text{ cyclic } \Leftrightarrow K/F : \text{ unramified or totally ramified}$$

Th. 1) *If* Gal(K/F) *is cyclic, the formal degree conjecture is valid for* $\pi_{\beta,\theta}$ *and* φ_ν*. In this case* $|\mathcal{A}_{\varphi_\nu}| = 2$*.*

2) *If* Gal(K/F) *is not cyclic, then* $|\mathcal{A}_{\varphi_\nu}| = 4$ *and*

$$\frac{1}{|\mathcal{A}_{\varphi_\nu}|} \cdot \left| \frac{\gamma(\mathrm{Ad} \circ \varphi_\nu, 0)}{\gamma(\varphi_0, \mathrm{Ad}, 0)} \right|$$

is $\frac{1}{2}$ *of the formal degree of* $\pi_{\beta,\theta}$ *with respect to the absolute value of the Euler-Poincaré measure.*

Sp_4 case

Th. *If* $\mathrm{Gal}(K/F) = \langle \rho \rangle$ *is cyclic*

1) $\mathrm{Ad} \circ \varphi_\nu = \bigoplus_{\substack{\chi \in \mathrm{Gal}(K/F)^{\widehat{}} \\ \chi(\tau) \neq 1}} \chi \oplus \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}^2 \oplus \mathrm{Ind}_{K^\times}^{W_{K/F}} \widetilde{\vartheta}_\rho$

2) $w(\mathrm{Ad} \circ \varphi_\nu) = \begin{cases} 1 & : \text{if } K/F \text{ is unramified,} \\ (-1)^{\frac{q-1}{4}} & : \text{if } K/F \text{ is totally ramified} \end{cases}.$