Twisted endoscopic character relation for Kaletha's regular supercuspidal L-packets

Masao Oi

Kyoto University (Hakubi center)

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Local Langlands correspondence

- F: a p-adic field.
- $\mathbf{G}$: a connected reductive group over F ($G := \mathbf{G}(F)$).
- $\Pi(\mathbf{G}) := \{$irreducible admissible representations of $G\}/\sim$,
- $\Phi(\mathbf{G}) := \{L$-parameters of $\mathbf{G}\}/\sim$.

Local Langlands correspondence (LLC)

There exists a natural map with finite fibers:

$$\mathrm{LLC}_{\mathbf{G}} \colon \Pi(\mathbf{G}) \to \Phi(\mathbf{G}).$$

In other words, there exists a natural partition of the set $\Pi(\mathbf{G})$ into subsets (which are finite, called L-packets) parametrized by L-parameters:

$$\Pi(\mathbf{G}) = \bigsqcup_{\phi \in \Phi(\mathbf{G})} \Pi_{\phi}^{\mathbf{G}} \quad (\Pi_{\phi}^{\mathbf{G}} := \mathrm{LLC}_{\mathbf{G}}^{-1}(\phi)).$$

- LLC is still conjectural in general, but a number of results have been obtained.

Motivation: comparison of different constructions

Approach 1: Specialize the group. For example,

- GL_N; Harris–Taylor
- quasi-split classical groups (Sp_{2n}, SO_N, U_N); Arthur, Mok

Approach 2: Specialize the class of representations. For example,

- regular supercuspidal representations; Kaletha
  (works for tamely ramified groups)
  (he dropped the regularity recently; `arXiv:1912.03274`)

Q. Do the two approaches give the same LLC (on their "intersection")?

Theorem (O.-Tokimoto, 2019)

We assume that $p \neq 2$. Kaletha's and Harris–Taylor's LLC coincide for regular supercuspidal representations of GL_N.

Q. Kaletha's LLC $=$ Arthur's (Mok's) LLC?

Key: Arthur's LLC is characterized by the twisted endoscopic character relation.

Twisted endoscopic character relation (TECR)

- $\theta \curvearrowright \mathbf{G}$: a "twist" (i.e., rational automorphism preserving a pinning)
- $\mathbf{H}$: an endoscopic group for $(\mathbf{G}, \theta)$
  $=$ a quasi-split connected reductive group $\mathbf{H}$ over F equipped with
  $\hat{\xi}\colon {}^L\mathbf{H} \hookrightarrow {}^L\mathbf{G}$ such that $\hat{\xi}(\hat{\mathbf{H}}) = \hat{\mathbf{G}}^{\mathrm{Int}(s)\circ\hat{\theta},0}$ for $s \in \hat{\mathbf{G}}$.
- The existences of $\mathrm{LLC}_{\mathbf{H}}$ and $\mathrm{LLC}_{\mathbf{G}}$ induces a lifting of L-packets:

$$
\begin{tikzcd}
\Pi_\phi^{\mathbf{G}} \arrow[r, dashed, leftrightarrow, "\text{LLC for } \mathbf{G}"] & {} & & {}^L\mathbf{G} \\
\Pi_\phi^{\mathbf{H}} \arrow[u, rightsquigarrow, "\text{endoscopic lifting}"] \arrow[r, dashed, leftrightarrow, "\text{LLC for } \mathbf{H}"] & {} & W_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\phi"'] \arrow[ur, "\hat{\xi}\circ\phi"] & {}^L\mathbf{H} \arrow[u, hook, "\hat{\xi}"']
\end{tikzcd}
$$

Twisted endoscopic character relation
$=$ an identity between (twisted) characters of representations in $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$,
which characterizes the map $\Pi_\phi^{\mathbf{H}} \mapsto \Pi_\phi^{\mathbf{G}}$

Goal of my ongoing project

Kaletha's LLC for regular supercuspidal representations satisfies the TECR.

Regular supercuspidal representations

- Assume: $\mathbf{G}$ is tamely ramified and $p \gg 0$.
- In 2001, Yu constructed a certain wide class of supercuspidal representations.
- Kaletha reparametrized "regular" Yu-data by "tame elliptic regular pairs"; $(\mathbf{S}, \vartheta)$ consists of
  - $\mathbf{S} \subset \mathbf{G}$: a tamely ramified elliptic maximal torus defined over F,
  - $\vartheta\colon \mathbf{S}(F) \to \mathbb{C}^{\times}$: a "regular" character.

$$\begin{tikzcd}
\{\text{Yu-data}\}/{\sim} \arrow[rr, "1:1"] & & \{\text{Yu s.c. rep'ns of } \mathbf{G}(F)\}/{\sim} \\
\{\text{reg. Yu-data}\}/{\sim} \arrow[u, hook] \arrow[rr, "1:1"] & & \{\text{reg. s.c. rep'ns of } \mathbf{G}(F)\}/{\sim} \arrow[u, hook] \\
\{\text{tame ell. reg. pairs}\}/\mathbf{G}(F)\text{-conj.} \arrow[u, leftrightarrow, "1:1"] \arrow[urr, "(\mathbf{S},\vartheta)\mapsto\pi_{(\mathbf{S},\vartheta)}"'] & &
\end{tikzcd}$$

Construction of L-parameters/L-packets

- $(\mathbf{S}, \vartheta)$: a tame elliptic regular pair of $\mathbf{G}$.

Construction of an L-parameter ϕ

- LLC for $\mathbf{S}$ gives $\phi_\vartheta \colon W_F \to {}^L\mathbf{S}$.
- Langlands–Shelstad construction gives ${}^Lj_\chi \colon {}^L\mathbf{S} \hookrightarrow {}^L\mathbf{G}$.
- Then we get an L-parameter ϕ of $\mathbf{G}$ ("regular supercuspidal L-parameter"):

$$\phi \colon W_F \xrightarrow{\phi_\vartheta} {}^L\mathbf{S} \xrightarrow{{}^Lj_\chi} {}^L\mathbf{G}.$$

Construction of an L-packet $\Pi_\phi^{\mathbf{G}}$

- Put $\mathcal{J}^{\mathbf{G}}$ to be the stable $\mathbf{G}$-conjugacy class of the embedding $\mathbf{S} \hookrightarrow \mathbf{G}$ and

$$\mathcal{J}_G^{\mathbf{G}} := \{ j \in \mathcal{J}^{\mathbf{G}} \mid \text{defined over } F \}/G\text{-conj}.$$

$\rightsquigarrow$ for each $j \in \mathcal{J}_G^{\mathbf{G}}$, we get a tame elliptic regular pair $(j(\mathbf{S}), \vartheta_j)$ of $\mathbf{G}$.

- Put

$$\Pi_\phi^{\mathbf{G}} := \{ \pi_j \mid j \in \mathcal{J}_G^{\mathbf{G}} \}, \quad \pi_j := \pi_{(j(\mathbf{S}), \vartheta_j)}.$$

Setting of the problem

- $\mathbf{G}$: a quasi-split connected reductive group with θ over F.
- $\mathbf{H}$: an endoscopic group for $(\mathbf{G}, \theta)$.
- Suppose: $\mathbf{G}$ and $\mathbf{H}$ are tamely ramified and $p \gg 0$.
- ϕ: a "toral" regular supercuspidal L-parameter of $\mathbf{G}$.
- Assume: ϕ factors through ${}^{L}\mathbf{H} \hookrightarrow {}^{L}\mathbf{G}$.

  $\rightsquigarrow$ ϕ is regarded as an L-parameter of $\mathbf{H}$ (again toral).

$$\begin{tikzcd}
\Pi_{\phi}^{\mathbf{G}} \arrow[r, leftrightarrow, "\text{Kaletha's LLC for } \mathbf{G}"] & {} \\
\Pi_{\phi}^{\mathbf{H}} \arrow[u, rightsquigarrow, "\text{want to show ECR}"] \arrow[r, leftrightarrow, "\text{Kaletha's LLC for } \mathbf{H}"] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
W_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\phi"] \arrow[dr, "\phi"'] & {}^{L}\mathbf{G} \\
& {}^{L}\mathbf{H} \arrow[u, hook, "\hat{\xi}"']
\end{tikzcd}$$

- $\Pi_{\phi}^{\mathbf{G}}$ is θ-stable, i.e., $\Pi_{\phi}^{\mathbf{G}} \circ \theta = \Pi_{\phi}^{\mathbf{G}}$ as sets.

Precise statement of main result

Main result at present (ECR for $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$)

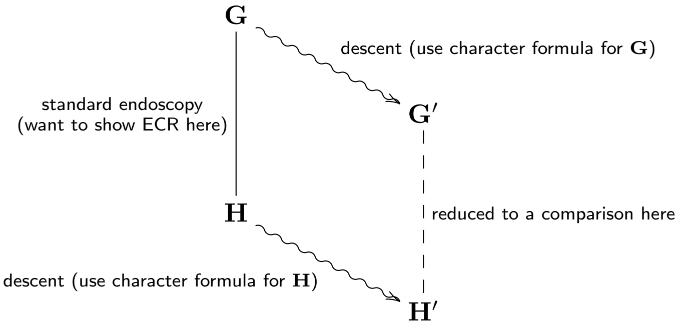
For an elliptic strongly regular s-s. element $\delta \in \tilde{G}$ having a norm in H, we have

$$\sum_{\pi \in \Pi_\phi^{\mathbf{G}}} \Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi)\tilde{\Theta}_\pi(\delta) = \sum_{\gamma \in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2} \Delta_{\mathbf{H},\mathbf{G}}(\gamma, \delta) \sum_{\pi_{\mathbf{H}} \in \Pi_\phi^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma).$$

- $\Theta_{\pi_{\mathbf{H}}}$ is the Harish-Chandra character of $\pi_{\mathbf{H}} \in \Pi_\phi^{\mathbf{H}}$.
- $\tilde{\Theta}_\pi$ is the θ-twisted character of π.
  (We follow the formalism of twisted space $(\mathbf{G}, \tilde{\mathbf{G}})$; $\tilde{\mathbf{G}} := \mathbf{G} \rtimes \theta$.)
- $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi) \in \mathbb{C}$ is an explicit constant ("spectral transfer factor").
  ($\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\pi) \neq 0$ if and only if π is θ-stable)
- γ runs over stable conjugacy classes of norms of δ in H.
- $D_{\mathbf{H}}$ and $D_{\tilde{\mathbf{G}}}$ denote the Weyl discriminants.
- $\Delta_{\mathbf{H},\mathbf{G}}(\gamma, \delta)$ is the Langlands–Shelstad–Kottwitz transfer factor.

Strategy: imitate Kaletha's proof in the standard case (i.e., $\theta = \mathrm{id}$).

Rough outline of Kaletha's proof



- Adler–DeBacker–Spice formula describes the characters in terms of a descended group $\mathbf{G}'$ (or $\mathbf{H}'$).
- Langlands–Shelstad descent gives $(\mathbf{G}', \mathbf{H}')$ a structure of endoscopy.
- Waldspurger–Ngô transfer gives a comparison between Lie algebras of $\mathbf{G}'$ and $\mathbf{H}'$. For this, we need
  - Kaletha's descent lemma, and
  - Kaletha's refinement of ADS formula.

Adler–DeBacker–Spice character formula

- Let $(\mathbf{S}, \vartheta)$ be a toral tame elliptic regular pair of depth $r \in \mathbb{R}_{>0}$.
- Let $\delta \in G$ be an elliptic strongly regular semisimple element.
- Take a normal r-approximation $\delta = \delta_{<r} \cdot \delta_{\geq r}$;
  - $\delta_{<r}$: "p-adically shallower than r"-part of δ,
  - $\delta_{\geq r}$: "p-adically deeper than r"-part of δ.
  - $\delta_{\geq r} \in \mathbf{G}_{\delta_{<r}} := \mathrm{Cent}_{\mathbf{G}}(\delta_{<r})^{\circ}$.

Prop (1st form of ADS formula)

$$\Theta_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ g\delta_{<r}g^{-1} \in S}} \Theta_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\big(\log(\delta_{\geq r})\big).$$

- $\pi_{(\mathbf{S},\vartheta)} = \text{c-Ind}_{K_{\sigma}}^{\mathbf{G}(F)} \sigma$ for certain open compact (mod-center) subgroup K_{σ}.
- $X^{*} \in (\mathrm{Lie}\,\mathbf{S})^{*}$ is an element representing $\vartheta|_{S_r}$.
- $\hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}$ is the Fourier transform of the orbital integral on $\mathrm{Lie}\,\mathbf{G}_{\delta_{<r}}$. (with respect to the $\mathbf{G}_{\delta_{<r}}(F)$-orbit of $g^{-1}X^{*}g$)

Langlands–Shelstad descent & reduction to Lie algebras

- Recall:

$$\Pi_\phi^{\mathbf{G}} \stackrel{1:1}{\longleftrightarrow} \mathcal{J}_G^{\mathbf{G}} := \{j \in \mathcal{J}^{\mathbf{G}} \mid j\colon \mathbf{S} \hookrightarrow \mathbf{G}\text{: defined over } F\}/{\sim_G},$$

$$\Pi_\phi^{\mathbf{H}} \stackrel{1:1}{\longleftrightarrow} \mathcal{J}_H^{\mathbf{H}} := \{j_{\mathbf{H}} \in \mathcal{J}^{\mathbf{H}} \mid j_{\mathbf{H}}\colon \mathbf{S}_{\mathbf{H}} \hookrightarrow \mathbf{H}\text{: defined over } F\}/{\sim_H}.$$

- For each $j \in \mathcal{J}_G^{\mathbf{G}}$ and $j_{\mathbf{H}} \in \mathcal{J}_H^{\mathbf{H}}$, we have

  $\mathbf{G}$-side: $\Theta_{\pi_j}(\delta) = \sum \Theta_{\sigma_j}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X_j^* g}(\log(\delta_{\geq r}))$

  $\mathbf{H}$-side: $\Theta_{\pi_{j_{\mathbf{H}}}}(\gamma) = \sum \Theta_{\sigma_{\mathbf{H},j_{\mathbf{H}}}}(h\gamma_{<r}h^{-1}) \cdot \hat{\mu}^{\mathbf{H}_{\gamma_{<r}}}_{h^{-1}X^*_{\mathbf{H},j_{\mathbf{H}}}h}(\log(\gamma_{\geq r}))$

The Langlands–Shelstad descent gives an endoscopic structure on $(\mathbf{H}_{\gamma_{<r}}, \mathbf{G}_{\delta_{<r}})$.

$$\begin{tikzcd}
\mathbf{G} \arrow[d, dash, "\text{standard endoscopy}"'] \arrow[dr, squiggly, "\text{descent}"] & \\
\mathbf{H} \arrow[dr, squiggly, "\text{descent}"'] & \mathbf{G}_{\delta_{<r}} =: \mathbf{G}' \arrow[d, dash, "\text{standard endoscopy}"] \\
 & \mathbf{H}_{\gamma_{<r}} =: \mathbf{H}'
\end{tikzcd}$$

Waldspurger–Ngô transfer on Lie algebras

Waldspurger–Ngô transfer (= Lie algebra version of SECR)

$$\gamma(\mathfrak{g}') \sum_{X'^* \sim_{\mathrm{st}} X^*} \Delta_{\mathbf{H}',\mathbf{G}'}(Y^*, X'^*)\hat{\mu}^{\mathbf{G}'}_{X'^*}(X)$$

$$= \gamma(\mathfrak{h}') \sum_{Y/\mathrm{st}} \Delta_{\mathbf{H}',\mathbf{G}'}(Y, X) \sum_{Y'^* \sim_{\mathrm{st}} Y^*} \hat{\mu}^{\mathbf{H}'}_{Y'^*}(Y)$$

- index sets: G' (resp. H')-conj. classes in a $\mathbf{G}'$ (resp. $\mathbf{H}'$)-conj. class.

Recall: $\mathbf{G}$-side of SECR

$$\sum_{j \in \mathcal{J}_G^{\mathbf{G}}} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j)\Theta_{\pi_j}(\delta) = \sum_{j \in \mathcal{J}_G^{\mathbf{G}}} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) \sum_{\substack{g \in j(S)\backslash G/G' \\ g\delta_{<r}g^{-1} \in j(S)}} \Theta_{\sigma_j}({}^{g}\delta_{<r}) \cdot \hat{\mu}^{\mathbf{G}'}_{X_j^*, g}(\log(\delta_{\geq r}))$$

$\rightsquigarrow$ We need to consider a descent and transfer of index sets!

Transfer of index sets; Kaletha's descent lemma

- The index set of ADS formula is understood as

$$\mathcal{J}_{G'}^{G}(j) := \{k \in j \mid k\colon \mathbf{S} \hookrightarrow \mathbf{G}' \subset \mathbf{G}\colon \text{ defined over } F\}/{\sim_{G'}}.$$

$$\{g \in j(S)\backslash G/G' \mid g\delta_{<r}g^{-1} \in j(S)\} \xleftrightarrow{1:1} \mathcal{J}_{G'}^{G}(j)\colon g \mapsto \mathrm{Int}(g^{-1}) \circ j$$

- Combine $\mathcal{J}_G^{\mathbf{G}}$ with $\mathcal{J}_{G'}^{G}$ and divide it again via stable $\mathbf{G}'$-conjugacy.

$$\begin{tikzcd}
& \mathbf{G} \arrow[dl, dash, "\mathcal{J}_G^{\mathbf{G}}"'] \arrow[dr, dash, "\mathcal{J}_{\mathbf{G}'}^{\mathbf{G}}"] \arrow[dd, dash, "\mathcal{J}_{G'}^{\mathbf{G}}"] & \\
G \arrow[dr, dash, "\mathcal{J}_{G'}^{G}"'] & & \mathbf{G}' \arrow[dl, dash, "\mathcal{J}_{G'}^{\mathbf{G}'}"] \\
& G' &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{H} \arrow[dl, dash, "\mathcal{J}_H^{\mathbf{H}}"'] \arrow[dr, dash, "\mathcal{J}_{\mathbf{H}'}^{\mathbf{H}}"] \arrow[dd, dash, "\mathcal{J}_{H'}^{\mathbf{H}}"] & \\
H \arrow[dr, dash, "\mathcal{J}_{H'}^{H}"'] & & \mathbf{H}' \arrow[dl, dash, "\mathcal{J}_{H'}^{\mathbf{H}'}"] \\
& H' &
\end{tikzcd}$$

- Kaletha's descent lemma relates $\mathcal{J}_{\mathbf{G}'}^{\mathbf{G}}$ to $\mathcal{J}_{\mathbf{H}'}^{\mathbf{H}}$.
- The Waldspurger–Ngô transfer relates $\mathcal{J}_{G'}^{\mathbf{G}'}$ to $\mathcal{J}_{H'}^{\mathbf{H}'}$.

Computation of the contributions of shallow parts

1st form of ADS formula

$$\Theta_{\pi_j}(\delta) = \sum_{\substack{g\in j(S)\backslash G/G' \\ g\delta_{<r}g^{-1}\in j(S)}} \Theta_\sigma(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}'}_{g^{-1}X_j^* g}\big(\log(\delta_{\geq r})\big).$$

- Adler–Spice ('09) computed Θ_σ explicitly.
- DeBacker–Spice ('18) sophisticated it based on a root-theoretic language.
- Kaletha ('19) rewrote it via endoscopic invariants such as transfer factors.

ADS formula rewritten by Kaletha

$$\Theta_{\pi_j}(\delta) = \frac{e(\mathbf{G})}{e(\mathbf{G}')} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}} - \mathbf{T}_{\mathbf{G}'^*})}{D_{\mathbf{G}}(\delta)}$$

$$\cdot \sum_{\substack{g\in j(S)\backslash G/G' \\ g\delta_{<r}g^{-1}\in j(S)}} \Delta_{\mathrm{II}}^{\mathbf{G},\mathrm{abs}}(g\delta_{<r}g^{-1}) \cdot \vartheta\circ j^{-1}(g\delta_{<r}g^{-1}) \cdot \hat{\iota}^{\mathbf{G}'}_{g^{-1}X_j^* g}\big(\log(\delta_{\geq r})\big).$$

$\rightsquigarrow$ Finally, by putting $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) := \langle \mathrm{inv}(j_{\mathfrak{w}}, j), s\rangle$, we get SECR.

Twisted version of the Adler–DeBacker–Spice formula

- Let $\delta \in \tilde{G}$ be an elliptic strongly regular semisimple element.
- Suppose: δ has a norm $\gamma \in H$.
  $\rightsquigarrow$ By transferring $\gamma = \gamma_{<r} \cdot \gamma_{\geq r}$ from H to $\tilde{G}$, get $\delta = \delta_{<r} \cdot \delta_{\geq r}$.
  (Note: $\delta_{<r} \in \tilde{G}$ and $\delta_{\geq r} \in G_{\delta_{<r}}$)

1st form of twisted ADS formula

$$\tilde{\Theta}_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in \tilde{S}}} \tilde{\Theta}_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\bigl(\exp^{-1}(\delta_{\geq r})\bigr).$$

- This is completely parallel to the standard case!
  $\rightsquigarrow$ can imitate Kaletha's proof if we have a twisted version of LS descent...?

twisted version of Langlands–Shelstad descent
$=$ Waldspurger's framework **"l'endoscopie tordue n'est pas si tordue"**.

L'endoscopie tordue n'est pas si tordue

- Waldspurger constructed another connected reductive group $\bar{\mathbf{H}}$ relating $\mathbf{G}_{\delta_{<r}}$ to $\mathbf{H}_{\gamma_{<r}}$ via standard and non-standard endoscopy.
- Then he proved that
  - Fourier transforms of orbital integrals are transferred between $\mathbf{H}'_{\mathrm{SC}}$ and $\bar{\mathbf{H}}_{\mathrm{SC}}$,
  - transfer factor for $(\mathbf{H}, \tilde{\mathbf{G}})$ is descended to $(\bar{\mathbf{H}}, \mathbf{G}'_{\mathrm{SC}})$.

$$
\begin{tikzcd}
\tilde{\mathbf{G}} \arrow[dr, rightsquigarrow, "\text{descent}"] \arrow[dd, no head, "\text{twisted endoscopy}"'] & & & \\
 & \mathbf{G}_{\delta_{<r}} =: \mathbf{G}' & \mathbf{G}'_{\mathrm{SC}} \arrow[l] \arrow[d, no head, "\text{standard endoscopy}"] & \\
\mathbf{H} \arrow[dr, rightsquigarrow, "\text{descent}"'] & & \bar{\mathbf{H}} & \bar{\mathbf{H}}_{\mathrm{SC}} \arrow[l] \arrow[d, no head, "\text{non-standard endoscopy}"] \\
 & \mathbf{H}_{\gamma_{<r}} =: \mathbf{H}' & & \mathbf{H}'_{\mathrm{SC}} \arrow[ll]
\end{tikzcd}
$$

Twisted version of Kaletha's descent lemma

- In the standard case, a map $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}'} \to \mathcal{J}^{\mathbf{G}}_{\mathbf{G}'}$ of Kaletha's descent lemma was constructed via admissible isomorphisms (in the sense of LSK):

$$
\begin{tikzcd}
 & \mathbf{S}_{\mathbf{H}} \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "j_{\mathbf{H}}"'] & \mathbf{S} \arrow[d, "j"'] & \\
\mathbf{H}' & \mathbf{T}^{\flat} \arrow[l, hook'] \arrow[r, leftrightarrow, "\text{adm. isom.}", "\cong"'] & \mathbf{T}^{\diamondsuit} \arrow[r, hook] & \mathbf{G}'
\end{tikzcd}
$$

Unfortunately, we cannot simply imitate this construction in the twisted case.

$$
\begin{tikzcd}
 & \mathbf{S}_{\mathbf{H}} \arrow[r, leftrightarrow, "\cong"'] \arrow[d, "j_{\mathbf{H}}"'] & \mathbf{S}_{\theta_{\mathbf{S}}} & \mathbf{S} \arrow[l, two heads] \arrow[d, dashed, "\exists ?"] \\
\mathbf{H}' & \mathbf{T}^{\flat} \arrow[l, hook'] \arrow[r, leftrightarrow, "\text{adm. isom.}", "\cong"'] & \mathbf{T}^{\diamondsuit}_{\theta_{\diamondsuit}} & \mathbf{T}^{\diamondsuit} \arrow[l, two heads]
\end{tikzcd}
$$

$\rightsquigarrow$ We need to "rigidify" admissible isomorphisms in some way.

Waldspurger's "diagram"

- We utilize Waldspurger's notion of a diagram: $(\mathbf{T}^\flat, \mathbf{T}_0, \mathbf{T}^\diamondsuit, \mathbf{T}^\natural, h, g_0, g_1)$.

$$\begin{tikzcd}
\mathbf{T}^\flat \arrow[r, leftrightarrow, "\text{adm. isom.}", "\cong"'] \arrow[dd, "\mathrm{Int}(h)"'] & \mathbf{T}^{\diamondsuit}_{\theta_{\diamondsuit}} \arrow[d] & \mathbf{T}^\diamondsuit \arrow[l, twoheadrightarrow] \arrow[d, "\mathrm{Int}(g_1)"] \\
 & \mathbf{T}_{0,\theta} \arrow[d] & \mathbf{T}_0 \arrow[l, twoheadrightarrow] \arrow[d, "\mathrm{Int}(g_0)"] \\
\mathbf{T}_{\mathbf{H}} \arrow[r, leftrightarrow] & \mathbf{T}_\theta & \mathbf{T} \arrow[l, twoheadrightarrow]
\end{tikzcd}$$

- A diagram encodes information about how an admissible isomorphism is given by conjugation (i.e., h, g_0, g_1).
- We can formulate a twisted version of Kaletha's descent lemma (which relates $\mathcal{J}^{\mathbf{H}}_{\mathbf{H}'}$ to $\mathcal{J}^{\mathbf{G}}_{\mathbf{G}'}$) via diagrams.

$\rightsquigarrow$ get a comparison of index sets

Computation of the contributions of shallow parts

1st form of twisted ADS formula

$$\tilde{\Theta}_{\pi_{(\mathbf{S},\vartheta)}}(\delta) = \sum_{\substack{g \in S\backslash G/G_{\delta_{<r}} \\ {}^{g}\delta_{<r} \in \tilde{S}}} \tilde{\Theta}_{\sigma}(g\delta_{<r}g^{-1}) \cdot \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}\bigl(\exp^{-1}(\delta_{\geq r})\bigr).$$

- The contribution of the head is eventually reduced to a computation of twisted characters of Weil representations of finite Heisenberg groups.
- We upgrade it to "twisted ADSK" by looking at the definition of Δ in the twisted case carefully.

Proposition (twisted version of ADSK formula)

$$\tilde{\Theta}_{\pi_j}(\delta) = \frac{e(\mathbf{G}_\theta)}{e(\mathbf{G}')} \cdot \frac{\varepsilon(\mathbf{T}_{\mathbf{G}_\theta} - \mathbf{T}_{\mathbf{G}'^{*}})}{D_{\tilde{\mathbf{G}}}(\delta)}$$

$$\cdot \sum_{\substack{g \in j(S)\backslash G/G' \\ g\delta_{<r}g^{-1} \in \tilde{S}_{\tilde{j}}}} \Delta^{\tilde{\mathbf{G}},\mathrm{abs}}_{\mathrm{II}}(g\delta_{<r}g^{-1}) \cdot \tilde{\vartheta} \circ \tilde{j}^{-1}(g\delta_{<r}g^{-1}) \cdot \hat{\iota}^{\mathbf{G}'}_{g^{-1}X^{*}_{j}g}\bigl(\log(\delta_{\geq r})\bigr).$$

Spectral transfer factors in twisted endoscopy

$$\sum_{\pi\in\Pi_\phi^{\mathbf{G}}} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi)\tilde{\Theta}_\pi(\delta) = \sum_{\gamma\in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2}\Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta) \sum_{\pi_{\mathbf{H}}\in\Pi_\phi^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma),$$

- We put

$$\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi_j) := \Delta_{\mathrm{I,III}}\big(\gamma_{<r}\cdot\exp(X^*_{\mathbf{H},j_{\mathbf{H}}}), \delta_{<r}\cdot\exp(X^*_j)\big)\cdot\frac{\vartheta_{\mathbf{H}}\circ j_{\mathbf{H}}^{-1}(\gamma_{<r})}{\tilde{\vartheta}\circ\tilde{j}^{-1}(\delta_{<r})}.$$

$\rightsquigarrow$ In fact, $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi)$ depends only on π.

- standard case: $\Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\pi) = \langle\mathrm{inv}(j_{\mathfrak{w}}, j), s\rangle$ by Kaletha

Remark. I borrowed this idea from the argument of Mezo in his proof of the twisted ECR for L-packets of real reductive groups (constructed by Langlands).