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On Ichino-Ikeda type formula of Bessel periods for $(SO(5), SO(2))$
(joint work with H. Furusawa)

~~Conj (Gross-Prasad)~~

π, σ : irr cusp aut rep^n of $SO(V), SO(W)$
s.t. $\dim V - \dim W$: odd

$B_{\pi,\sigma}$: Bessel period of (π, σ)

~~$B_{\pi,\sigma} \neq 0$~~

Gross - Prasad conj a relationship between

$$B_{\pi,\sigma} \neq 0 \quad \text{and} \quad L(\tfrac{1}{2}, \pi \times \sigma) \neq 0$$

(for example $B_{\pi,\sigma} \neq 0 \Rightarrow L(\tfrac{1}{2}, \pi \times \sigma) \neq 0$)

Note Gan - Gross - Prasad extend this conj
to classical gps and metaplectic gps

" Ichino - Ikeda type formula of Bessel periods
= refinement of Gross - Prasad conj "

Suppose π, σ : tempered

when $\dim V - \dim W = 1$. by Ichino - Ikeda
general by Liu

the following explicit formula of $B_{\pi,\sigma}$ is conjectured.

Conj $\varphi_\pi \in \pi$, $\varphi_\sigma \in \sigma$ decomposable $\neq 0$

$$\frac{|B_{\pi,\sigma}(\varphi_\pi, \varphi_\sigma)|^2}{\langle \varphi_\pi, \varphi_\pi \rangle \langle \varphi_\sigma, \varphi_\sigma \rangle} = 2^{-S} \cdot C_{V,W} \cdot \frac{L(\tfrac{1}{2}, \pi \times \sigma)}{L(1, \pi, Ad) L(1, \sigma, Ad)} \times \prod_v (\text{local Bessel period})$$

Rmk • local Bessel period

$=$ certain integral of matrix coeff.

• Suppose Arthur's endoscopic classification holds for $SO(V)$, $SO(W)$

$$\Rightarrow \quad 2^{S} = |\mathcal{S}_{\phi_{\pi}}| \, |\mathcal{S}_{\phi_{\sigma}}|$$

Known cases : $(SO(3), SO(2))$: Waldspurger

: $(SO(4), SO(3))$: Harris-Kudla, Ichino.

$(\overset{\pi}{SO(5)}, \overset{\sigma}{SO(2)})$: π : endoscopic Liu

π : non-endo Tsuda Corbett
theta lift from $GSO_{3,1}$

$(SO(5), SO(4))$: ~~SO~~ π : endo Gan-Ichino

$(SO(2n+1), SO(2))$: σ : trivial, π_∞ : dis ser Furusawa-M.

Note Ichino-Ikeda type formula ~~of~~ for $(U(n+1), U(n))$

proved by • Beuzart-Plessis-~~Chau~~ Chaudouard-Zydor (endo)

• ——— -Liu-Zhang-Zhu (stable)

Thm Conj holds for $(SO(5), SO(2))$
(Furusawa-M.)

for any temp π, σ.

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<u>Rmk</u> ~~π~~ Suppose π has σ-Bessel, $(SO(2) = SO(E))$

"π has weak lift to $GL_4(E)$ (E/F: quad)" w.r.t $BC \circ std$.

explain later

This lift $= \Pi_1 \boxplus \cdots \boxplus \Pi_\ell$

$$\Rightarrow 2^s = 2^\ell.$$

<u>Generalized Böcherer conj</u>

holo Siegel cusp form Φ deg 2, wt $k \geq 3$, level $Sp_4(\mathbb{Z})$ Hecke eigenform $\rightsquigarrow$ cusp form φ_Φ on $SO(3,2)$

$E/\mathbb{Q}$: imag quad ext $\rightsquigarrow$ $SO(2) = SO(E)$

disc D_E, ideal class gp Cl_E

$$\Phi = \sum_{S>0} a(S, \Phi)\, e^{2\pi i \operatorname{Tr}(S)}$$

(Note $a({}^t\gamma S \gamma, \Phi) = a(S, \Phi)$ $\gamma \in SL_2(\mathbb{Z})$)

$c \in Cl_E$ $a(c, \Phi)$ makes sense

Λ : char of Cl_E $\rightsquigarrow$ rep^n of $SO(2)$

$$B_\Lambda(\Phi) := w(E)^{-1} \sum_{c \in Cl_E} \Lambda^{-1}(c)\, a(c, \Phi) \rightsquigarrow \text{Bessel period}$$

$w(E)$: #{root of unity in E}

Thm + local computation (Dickson - Pitale - Saha - Schmidt)

<u>Cor</u>

$$\frac{|B_\Lambda(\Phi)|^2}{\langle \Phi, \Phi \rangle} = 2^{2k-4}\, D_E^{k-1}\, \frac{L(1/2, \pi(\Phi) \times \theta(\Lambda))}{L(1, \pi(\Phi), \mathrm{Ad})}$$

$\theta(\Lambda)$: auto rep^n of $GL_2(\mathbb{A}_\mathbb{Q})$ ass to Λ.

<u>Idea of proof</u>

<u>Previous case</u> ($SO(2)$ - trivial)

- Used theta lift $SO(2n+1) \to Mp_{2n}$.

$\nexists$ special Bessel period $(\sigma = I)$ $\rightsquigarrow$ Whittaker period

- $\exists$ Lapid-Mao's formula of Whittaker periods

$\Rightarrow$ get explicit formula of Bessel period

<u>Today's case</u> ($SO(2)$ - non-trivial , $SO(5)$)

For simplicity consider only the case

- $SO(5) \simeq SO(3,2) \simeq PGSp_4$
- Bessel period $\neq 0$

Take $S \in \mathrm{Sym}^2(F)$ s.t. $SO(S) = SO(E)$

ψ : char of $\mathbb{A}_F/F$

Λ : char of $\mathbb{A}_E^\times/E^\times$ s.t. $\Lambda|_{\mathbb{A}_F^\times} = \mathrm{triv}$ regard as a char of $SO(S)$

$$\mathcal{B}_\Lambda(\varphi) = \int_{g \in SO(S)} \begin{pmatrix} g & 0 \\ 0 & \det g\, {}^t g \end{pmatrix} \begin{pmatrix} 1 & X \\ & 1 \end{pmatrix} \psi(\mathrm{tr}\, SX) \Lambda(g)\, dg\, dx$$

(1) Recall $PGSp_4 \simeq SO(3,2)$

consider theta lift $GSp_4 \to GSO_{4,2}$

<u>Rmk</u> Indeed, we should consider $GSp_4^+ = \{ g \in GSp_4 \mid \nu(g) \in \mu(GSO_{4,2}) \}$

$GSp_4^+ \supset R_E \Rightarrow \exists ! \pi_0$: irr consti of $\pi|_{GSp_4^+}$

s.t. π_0 has Bessel period

~~Prp Define Λ-Bessel period for GSO4~~

Recall $PGSO_{4,2} \simeq PGU(2,2)$

Define Λ-Bessel period B_Λ^U for $GU(2,2)$

where Λ regarded as a char of $GU(1)$

Prp φ $B_\Lambda^U(\theta(\varphi)) = \int_\square B_\Lambda(\pi(g)\varphi) f(-)\, dg$

$\varphi \in \pi$

As we noted $GSp_4 \xrightarrow{\theta} GSO_{3,1} \neq 0$

~~this~~ our thm is proved by Corbett.)

$B \neq 0 \Rightarrow GSp_4 \xrightarrow{\theta} GSO_{4,2} \neq 0 \quad \theta(\pi) = \sigma$

irr cuspidal

Thus $\boxed{\exists \text{ explicit formula for } B_\Lambda^U}$

$\Rightarrow \exists$ explicit formula for B_Λ

(2) $(GU(2,2), GU(2,2))$ Consider theta

~~Prp $N(\theta(\varphi_\sigma))$ =~~ Note theta lift dep on

~~$\varphi_\sigma \in \sigma$~~ a choice of Λ

Prp $\mathcal{W}(\theta_\Lambda(\varphi_\sigma)) = \int_{\square} B^{\circ}_{\Lambda}(\cancel{\Phi}(g)\varphi_\sigma)$ ---

Whittaker period

As in step I

$\boxed{\exists \text{ explicit formula for } \mathcal{W} \text{ on } \theta(\sigma)}$

$\Rightarrow$ $\exists$ explicit formula for B°_{Λ} on σ

(3) Recall $PGU(2,2) \simeq PGSO_{4,2}$. $\theta(\sigma) = \Sigma$

$GSO_{4,2} \longrightarrow GSp_6$

- conservation relation local theta $\neq 0$
- $L(1, \theta(\sigma), \mathrm{std}) = \infty$ (in particular $\neq 0$)

$\xrightarrow[\text{Takeda}]{}$ $\neq 0$

$\downarrow$

$GSO_{4,2} \longrightarrow GSp_4$ $\neq 0$

π_0

Prp
Suppose $\theta(\pi_0) = \Sigma$.

$$\mathcal{W}(\theta(\varphi_{\pi_0})) = \cancel{\varphi} \int \mathcal{W}(\pi_0(g)\varphi_{\pi_0}) \text{ ---}$$

As in Step 1.2.

$\boxed{\exists \text{ explicit formula for } W \text{ on } \pi_0}$

$\Rightarrow$ $\exists$ explicit formula for W on Σ

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(4)

$$\begin{array}{ccc} & \xrightarrow{\theta_1} & PGSO(3.3) \simeq PGL_4 \\ PGSp_4 & & \\ \pi_0 & \xrightarrow{\theta_2} & PGSO(2.2) \simeq PGL_2 \times PGL_2 \\ & & \sigma_0 \end{array}$$

$$\theta(\sigma_0) = \pi_0$$

~~Now Lapid - Mao prod explicit for~~

__Prp__ (Soudry. JPSS)

$$W(\theta_1(\varphi_{\pi_0})) = \int W(\pi_0(g)\varphi_{\pi_0}) \text{ --- }$$

__Prp__

$$W(\theta_2(\varphi_{\sigma_0})) = \int W(\sigma_0(g)\varphi_{\sigma_0}) \text{ --- }$$

$$\quad \| $$

$$W(\varphi'_{\pi_0})$$

$\Rightarrow$ $\exists$ explicit for for W for $PGSO_{3.3}$, $PGSO_{2.2}$

$\Rightarrow$ our thm follows //

Lapid - Mao prod

__Rmk__ inner form case

$SO(4.1) \simeq PGU_{1.1}(D)$ $\qquad$ $PGSU_{3.D} \simeq PGU(3.1)$

$GU_{1.1}(D) \to GU_3(D) \xrightarrow{\sim} GU(3.1) \to GU(2.2) \to GSp_4$

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split case

$$\begin{tikzcd} PGSp_4 \arrow[r] & PGSO_{3,3} \arrow[r, phantom, "\simeq"] \arrow[d] & PGL_4 \\ & PGL_4 & \end{tikzcd}$$