paper is available at math.berkeley.edu/~swshin,
notes for dim 1 case at ms.u-tokyo.ac.jp/~mieda/lectures202101.html

I. Intro

joint work with Arno Kret

1968 Igusa curve "level Np^m" (smooth over ord. locus)

$\downarrow$

modular curve $/\mathbb{F}_p$ level N $(N,p)=1$.

generalizes to ① Igusa towers $\subset$ ② Igusa varieties over mod p Shimura varieties.

① by Ribet, Faltings-Chai, Hida, ..., Eischen-Mantovan
$\rightsquigarrow$ p-adic auto. forms. "irreducibility". cf. van Hoften's talk.

② by Harris-Taylor, Oort, Mantovan, Hamacher, C. Zhang, Hamacher-Kim, Caraiani-Scholze.
$\rightsquigarrow$ product str, ℓ-adic cohomology, Langlands.

A long-term goal (in spirit of ②)
Compute ℓ-adic cohom. of Igusa var. $\circlearrowleft$ G_p action. (torsion)

This talk Compute H^0(Igusa) via auto. reps.
1-dim'l.
($\rightsquigarrow$ irreducibility of central leaves, Igusa var ~~extend~~ ①)

van Hoften, partly with L.X. Xiao. proved independently.

$\boxed{\text{II. Igusa var.}}$

Setup • (G, X) Hodge-type Shimura datum.

$\quad$ conn. reductive $/\mathbb{Q}$.

• p fixed prime > 2 (expected: $p=2$ ok)

• $K_p < G(\mathbb{Q}_p)$ hyperspecial

Mod p of Kisin's canonical integral model gives.

abelian scheme. $\mathcal{A}$ $\leftarrow$ with "G-str."

$$\downarrow$$

$$G(\mathbb{A}^{\infty,p}) \circlearrowright \mathcal{S}_{K_p} = \{ \mathcal{S}_{K_p K^p} \}_{K^p \subset G(\mathbb{A}^{\infty,p})} / \mathbb{F}_{p^s} \quad \text{smooth.}$$

Hecke. $\quad$ neat $\quad$ suppress.

Let $\Sigma/\overline{\mathbb{F}}_p$ be a p-divisible gp. $(+ G_{\mathbb{Q}_p}\text{-str})$

central leaf.

$$C_\Sigma := \{ x \in \mathcal{S}_{K_p} : \mathcal{A}_x[p^\infty] \text{ is isom. to } \Sigma \} \subset \mathcal{S}_{K_p, \overline{\mathbb{F}}_p}$$

loc. closed, smooth, equidim'l.

May assume $\neq \phi$.

Igusa var.

$$\mathrm{Ig}_\Sigma := \mathrm{Isom}(\Sigma, \mathcal{A}[p^\infty]) \circlearrowleft G(\mathbb{A}^{\infty,p}) \times \underbrace{J_\Sigma(\mathbb{Q}_p)}_{\overset{\text{def}}{=} \mathrm{Aut}^\circ_{\overline{\mathbb{F}}_p}(\Sigma)}$$

$\downarrow$ $\mathrm{Aut}(\Sigma)$-torsor

C_Σ

analog of Hecke action at p.

III. Main Thm

Fix $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$

("non-supersingular")

Assume · G^{ad} is $\mathbb{Q}$-simple, $\mathcal{I}$ is non-basic

Thm As $G(\mathbb{A}^{\infty,p}) \times J_{\mathcal{I}}(\mathbb{Q}_p)$-mod,

$$H^0(Ig_{\mathcal{I}}, \overline{\mathbb{Q}}_\ell) = \bigoplus_{\pi \in \mathcal{A}_1(G)} \pi^{\infty,p} \otimes \pi_p .$$

$\pi_p \circlearrowleft \circledast\ J_{\mathcal{I}}(\mathbb{Q}_p)$

inner form of Levi of $G_{\mathbb{Q}_p}$

where

- $\mathcal{A}_1(G) = \{$1-dim'l auto. reps of $G(\mathbb{A})$, "triv. at ∞"$\}$
- $\circledast$ is via $J_{\mathcal{I}}(\mathbb{Q}_p) \to J_{\mathcal{I}}(\mathbb{Q}_p)^{ab} \xrightarrow[\text{canon.}]{\sim} G(\mathbb{Q}_p)^{ab} \xrightarrow{\pi_p} \mathbb{C}^\times$

Rmk $\mathcal{I}$ is basic $\Leftrightarrow$ $\dim C_{\mathcal{I}} = \dim Ig_{\mathcal{I}} = 0$.

Then $H^0(Ig_{\mathcal{I}}, \overline{\mathbb{Q}}_\ell) \doteq \{$auto. forms on "global $J_{\mathcal{I}}$"$\}$

(inner form of G)

Cor (i) irred. of Igusa var. (beyond μ-ord)

(follows from $\circledast$)

take $\mathrm{Aut}(\mathcal{I})$-inv. on H^0

(ii) irred of $C_{\mathcal{I}}$. (a.k.a. disc. HO conj).

independently by van Hoften, + L.X. Xiao.

IV. Sketch of proof (geometry)

via automorphic harmonic analysis (AHA)

1. Reduce to: Σ is completely slope divisible.

$\rightsquigarrow$ $Ig_\Sigma \overset{/}{=} \varprojlim Ig_{\Sigma,m} = \varprojlim Isom(gr\,\Sigma[p^m], gr\,A[p^m])$.

up to perfection. (ignore)

m: finite level.

$\downarrow$

C_Σ.

$/\mathbb{F}_q$.

$d := \dim Ig_\Sigma$. $H^0(Ig_\Sigma) \longleftrightarrow H_c^{2d}(Ig_\Sigma)$.

2. Langlands–Kottwitz method for Igusa var.

... Describe $Ig_\Sigma(\overline{\mathbb{F}}_p) \circlearrowleft \mathcal{H}_p$, apply LTF, ... to obtain:

Thm ('21 thesis) (Mack-Crane) $\Leftarrow$ Kisin-S.-Zhu $\Leftarrow$ Kisin

$\forall\ \phi^{\infty,p} \times \phi_p \in C_c^\infty(G(\mathbb{A}^{\infty,p}) \times J_\Sigma(\mathbb{Q}_p))$,

$\exists$ tech. condition (harmless)

$$tr\left(\phi^{\infty,p} \times \phi_p \times \underbrace{\mathrm{Frob}_q^j}_{Z(J_\Sigma(\mathbb{Q}_p))} \middle| H_c^*(Ig_\Sigma)\right) = T_{ell}^G(\phi^{\infty,p} \times f_p^{(j)} \times f_\infty)$$

$+$ (endoscopic).

Lang-Weil estimate: order q^{dj}.

$$tr(\cdots | H_c^{2d}(Ig_\Sigma)) = (\text{leading term in } j)$$

3. (global HM) show by induction + ⋯,

- endoscopic $= o(q^{d_j})$
- $T^G_{ell} = T^G_{disc} + o(q^{d_j})$. ★★

(GKM, most technical)

our strategy ⇒ spectral interpret. of difference

<u>Outcome</u> $\mathrm{tr}\left(\phi^{\infty,p} \times \phi_p \times \mathrm{Frob}_q^j \,\middle|\, H_c^{2d}(Ig_{\Sigma})\right)$

$$= \mathrm{tr}\left(\phi^{\infty,p} \times f_p^{(j)} \times f_\infty \,\middle|\, L^2_{disc}([G])\right)$$

⊂ auto. spec of G

4. (local HM at p)

As $j \to \infty$, $\mathrm{tr}(f_p^{(j)} \mid \Pi_p)$ is controlled by central char of $\mathrm{Jac}(\Pi_p)$ at $\mathrm{Frob}_q^j \in Z(J_\Sigma(\mathbb{Q}_p))$.

By Howe-Moore + Casselman, leading term ⟺ $\dim \Pi_p = 1$.
(use non-basic)

strong approx. ⟺ $\dim \Pi = 1$

<u>Conclusion</u> $H_c^{2d}(Ig_\Sigma) = $ 1-dim'l part of $L^2_{disc}([G])$ triv. at ∞.

$G(\mathbb{Q}_p)$-action res to $J_\Sigma(\mathbb{Q}_p)$