Zeta morphisms for rank two universal deformations

Kentaro Nakamura

Saga University

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Notation

- p: a prime number.
- $\iota_\infty\colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad \iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$: fixed embeddings.
- L: a (sufficiently large) finite extension of $\mathbb{Q}_p$, $\quad \mathcal{O} = \mathcal{O}_L$,
  $\varpi \in \mathcal{O}$: a uniformizer, $\quad \mathbb{F} = \mathcal{O}/(\varpi)$.
- $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}), \quad \Lambda = \mathcal{O}[[\Gamma]]$: the Iwasawa algebra of Γ.
- For a field F, we set $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Kato's zeta morphisms

- $f = \sum_{n=1}^{\infty} a_n q^n \in S_k^{\mathrm{new}}(\Gamma_1(N_f))$: a normalized Hecke eigen cusp newform of level $N_f \geqq 1$, weight $k \in \mathbb{Z}_{\geqq 2}$.
- $\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$: a Galois representation associated to f, i.e. odd and unramified outside $\Sigma_f = \mathrm{prime}(N_f) \cup \{p\}$ satisfying

$$\mathrm{tr}(\rho_f(\mathrm{Frob}_\ell)) = a_l$$

  for all $\ell \notin \Sigma_f$.
- $\mathbf{H}^i(\rho_f^*(1)) := \varprojlim_{k \geqq 0} H^i(\mathbb{Z}[1/p, \zeta_{p^k}], (j_k)_*(\rho_f^*)(1))$: Iwasawa cohomology of $\rho_f^*(1)$. This is a Λ-module.
  ($j_k \colon \mathrm{Spec}(\mathbb{Z}[1/\Sigma_f, \zeta_{p^k}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^k}])$: the canonical inclusion)

Kato defined a non zero Euler system, i.e. a collection of elements

$$z_{np^k} \in H^1(\mathbb{Q}(\zeta_{np^k}), \rho_f^*(1))$$

for $k \geqq 0$, $n \geqq 1$ such that $(n, \Sigma_f) = 1$ satisfying the Euler system norm relation, and proved the following :

Theorem (12.4 of Kato (04))

- $\mathbf{H}^2(\rho_f^*(1))$ is a torsion Λ-module.
- $\mathbf{H}^1(\rho_f^*(1))[1/p]$ is a free $\Lambda[1/p]$-module of rank one.

We can define an element

$$\{z_{p^k}\}_{k \geqq 1} \in \mathbf{H}^1(\rho_f^*(1)),$$

but it is not canonical since $\{z_{np^k}\}_{n,k}$ depends on many choices $c, d \geqq 2$ such that $(cd, 6pN_f) = 1$, $1 \leqq j \leqq k-1$ and $\alpha \in \mathrm{SL}_2(\mathbb{Z})$, etc. appearing in Kato's article.

Dividing its dependent factors (and the L-factors at the bad primes $\ell \neq p$), he constructed the following map which we call the zeta morphism for ρ_f.

Theorem (12.5 of Kato (04))

There is a canonical $\mathcal{O}$-linear map

$$\mathbf{z}(f)\colon \rho_f^* \to \mathbf{H}^1(\rho_f^*(1))[1/p]$$

interpolating, via Bloch-Kato's dual exponentials, all the critical values of

$$L_{\{p\}}(f,\chi,s) = \sum_{n=1,(n,p)=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

for all the finite characters $\chi\colon \Gamma(\xrightarrow{\sim} \mathbb{Z}_p^\times) \to \mathbb{C}^\times$. If p is odd and $\overline{\rho}_f = \rho_f \pmod{\varpi}$ is absolutely irreducible, one has the inclusion

$$\mathrm{Char}_\Lambda(\mathbf{H}^1(\rho_f^*(1))/\Lambda \cdot \mathrm{Im}(\mathbf{z}(f))) \subseteq \mathrm{Char}_\Lambda(\mathbf{H}^2(\rho_f^*(1)))$$

Conjecture (12.10 of Kato (04), Kato (93))

(1) (Kato main conjecture, KMC)

$$\mathrm{Char}_\Lambda(\mathbf{H}^1(\rho_f^*(1))/\Lambda \cdot \mathrm{Im}(\mathbf{z}(f))) = \mathrm{Char}_\Lambda(\mathbf{H}^2(\rho_f^*(1)))$$

(2) (Roughly speaking,) Such zeta morphisms exist for all the families of p-adic representations of $G_{\mathbb{Q}}$ which are unramified outside a finite set of primes.

- When f is ordinary at p, KMC is equivalent to the usual Iwasawa main conjecture, i.e. the equality

$$(p\text{-adic } L\text{-function}) = \mathrm{Char}_\Lambda((\text{cyclotomic Selmer group})^\vee),$$

  which is (up to now) formulated only for f whose p-component $\pi_p(f)$ is principal series (B. Mazur, R. Greenberg, S. Kobayashi, Lei-Loeffler-Zerbes).
- We can consider KMC for arbitrary f, e.g. even for f whose $\pi_p(f)$ is supercuspidal.

Zeta morphisms for rank two universal deformations

- Σ: a finite set of primes containing p, $\Sigma_0 = \Sigma \setminus \{p\}$.
- $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F})$: odd, absolutely irreducible, unramified outside Σ.
- $\mathrm{Comp}(\mathcal{O})$: the category of commutative local Noetherian complete $\mathcal{O}$-algebras with finite residue field.
- $\rho_{\Sigma}^{\mathrm{univ}}\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(R_{\Sigma,\overline{\rho}})$: the universal deformation for the deformations $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ ($A \in \mathrm{Comp}(\mathcal{O})$) of $\overline{\rho}$ which are unramified outside Σ (no condition at the primes in Σ).
- $\mathfrak{X}_{\Sigma}(\overline{\rho}) = \mathrm{Spec}(R_{\Sigma,\overline{\rho}}[1/p])_0$: the set of closed points.
- $\mathfrak{X}_{\Sigma}^{\mathrm{mod}}(\overline{\rho})$: the subset of modular points.
- For $f \in S_k^{\mathrm{new}}(\Gamma_1(N))$, we set

$$P_{f,\ell}(T) = \det(1 - \mathrm{Frob}_{\ell} \cdot T \mid \rho_f^{I_\ell}) \in \mathcal{O}[T].$$

Theorem (Main Theorem, Nakamura (20))

Assume the following:

(i) $\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is absolutely irreducible, (ii) $p \geqq 3$, (iii) $\mathrm{End}_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\overline{\rho}) = \mathbb{F}$,

(iv) $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is not of the form $\begin{pmatrix} \overline{\chi}_p^{\pm 1} & * \\ 0 & 1 \end{pmatrix} \otimes \eta \quad (\eta : G_{\mathbb{Q}_p} \to \mathbb{F}^\times)$.

Then, there exists a $R_{\Sigma,\overline{\rho}}$-linear map

$$Z_{\Sigma,\overline{\rho}} \colon (\rho_\Sigma^{\mathrm{univ}})^* \to \mathbf{H}^1((\rho_\Sigma^{\mathrm{univ}})^*(1))$$

satisfying the following: for any $x_f \in \mathfrak{X}_\Sigma^{\mathrm{mod}}(\overline{\rho})$, one has

$$x_f^*(Z_{\Sigma,\overline{\rho}}) = \prod_{\ell \in \Sigma_0} P_{f,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f)$$

Namely, $Z_{\Sigma,\overline{\rho}}$ interpolates $\prod_{\ell \in \Sigma_0} P_{f,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f)$ $(x_f \in \mathfrak{X}_0^{\mathrm{mod}}(\overline{\rho}))$ which are related with $L_\Sigma(f, \chi, s) = \sum_{n=1,(n,\Sigma)=1}^\infty \frac{a_n \chi(n)}{n^s}$.

Related works on the construction of zeta morphisms (or Euler systems) for families.

- Ochiai, Fukaya-Kato: Hida families (of ordinary p-adic modular forms).
- Hansen, Ochiai, S. Wan: Coleman-Mazur eigencurves (families of overconvergent of p-finite slope modular forms).
- Fouquet, S. Wan: universal deformations.

Remark

Fukaya-Kato generalized the construction of zeta morphisms to Hida families. We construct our zeta morphisms $Z_{\Sigma,\overline{\rho}}$ combining (a generalization of) Fukaya-Kato's method and many deep results in the theory of p-adic Langlands correspondence for $\mathrm{GL}_{2/F}$ for $F = \mathbb{Q}, \mathbb{Q}_p, \mathbb{Q}_\ell$ (Colmez (10), Emerton (11), Paškūnas (13), Emerton-Helm (14)).

Application to KMC

For $f_i = \sum_{n=1}^{\infty} a_n(f_i)q^n \in S_{k_i}^{\mathrm{new}}(N_i)$ $(i = 1, 2)$, we say that f_1 and f_2 are congruent if

$$a_\ell(f_1) \equiv a_\ell(f_2) \pmod{\varpi}$$

for all but finitely many primes ℓ.

Theorem (Na)

Assume f_1 and f_2 are congruent, $\overline{\rho}_{f_1}$ satisfies all the assumptions in our main theorem, and

$$\mathbf{z}(f_1) \pmod{\varpi} \neq 0.$$

Then one also has

$$\mathbf{z}(f_2) \pmod{\varpi} \neq 0.$$

Moreover, one has the following equivalence

$$\text{KMC for } f_1 \text{ holds} \iff \text{KMC for } f_2 \text{ holds.}$$

- It is expected that the assumption $\mathbf{z}(f_1) \pmod{\varpi} \neq 0$ always holds.

Known results (Assume that $\overline{\rho}_{f_1}$ is absolutely irreducible and $\mu(f_1) = 0$)

- Greenberg-Vatsal (00): congruent elliptic curves E_1 and E_2 with good ordinary reduction at p (i.e. of weight two).
- Emerton-Pollack-Weston (06): congruent eigenforms which are ordinary at p (of arbitrary weights).
- many related results in many related settings $\cdots$
- Kim-Lee-Ponsinet (19): congruent eigenforms which are of finite slope (not ordinary in general) but with a fixed weight $2 \leqq k \leqq p-1$.
- (Na): all the congruent eigenforms with arbitrary levels and weights.

Therefore, we can compare (under the assumption that $\mathbf{z}(f_1) \bmod \varpi \neq 0$)

known IMC (=KMC) for ordinary case (Kato, Skinner-Urban),

or of finite slope case (Kato, X.Wan)

with

unknown KMC, e.g. for supercuspidal case.

Outline of 「Main theorem $\Rightarrow$ Theorem on KMC」

- a congruence between f_1 and f_2 and the irreducibility of $\overline{\rho}_{f_1}$ imply $\overline{\rho}_{f_1} \overset{\sim}{\to} \overline{\rho}_{f_2} =: \overline{\rho}$, and we can consider

$$x_{f_1}, x_{f_2} \in \mathfrak{X}^{\mathrm{mod}}_{\Sigma}(\overline{\rho}) \text{ for sufficiently large } \Sigma.$$

- By our main theorem, one obtains a congruence

$$\prod_{\ell \in \Sigma_0} P_{f_1,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f_1) \equiv \prod_{\ell \in \Sigma_0} P_{f_2,\ell}(\mathrm{Frob}_\ell) \cdot \mathbf{z}(f_2) \, (\mathrm{mod} \varpi).$$

- Kim-Lee-Ponsinet proved that a congruence between zeta morphisms as above implies the equivalence of KMC.

Remark

We can also formulate KMC for arbitrary points in $\mathfrak{X}(\overline{\rho}) = \bigcup_{\Sigma} \mathfrak{X}_{\Sigma}(\overline{\rho})$, and obtain the equivalence of KMC between (almost) all the points in $\mathfrak{X}(\overline{\rho})$.

The proof of the main theorem

We mainly explain how to construct our zeta morphism

$$Z_{\Sigma,\overline{\rho}} \colon (\rho_\Sigma^{\mathrm{univ}})^* \to \mathbf{H}^1((\rho_\Sigma^{\mathrm{univ}})^*(1)).$$

Notation

- For each open compact subgroup K of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, we set $Y(K)(\mathbb{C}) = \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{H}^\pm \times (\mathrm{GL}_2(\mathbb{A}_f)/K)$ and $H^1(K) = H^1(Y(K)(\mathbb{C}), \mathcal{O})$.
- For $N \geqq 1$, we set
$$K(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \;\middle|\; g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N \right\}$$ and
$$K_1(N) = \left\{ g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \;\middle|\; g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$
- $G_\ell = \mathrm{GL}_2(\mathbb{Q}_\ell)$, $G_\Sigma = \prod_{\ell\in\Sigma} G_\ell$, $G_{\Sigma_0} = \prod_{\ell\in\Sigma_0} G_\ell$.

Fukaya-Kato's construction for Hida families

Theorem (Fukaya-Kato (12))

For each $N \geqq 4$, there exists a canonical Hecke equivariant $\mathcal{O}$-linear map

$$\mathbf{z}_{1,N} \colon H^1(K_1(N)) \to \mathbf{H}^1(H^1(K_1(N))(1)) \otimes_\Lambda \mathrm{Frac}(\Lambda)$$

interpolating the operator valued L-functions $\sum_{n\geqq 1,(n,p)=1} \frac{T_n \cdot \chi(n)}{n^s}$ acting on $H^1(Y_1(N)(\mathbb{C}), \mathbb{C})$.

The set of maps $\{\mathbf{z}_{1,Np^k}\}_{k\geqq 0}$ is compatible with the corestrictions $H^1(K_1(Np^{k+1})) \to H^1(K_1(Np^k))$ ($k \geqq 0$), and we can control the denominators of the images, so we can take the limit

$$\mathbf{z}_{1,Np^\infty} : \varprojlim_{k\geqq 0} H^1(K_1(Np^k)) \to \mathbf{H}^1(\varprojlim_{k\geqq 0} H^1(K_1(Np^k))(1)) \otimes_\Lambda \mathbf{\Lambda}[1/\lambda]$$

for some $\lambda \in \mathbf{\Lambda} = \Lambda[[\varprojlim_{k\geqq 1}(\mathbb{Z}/Np^k)^\times]]$. Applying Hida's ordinary projection defined using a Hecke operator at p (U_p-operator), we can obtain the zeta morphisms for Hida families.

Strategy for the construction of $Z_{\Sigma,\overline{\rho}}$

- $(\rho_\Sigma^{\mathrm{univ}})^*$ appears in $\varprojlim_{k\geqq 0} H^1(K(Np^k))$ by (Pontryagin dual of) Emerton's theory of the completed cohomology.
- We generalize Fukaya-Kato's map $\mathbf{z}_{1,N}$ to $H^1(K(N))$.
- (a subtle part) To control denominators, we need to restrict it to the $\overline{\rho}$-part (non-Eisenstein part).
- Taking its limit, we obtain a equivariant zeta map for the $\overline{\rho}$-part of $\varprojlim_{k\geqq 0} H^1(K(Np^k))$.
- Using deep results in family versions of p-adic local Langlands correspondence for G_ℓ (Paškūnas for $\ell = p$, Emerton-Helm for $\ell \in \Sigma_0$), we can factor out the $(\rho_\Sigma^{\mathrm{univ}})^*$-part.
- To compare our zeta morphisms $Z_{\Sigma,\overline{\rho}}$ with Kato's ones $\mathbf{z}(f)$, we need Paškūnas' another result (Paškūnas (15)). This part is the most technical part of the article, but we omit to explain it in this talk.

A refined local-global compatibility (Emerton)

For each $N_0 \geqq 1$ such that $\mathrm{prime}(N_0) = \Sigma_0$, we set

$$\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0)) := \varprojlim_{k \geqq 0} H^1(K(N_0p^k))(1)$$

w.r.t. the corestrictions $H^1(K(N_0p^{k+1}))(1) \to H^1(K(N_0p^k))(1)$ ($k \geqq 1$), and

$$\widetilde{H}_{1,\Sigma}^{BM} := \varinjlim_{N_0} \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))$$

w.r.t. the restrictions $\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0)) \to \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0'))$ for $N_0 | N_0'$. $\widetilde{H}_{1,\Sigma}^{BM}$ is equipped with actions of $G_{\mathbb{Q}}$, G_p and $G_{\Sigma_0} = \prod_{\ell \in \Sigma_0} G_\ell$, and (homological) Hecke actions at the primes $\ell \notin \Sigma$. Using its Hecke actions, one can define its $\overline{\rho}$-part

$$\widetilde{H}_{1,\Sigma,\overline{\rho}}^{BM}$$

which is a topological $R_{\Sigma,\overline{\rho}}[G_{\mathbb{Q}} \times G_\Sigma]$-module.

The following is the dual version of Emerton's theorem.

Theorem (A refined local-global compatibility, Emerton (11))

There exists a topological $R_{\Sigma,\overline{\rho}}[G_{\mathbb{Q}} \times G_\Sigma]$-linear isomorphism

$$\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \xrightarrow{\sim} (\Pi_p)^* \otimes_{R_{\Sigma,\overline{\rho}}} (\rho^{\mathrm{univ}}_\Sigma)^* \otimes_{R_{\Sigma,\overline{\rho}}} \widetilde{\pi}_{\Sigma_0},$$

where

- Π_p is the representation of G_p corresponding to $\rho^{\mathrm{univ}}_\Sigma|_{G_{\mathbb{Q}_p}}$,
- π_{Σ_0} is the representation of G_{Σ_0} corresponding to $\{\rho^{\mathrm{univ}}_\Sigma|_{G_{\mathbb{Q}_\ell}}\}_{\ell\in\Sigma_0}$

by the family version of p-adic local Langlands correspondence defined by Colmez (10) (+many people) for Π_p and Emerton-Helm (14) for π_{Σ_0}.

G_Σ-equivariant zeta morphisms

Theorem (Na)

For each $N_0 \geqq 1$ and $k \geqq 1$ as before, there exists a canonical Hecke equivariant $\mathcal{O}$-linear map

$$\mathbf{z}_{N_0p^k,\overline{\rho}} \colon H^1(K(N_0p^k))_{\overline{\rho}}(1) \to \mathbf{H}^1(H^1(K(N_0p^k))_{\overline{\rho}}(2))$$

characterized by a similar interpolation property using the L-functions removing its Euler factors at $\ell \in \Sigma$, which is compatible with corestrictions for $k \geqq 1$ and restrictions for N_0.

- (A subtle point) We can define the map $\mathbf{z}_{N_0p^k,\overline{\rho}}$ over Λ (not over $\mathrm{Frac}(\Lambda)$) after taking the $\overline{\rho}$-part.

By this integrality and the compatibilities, we can define the following maps.

We set

$$\mathbf{z}_{N_0p^\infty,\overline{\rho}} := \varprojlim_{k\geqq 1} \mathbf{z}_{N_0p^k,\overline{\rho}} \colon \widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))_{\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}_1^{BM}(K_{\Sigma_0}(N_0))_{\overline{\rho}}(1))$$

and

$$\mathbf{z}_{\Sigma,\overline{\rho}} := \varinjlim_{N_0} \mathbf{z}_{N_0p^\infty,\overline{\rho}} \colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)).$$

Theorem (Na)

The map $\mathbf{z}_{\Sigma,\overline{\rho}}$ is continuous and $R_{\Sigma,\overline{\rho}}[G_\Sigma]$-linear.

- All the equivariances for Fukaya-Kato's and our maps follow from the interpolation property, which follows from Kato's very deep result, i.e. the explicit reciprocity law.
- Sharifi-Venkatesh recently studies a different, a motivic approach to define zeta morphisms (for $K_1(N)$). I hope their idea enables us to prove the equivariances more directly.

Factoring out the $(\rho_\Sigma^{\mathrm{univ}})^*$-part from $\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}$

Since one has an isomorphism

$$\psi_1\colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \xrightarrow{\sim} (\Pi_p)^* \otimes_{R_{\Sigma,\overline{\rho}}} (\rho_\Sigma^{\mathrm{univ}})^* \otimes_{R_{\Sigma,\overline{\rho}}} \widetilde{\pi}_{\Sigma_0},$$

it suffices to remove $(\Pi_p)^*$-and $\widetilde{\pi}_{\Sigma_0}$-parts.
Removing $\widetilde{\pi}_{\Sigma_0}$-part: For π a smooth admissible representation of G_ℓ $(\ell \neq p)$ defined over $\overline{\mathbb{Q}}_p$, we set $\Psi_\ell(\pi)$ the largest quotient on which $U_\ell = \begin{pmatrix} 1 & \mathbb{Q}_\ell \\ 0 & 1 \end{pmatrix}$ acts by a fixed non-trivial additive character $U_\ell \to \overline{\mathbb{Q}}_p$.
Emerton-Helm extended this functor for smooth admissible representations of G_ℓ defined over more general $\mathbb{Z}_p$-algebras, e.g. for $\widetilde{\pi}_{\Sigma_0}$. We set

$$\Psi_{\Sigma_0}(\widetilde{\pi}_{\Sigma_0}) := \Psi_{\ell_1} \circ \cdots \circ \Psi_{\ell_d}(\widetilde{\pi}_{\Sigma_0})$$

for $\Sigma_0 = \{\ell_1, \cdots, \ell_d\}$. By the characterization property of their correspondence, one has a $R_{\Sigma,\overline{\rho}}$-linear map

$$\psi_2\colon \Psi_{\Sigma_0}(\widetilde{\pi}_{\Sigma_0}) \xrightarrow{\sim} R_{\Sigma,\overline{\rho}}$$

(genericity of $\widetilde{\pi}_{\Sigma_0}$).

Removing $(\Pi_p)^*$-part:

- $\mathfrak{C}(\mathcal{O})$: the category which is the Pontryagin dual of the category of locally admissible G_p-representations on torsion $\mathcal{O}$-modules (Emerton).
- $\rho_p\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R_p)$: the universal deformation of $\overline{\rho}|_{G_{\mathbb{Q}_p}}$.
- Π_p^{univ}: the representation of G_p over R_p corresponding to ρ_p.

Theorem (Paškūnas (13), a very rough form)

- $P := (\Pi_p^{\mathrm{univ}})^*$ is a projective object in $\mathfrak{C}(\mathcal{O})$.
- $R_p = \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(P)$.

By the universality for ρ_p, one has $R_p \to R_{\Sigma,\overline{\rho}}$ and

$$(\Pi_p)^* \xrightarrow{\sim} P\widehat{\otimes}_{R_p} R_{\Sigma,\overline{\rho}}.$$

Hence, one also has

$$\psi_1\colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \xrightarrow{\sim} P\widehat{\otimes}_{R_p}(\rho_\Sigma^{\mathrm{univ}})^* \otimes_{R_{\Sigma,\overline{\rho}}} \widetilde{\pi}_{\Sigma_0}.$$

Definition of $Z_{\Sigma,\overline{\rho}}$

The isomorphisms ψ_1 and ψ_2 induce the following isomorphisms.

Corollary

- *One has* $\Psi_{\Sigma_0}(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}) \in \mathfrak{C}(\mathcal{O})$*, and*

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}})) \xrightarrow{\sim} (\rho^{\mathrm{univ}}_{\Sigma})^*.$$

- *One has* $\Psi_{\Sigma_0}(\mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1))) \in \mathfrak{C}(\mathcal{O})$*, and*

$$\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)))) \xrightarrow{\sim} \mathbf{H}^1((\rho^{\mathrm{univ}}_{\Sigma})^*(1)).$$

Applying $\mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(-))$ to the continuous $R_{\Sigma,\overline{\rho}}[G_\Sigma]$-linear map

$$\mathbf{z}_{\Sigma,\overline{\rho}} \colon \widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}} \to \mathbf{H}^1(\widetilde{H}^{BM}_{1,\Sigma,\overline{\rho}}(1)),$$

we can finally define

$$Z_{\Sigma,\overline{\rho}} := \mathrm{Hom}_{\mathfrak{C}(\mathcal{O})}(P, \Psi_{\Sigma_0}(\mathbf{z}_{\Sigma,\overline{\rho}})) \colon (\rho^{\mathrm{univ}}_{\Sigma})^* \to \mathbf{H}^1((\rho^{\mathrm{univ}}_{\Sigma})^*(1)).$$

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