Mod p points on Shimura varieties of parahoric level

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Structure of the talk

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Introduction to the Langlands-Rapoport conjecture and a quick survey of previous work.

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Idea of the proof.

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Understanding these integral models has interesting applications, e.g. construction of Galois representations (Deligne, Langlands), Ribet's proof of the ϵ-conjecture.

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Then, we describe elliptic curves lying in a single isogeny class $\mathscr{I}_\phi$ using (semi) linear algebra.

If we fix $E_0 \in \mathscr{I}_\phi$ and $E \sim E_0$ via a prime-to-p isogeny, then E corresponds to a $\widehat{\mathbb{Z}}^p$-lattice in $T^p E_0 \otimes \mathbb{A}_f^p$. If $E \sim E_0$ via a p-power isogeny, then E corresponds to a Dieudonné-lattice in the p-adic Dieudonné-module of E_0.

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$$X^p(\phi) = \{\text{Lattices in } T^p E_0\}$$
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Then $\mathscr{I}_\phi$ is almost equal to their product. Precisely

$$\mathscr{I}_\phi \simeq I_\phi(\mathbb{Q}) \backslash X_p(\phi) \times X^p(\phi)/K^p, \qquad (1)$$

where I_ϕ is the automorphism group of E_0, up to isogeny.

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For example $G = \mathrm{GL}_2, X = \mathbb{H}^{\pm}$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ or $K_p = \Gamma_0(p)$, then $E = \mathbb{Q}$ and the integral models from the previous slide are 'canonical'.

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Here $X_p(\phi)$ parametrises p-power isogenies, $X^p(\phi)$ parametrises prime-to-p isogenies and $I_\phi(\mathbb{Q})$ is the group of self quasi-isogenies.

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Let (G, X) be a Shimura datum of abelian type, let $p > 2$ and suppose that $G_{\mathbb{Q}_p}$ is unramified and that K_p is hyperspecial. Then the Langlands-Rapoport conjecture* holds for (G, X, p).

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Theorem (Zhou, 2017)

Let (G, X) be a Shimura datum of Hodge type, let $p > 2$ and suppose that $G_{\mathbb{Q}_p}$ is residually split, then isogeny classes have Rapoport-Zink uniformisation for arbitrary parahorics K_p.

Main Results I

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Remark

We can also prove Theorem 1 when $G_{\mathbb{Q}_p}$ is quasi-split and tamely ramified, under the assumption that $\mathrm{Sh}_K(G, X)$ is proper.

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For Hodge type Shimura varieties, the integral models do not have a moduli interpretation, which makes it difficult to make the above strategy work. We can still associate a p-divisible group with extra structures X to an $\overline{\mathbb{F}}_p$-point, but it is no longer clear that the fiber only depends on this X.

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The LR conjecture holds for the Shimura variety in the top left corner if and only if the diagram is Cartesian.

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This last result is new even for $S_{K',\overline{\mathbb{F}}_p}(G, X)$!

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This isn't entirely true, but we can deal with the other cases (those contained in the basic locus) using results of Zhou.

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- Monodromy arguments à la Chai

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Thank you for your attention!