Nonvanishing of Central Values of Rankin-Selberg L-functions

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Motivation

Question:

How likely $L\left(*, \frac{1}{2}\right) \neq 0$?

Some Results (GL_1)

Jutila (1981): There are infinitely many characters χ (quadratic with prime conductor) for which $L(\chi, \frac{1}{2}) \neq 0$.

Balasubramanian-Murty (1992): $L(\chi, \frac{1}{2}) \neq 0$ for at least 1/25 of the Dirichlet characters χ with prime conductor.

Iwaniec-Sarnak (1997): $L(\chi, \frac{1}{2}) \neq 0$ for at least 1/3 of the even primitive characters.

Soundararajan (2000) $L(\chi, \frac{1}{2}) \neq 0$ for at least 7/8 of the quadratic characters with conductor divisible by 8.

Some Results (GL_2)

Duke (1995): There is a positive absolute constant C such that there are at least $CN/\log^2 N$ newforms $f \in S_2(\Gamma_0(N))$ for which $L(f, \frac{1}{2}) \neq 0$, provided $N = 11$ or $N > 13$ is prime.

Iwaniec-Sarnak (2000): Let $k \in 2\mathbb{Z}$ be fixed. As $N \to \infty$, at least 50% of the values $L(f, \frac{1}{2})$ are nonvanishing while f varying among the newforms in $S_k(\Gamma_0(N))$ whose functional equations are even.

Lau-Tsang (2005): There is a positive absolute constant C such that there are at least $Ck/\log^2 k$ newforms $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ for which $L(f, \frac{1}{2}) \neq 0$, as $k \to \infty$ with $k \equiv 0 \mod 4$.

Luo (2015): There is a positive absolute constant C such that there are at least Ck newforms $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ for which $L(f, \frac{1}{2}) \neq 0$, as $k \to \infty$ with $k \equiv 0 \mod 4$ (a positive proportion result).

Some Result ($\mathrm{GL}_2 \times \mathrm{GL}_2$)

Rankin-Selberg L-functions: $L(f \otimes g, s)$

$$L(f \otimes g, s) = \zeta^*(2s) \sum_{n=1}^{\infty} \frac{c_f(n) c_g(n)}{n^s}$$

Kowalski-Michel-Vanderkam (2002): Let g be a newform in $\mathcal{S}_l(\Gamma_0(N))$ with l even and N square-free. There exists $c > 0$ such that, given g, $k < 12$ and any sufficiently large prime q, then

$$\frac{\#\{f \in S_k^*(q) : L(f \otimes g, \frac{1}{2}) \neq 0\}}{\# S_k^*(q)} \geq c + o_g(1).$$

Question

Can we establish similar nonvanishing results in the setting of Hilbert modular forms (in the weight aspect)?

Setting

- F: totally real number field of degree n such that $\zeta_F(s)$ has no Landau-Siegel zero.
- $\mathcal{O}_F$: ring of integers in F, $\mathfrak{d}_F$: different ideal of F, d_F: discriminant of F
- h_F: the narrow class number
  - $Cl^+(F) = I(F)/P^+(F)$
  - $\{\mathfrak{a}_\nu\}_{\nu=1}^{h_F}$: a set of representatives of the narrow class group
- embeddings of F: $\{\sigma_1, \cdots, \sigma_n\}$.
  - For $x \in F$ and $j \in \{1, \ldots, n\}$, we set $x_j = \sigma_j(x)$

Hilbert Modular Form

$f_\nu \in \mathcal{S}_k(\Gamma_{\mathfrak{a}_\nu}(\mathfrak{n}))$

- $k = (k_1, \ldots, k_n)$ with $k_1 \equiv \cdots \equiv k_n \equiv 0 \mod 2$
- $f_\nu : \mathfrak{h}^n \to \mathbb{C}$
- $f_\nu|_k\gamma = f_\nu$ for all $\gamma \in \Gamma_{\mathfrak{a}_\nu}(\mathfrak{n})$
- Fourier Expansion: $f_\nu(z) = \sum_{\substack{0 \ll \xi \in \mathfrak{a}_\nu \\ \xi = 0}} a_\nu(\xi) \exp\left(2\pi i \mathrm{Tr}(\xi z)\right)$

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- $\mathbf{f} := (f_1, \ldots, f_{h_F})$
- (Adelic function) $\mathbf{f}(\gamma x_\nu g_\infty k_0) = (f_\nu|_k g_\infty)(\mathbf{i})$ where

$$\mathrm{GL}_2(\mathbb{A}_F) = \sqcup_{\nu=1}^{h_F} \mathrm{GL}(F) \begin{pmatrix} 1 & \\ & t_\nu \end{pmatrix} \mathrm{GL}_2^+(F_\infty) \prod_{v<\infty} K_v.$$

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- $\mathbf{f} := (f_1, \ldots, f_{h_F})$
- Fourier coefficient at $\mathfrak{m} \subset \mathcal{O}_F$: $C_{\mathbf{f}}(\mathfrak{m}) = a_\nu(\xi)\xi^{-k/2}\mathrm{N}(\mathfrak{m})^{1/2}$

Rankin-Selberg L-function:

$$L(\mathbf{f} \otimes \mathbf{g}, s) = \zeta_F^*(2s) \sum_{\mathfrak{m} \subset \mathcal{O}_F} \frac{C_{\mathbf{f}}(\mathfrak{m}) C_{\mathbf{g}}(\mathfrak{m})}{\mathrm{N}(\mathfrak{m})^s}$$

Main Theorem

Theorem (Hamieh, T., 2020)

Let $k_j \equiv 0 \mod 2$. For any fixed $\mathbf{g} \in \Pi_k(\mathfrak{n})$, there exists an absolute constant $c > 1$ such that

$$\#\left\{\mathbf{f} \in \Pi_k(\mathcal{O}_F) \ : \ L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) \neq 0\right\} \gg \frac{k}{\log^c k}.$$

- $\dfrac{k}{\log^c k} = \dfrac{\prod k_j}{\log^c(\prod k_j)}$
- $\max\{k_j\} \to \infty$ while $\min\{k_j\} > M$ for sufficiently large M

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- (Liu-Masri, 2014) Let F be a totally real number field of degree n and narrow class number 1. For any fixed $\mathbf{g} \in \Pi_k(\mathcal{O}_F)$ with k being even parallel weight, we have

$$\#\left\{\mathbf{f} \in \Pi_k(\mathcal{O}_F) \,:\, L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) \neq 0\right\} \gg k^{1-\epsilon}$$

Idea of the Proof

- Find a lower bound for the first moment:

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) \right|$$

- Find an upper bound for the second moment:

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) \right|^2$$

- Cauchy-Schwarz inequality:

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) \right| \leq \left(\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F) \\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1 \right)^{1/2} \left(\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) \right|^2 \right)^{1/2}$$

Twisted First Moment

We consider twisted first moment:

$$\sum_{\mathbf{f} \in \Pi_k(\mathcal{O}_F)} L\left(\mathbf{f} \otimes \mathbf{g}, \frac{1}{2}\right) \omega_{\mathbf{f}} \text{ where } \omega_{\mathbf{f}} = \frac{\prod \Gamma\left(k_j - 1\right)}{(4\pi)^{\sum (k_j - 1)} |d_F|^{1/2} \langle \mathbf{f}, \mathbf{f} \rangle_{\mathcal{O}_F}}.$$

- approximate functional equation
- Petersson trace formula

Approximate Functional Equation

$$I(s,X) = \frac{1}{2\pi i}\int_{(3/2)} X^u \Lambda(\mathbf{f}\otimes\mathbf{g}, s+u)G(u)\frac{du}{u}$$

with a "nice" function $G(u)$.

- $\Lambda(\mathbf{f}\otimes\mathbf{g}, s) = I(s,X) + I(1-s, X^{-1})$
- $I(s,X) = \mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})^s L_\infty(\mathbf{f}\otimes\mathbf{g}, s)\sum_{m=1}^{\infty}\frac{b_m^{\mathfrak{n}}(\mathbf{f}\otimes\mathbf{g})}{m^s} V_s\left(\frac{4^n\pi^{2n}m}{X\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)$

  where

  - $b_m^{\mathfrak{n}}(\mathbf{f}\otimes\mathbf{g}) = \sum_{d^2|m}\left(a_d(\mathfrak{n})\sum_{\mathrm{N}(\mathfrak{m})=m/d^2} C_{\mathbf{f}}(\mathfrak{m})C_{\mathbf{g}}(\mathfrak{m})\right)$
  - $V_s(y) = \frac{1}{2\pi i}\int_{(3/2)} y^{-u}G(u)\frac{L_\infty(\mathbf{f}\otimes\mathbf{g}, s+u)}{L_\infty(\mathbf{f}\otimes\mathbf{g}, s)}\frac{du}{u}$

Approximate Functional Equation

Proposition

$$L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right)=2\sum_{m=1}^{\infty}\frac{b_m^{\mathfrak{n}}(\mathbf{f}\otimes\mathbf{g})}{m^{1/2}}V_{1/2}\left(\frac{4^n\pi^{2n}m}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)$$

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)}L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right)\omega_{\mathbf{f}}$$

$$=2\sum_{\mathfrak{m}\subset\mathcal{O}_F}\frac{C_{\mathbf{g}}(\mathfrak{m})}{\sqrt{\mathrm{N}(\mathfrak{m})}}\sum_{d=1}^{\infty}\frac{a_d(\mathfrak{n})}{d}V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)}C_{\mathbf{f}}(\mathfrak{m})\omega_{\mathbf{f}}$$

Petersson Trace Formula (Trotabas, 2011)

Let $k \in 2\mathbb{Z}^n_{>0}$, and let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of F. If $\alpha \in \mathfrak{a}^{-1}$ and $\beta \in \mathfrak{b}^{-1}$, we have

$$\sum_{\mathbf{f} \in \Pi_k(\mathfrak{q})} C_{\mathbf{f}}(\alpha\mathfrak{a})\overline{C_{\mathbf{f}}(\beta\mathfrak{b})}\omega_{\mathbf{f}} + (Oldforms) = \mathbb{1}_{\alpha\mathfrak{a}=\beta\mathfrak{b}}$$

$$+ \quad * \sum_{\substack{\mathfrak{c}^2 \sim \mathfrak{a} \\ c \in \mathfrak{c}^{-1} \setminus \{0\} \\ \epsilon \in \mathcal{O}_F^{\times +}/\mathcal{O}_F^{\times 2}}} \frac{Kl(\epsilon\alpha, \mathfrak{a}; \beta, \mathfrak{b}; c, \mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} J_{k-1}\left(\frac{4\pi\sqrt{\epsilon\alpha\beta[\mathfrak{a}\mathfrak{b}\mathfrak{c}^{-2}]}}{|c|}\right)$$

- $\mathfrak{m}$: integral ideal of $F \implies \mathfrak{m} = \xi\mathfrak{a}_\nu$ with $0 \ll \xi \in \mathfrak{a}_\nu^{-1}$
- $J_{k-1}(x) = \prod J_{k_j-1}(x_j)$

Twisted First Moment

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right)\omega_{\mathbf{f}} = 2M(k) + \frac{2C}{|d_F|^{1/2}} E_{\mathbf{g}}(k),$$

where

$$M(k) = \sum_{d=1}^{\infty} \frac{a_d(\mathfrak{n})}{d} V_{1/2}\left(\frac{4^n\pi^{2n}d^2}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right),$$

and

$$E_{\mathbf{g}}(k) = \sum_{\{\mathfrak{a}_\nu\}} \sum_{\xi\in(\mathfrak{a}_\nu^{-1})^+/\mathcal{O}_F^{\times+}} \frac{C_g(\xi\mathfrak{a}_\nu)}{\sqrt{\mathrm{N}(\xi\mathfrak{a}_\nu)}} \sum_{d=1}^{\infty} \frac{a_d(\mathfrak{n})}{d} V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\xi\mathfrak{a}_\nu)d^2}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)$$

$$\times \sum_{\substack{\mathfrak{c}^2\sim\mathfrak{a}_\nu \\ c\in\mathfrak{c}^{-1}\setminus\{0\} \\ \epsilon\in\mathcal{O}_F^{\times+}/\mathcal{O}_F^{\times 2}}} \frac{Kl(\epsilon\xi,\mathfrak{a}_\nu;1,\mathcal{O}_F;c,\mathfrak{c})}{\mathrm{N}(c\mathfrak{c})} J_{k-1}\left(\frac{4\pi\sqrt{\epsilon\xi\left[\mathfrak{a}_\nu\mathfrak{c}^{-2}\right]}}{|c|}\right).$$

Main Term for the First Moment

$$\sum_{d=1}^{\infty} \frac{a_d(\mathfrak{n})}{d} V_{1/2}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})d^2}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)$$

$$= \frac{1}{2\pi i}\int_{(3/2)} G(u)\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)^{-u} \frac{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2+u)}{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2)}\zeta_F^{\mathfrak{n}}(2u+1)\,\frac{du}{u}$$

$$= \mathop{Res}_{u=0}\left(\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)^{-u} \frac{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2+u)}{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2)}\,\frac{G(u)\zeta_F^{\mathfrak{n}}(2u+1)}{u}\right)$$

$$+ \frac{1}{2\pi i}\int_{(-1/4)} \left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{p})}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{n})}\right)^{-u} \frac{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2+u)}{L_\infty(\mathbf{f}\otimes\mathbf{g}, 1/2)} G(u)\zeta_F^{\mathfrak{n}}(2u+1)\,\frac{du}{u}$$

$$M(k) = C\log k + O(1)$$

Some tools for Error Term Computations

- (Weil bound)

  $$|Kl(\alpha, \mathfrak{n}; \beta, \mathfrak{m}; \gamma, \mathfrak{c})| \ll_F N(((\alpha)\mathfrak{n}, (\beta)\mathfrak{m}, (\gamma)\mathfrak{c}))^{1/2} \tau((\gamma)\mathfrak{c}) \mathrm{N}(\gamma\mathfrak{c})^{1/2}$$

  where $\tau((\gamma)\mathfrak{c}) = |\{I \subset \mathcal{O}_F : (\gamma)\mathfrak{c}I^{-1} \subset \mathcal{O}_F\}|$

- $J_v(x) \ll \left(\frac{x}{2(v+1)}\right)^{M-\delta}$ $\quad 0 \leq \delta < 1$ and $\delta < M \leq v$

- $J_v(x) \ll x^{-\frac{1}{2}+w}$ $\quad 0 \leq w < 1/2$

- (Luo) $\sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{|\eta_j|<1} |\eta_j|^w < \infty$ and $\sum_{\eta \in \mathcal{O}_F^{\times +}} \prod_{|\eta_j|>1} |\eta_j|^{-\delta} < \infty$

- Opening up $Kl(*)$ and $J_{k-1}(*)$

$$E_{\mathbf{g}}(k) = o(1)$$

Lower Bound for the First Moment

- $$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right)\omega_{\mathbf{f}} = C\log k + O(1)$$
- $$\omega_{\mathbf{f}} = \frac{\prod\Gamma\left(k_j-1\right)}{(4\pi)^{\sum(k_j-1)}|d_F|^{1/2}\left\langle\mathbf{f},\mathbf{f}\right\rangle_{\mathcal{O}_F}} \ll \frac{\log k}{k}$$
  - $\langle\mathbf{f},\mathbf{f}\rangle_{\mathcal{O}_F} = \dfrac{[\mathcal{O}_F^{\times+}:\mathcal{O}_F^{\times 2}]\Gamma(k)}{2^{n-1}R_F(4\pi)^k}\operatorname*{Res}_{s=1} L(\mathbf{f}\otimes\mathbf{f},s)$ (Shimura 1978)
  - $\operatorname*{Res}_{s=1} L(\mathbf{f}\otimes\mathbf{f},s) \gg (\log k)^{-1}$ (Hoffstein-Lockhart 1994, Goldfeld-Hoffstein-Lieman 1994)

Proposition

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left|L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right)\right| \gg k$$

Second Moment

$$I(s,X) = \frac{1}{2\pi i}\int_{(3/2)} X^u \Lambda^2(\mathbf{f}\otimes\mathbf{g}, s+u) G(u) \frac{du}{u}$$

$$\vdots$$

$$L^2\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) = 2\sum_{t=1}^{\infty} \frac{B_t}{t^{1/2}} W_{1/2}\left(\frac{(2\pi)^{4n}t}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{c})^2}\right)$$

$$\Rightarrow \sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L^2\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right)\omega_{\mathbf{f}} = \cdots$$

Second Moment Unfolding Method (Blomer 2012)

- $E_\nu(z,s) := \sum_{\substack{(c,d)\mathcal{O}_F^\times \\ (c,d)\in\mathfrak{a}_\nu\mathfrak{n}\mathfrak{d}_F\times\mathcal{O}_F}} \frac{y^s}{|cz+d|^{2s}}$ for $\nu = 1,\ldots,h_F$
- $v := (v_1,\ldots,v_{h_F})$ with

$$v_\nu = v_\nu(z,s) := \mathrm{N}(\mathfrak{a}_\nu)^s g_\nu(z) E_\nu(z,s)$$

- Interpret $\|v(z,1/2)\|^2$ in two different ways:

1. $\|v(z,s)\|^2 = \sum_{\nu=1}^{h_F} \|v_\nu(z,s)\|^2$
$$= \sum_{\nu=1}^{h_F} \frac{1}{\mu\left(\Gamma_\nu\backslash\mathfrak{h}^n\right)} \int_{\Gamma_\nu\backslash\mathfrak{h}^n} |v_\nu(z,s)|^2 y^k \, d\mu(z)$$
2. $\|v(z,s)\|^2 \geq \sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \frac{1}{\langle\mathbf{f},\mathbf{f}\rangle} |\langle v(z,s),\mathbf{f}\rangle|^2$ (Bessel's inequality)

Second Moment Unfolding Method (Blomer, 2012)

1. $\|v(z,1/2)\|^2 \ll (4\pi)^{-k}\Gamma\left(k+1/\log k\right)(\log k)^{c_1}$
2. $\|v(z,1/2)\|^2 \gg \frac{\Gamma(k-1/2)^2}{(4\pi)^k\Gamma(k)} \sum_{\mathbf{f}} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right)\right|^2 (\log k)^{-c_1}$

Proposition

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right)\right|^2 \ll k \log^c k$$

Completing the Proof

$$k \ll \left(\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F) \\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1 \right)^{1/2} \left(\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} \left| L\left(\mathbf{f}\otimes\mathbf{g}, \frac{1}{2}\right) \right|^2 \right)^{1/2}$$

$$\ll \left(\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F) \\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1 \right)^{1/2} \left(k \log^c k\right)^{1/2}$$

Theorem

For $\mathbf{g} \in \Pi_k(\mathfrak{n})$,
$$\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F) \\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1 \gg \frac{k}{\log^c k}$$

More Questions...

- weight aspect with $k \neq l$?
- level aspect?
- improving the current bound?

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Recall

Lau-Tsang (2005): There is a positive absolute constant C such that there are at least $Ck/\log^2 k$ newforms $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ for which $L(f, \frac{1}{2}) \neq 0$, as $k \to \infty$ with $k \equiv 0 \mod 4$.

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Mollification Method

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M_{\mathbf{f}\otimes\mathbf{g}}\cdot\omega_{\mathbf{f}}$$

$$\ll \left(\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F)\\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1\right)^{1/4} \left(\sum_{\mathbf{f}\in\Pi_K(\mathcal{O}_F)} \omega_{\mathbf{f}}^2\right)^{1/4}$$

$$\times\left(\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L^2\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M_{\mathbf{f}\otimes\mathbf{g}}^2\omega_{\mathbf{f}}\right)^{1/2}$$

$$\ll \frac{1}{k^{1/4}}\left(\sum_{\substack{\mathbf{f}\in\Pi_k(\mathcal{O}_F)\\ L(\mathbf{f}\otimes\mathbf{g},1/2)\neq 0}} 1\right)^{1/4} \left(\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L^2\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M_{\mathbf{f}\otimes\mathbf{g}}^2\omega_{\mathbf{f}}\right)^{1/2}$$

Mollifier

The mollifier $M_{\mathbf{f}\otimes\mathbf{g}}$ is defined as

$$M_{\mathbf{f}\otimes\mathbf{g}} = \sum_{\mathfrak{m}\subseteq\mathcal{O}_F} \frac{C_{\mathbf{f}}(\mathfrak{m})C_{\mathbf{g}}(\mathfrak{m})\mu(\mathfrak{m})P_{\mathfrak{m}}}{\mathrm{N}(\mathfrak{m})^{1/2}},$$

where

$$P_{\mathfrak{m}} = \begin{cases} \frac{1}{2}\frac{\log^2\left(k^{\delta}/\mathrm{N}(\mathfrak{m})\right) - \log^2\left(k^{\delta/2}/\mathrm{N}(\mathfrak{m})\right)}{\log k^{\delta/2}} & \mathrm{N}(\mathfrak{m}) \leq k^{\delta/2} \\ \frac{1}{2}\frac{\log^2\left(k^{\delta}/\mathrm{N}(\mathfrak{m})\right)}{\log k^{\delta/2}} & k^{\delta/2} \leq \mathrm{N}(\mathfrak{m}) \leq k^{\delta} \\ 0 & \mathrm{N}(\mathfrak{m}) \geq k^{\delta}. \end{cases}$$

with a fixed $\delta \in (0, 1)$.

Mollified First Moment

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M_{\mathbf{f}\otimes\mathbf{g}}\omega_{\mathbf{f}}$$

$$= 2\sum_{\mathfrak{m}\subset\mathcal{O}_F}\sum_{\mathfrak{n}\subset\mathcal{O}_F}\frac{C_{\mathbf{g}}(\mathfrak{m})C_{\mathbf{g}}(\mathfrak{n})\mu(\mathfrak{n})P_{\mathfrak{n}}}{\mathrm{N}(\mathfrak{m})^{1/2}\mathrm{N}(\mathfrak{n})^{1/2}}$$

$$\times\sum_{d=1}^{\infty}\frac{a_d(\mathfrak{c})}{d}V_{\frac{1}{2}}\left(\frac{4^n\pi^{2n}\mathrm{N}(\mathfrak{m})d^2}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{c})}\right)\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} C_{\mathbf{f}}(\mathfrak{m})C_{\mathbf{f}}(\mathfrak{n})\omega_{\mathbf{f}}$$

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M_{\mathbf{f}\otimes\mathbf{g}}\omega_{\mathbf{f}}$$

$$=\psi_1(0)\frac{\zeta_F^{\mathfrak{n}}(2)}{Res_{s=1}\,L(\mathbf{g}\otimes\mathbf{g},s)}+o\left(\frac{1}{\log^A k}\right)$$

Mollified Second Moment

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L^2\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M^2_{\mathbf{f}\otimes\mathbf{g}}\omega_{\mathbf{f}}$$

$$= \sum_{\substack{\mathfrak{a},\mathfrak{b}\\ \mathrm{N}(\mathfrak{b})\le k^{\delta}}} \frac{1}{\mathrm{N}(\mathfrak{a}\mathfrak{b})} \sum_{\mathfrak{m},\mathfrak{n}} \frac{A_{\mathbf{g}}(\mathfrak{a},\mathfrak{m})\Psi_{\mathfrak{b},\mathfrak{n}}}{\mathrm{N}(\mathfrak{m}\mathfrak{n})^{1/2}} W_{1/2}\left(\frac{(2\pi)^{4n}\mathrm{N}(\mathfrak{a}^2\mathfrak{m})}{\mathrm{N}(\mathfrak{d}_F^2\mathfrak{c})^2}\right)$$

$$\times \sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} C_{\mathbf{f}}(\mathfrak{m})C_{\mathbf{f}}(\mathfrak{n})\omega_{\mathbf{f}}$$

$$\sum_{\mathbf{f}\in\Pi_k(\mathcal{O}_F)} L^2\left(\mathbf{f}\otimes\mathbf{g},\frac{1}{2}\right) M^2_{\mathbf{f}\otimes\mathbf{g}}\omega_{\mathbf{f}} \overset{?}{\sim} \psi_2(0)\frac{\zeta_F^{\mathfrak{n}}(2)^2}{Res^2_{s=1}\, L(\mathbf{g}\otimes\mathbf{g},s)}$$

Thank you!