Local Saito-Kurokawa A-packets and ℓ-adic cohomology of Rapoport-Zink tower for GSp(4) (joint work with Tetsushi Ito)

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Local Langlands correspondence

G: connected reductive group over $\mathbb{Q}_p$, split for simplicity.
LLC for G is a conjectural map between

- isom. classes of irreducible smooth representations of $G(\mathbb{Q}_p)$
- conj. classes of L-parameters $W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$

with finite fiber. Here

- $W_{\mathbb{Q}_p}$ is the Weil group of $\mathbb{Q}_p$.
- $\widehat{G}$ is the dual group of G over $\mathbb{C}$.

The fiber Π_ϕ of $\phi\colon W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ is called the L-packet.

If $G = \mathrm{GL}_n$, then $\widehat{G} = \mathrm{GL}_n(\mathbb{C})$ and LLC is known (Harris-Taylor).
In this case, every L-packet is a singleton, i.e., LLC is bijective.

Lubin-Tate tower

LLC for GL_n was constructed by means of p-adic geometry.

$\exists$ Lubin-Tate tower $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)\text{: compact open}}$:

- Each M_K is an $(n-1)$-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$.
- $M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is the $(n-1)$-dimensional open unit disc over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$.
- $M_K/M_{\mathrm{GL}_n(\mathbb{Z}_p)}$ is a finite étale covering.
  If $K \lhd \mathrm{GL}_n(\mathbb{Z}_p)$, it is Galois with Galois group $\mathrm{GL}_n(\mathbb{Z}_p)/K$.
- $\mathrm{GL}_n(\mathbb{Q}_p)$ acts on $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$ (Hecke action).
- $D^\times$ acts on $\{M_K\}_{K\subset \mathrm{GL}_n(\mathbb{Z}_p)}$,
  where D is the central division algebra over $\mathbb{Q}_p$ with $\mathrm{inv}(D) = 1/n$.

Fix a prime $\ell \neq p$ and an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$.

Consider ℓ-adic étale cohomology $H^i_{\mathrm{LT}} := \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$.

This is a representation of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$.

Lubin-Tate tower and LLC

$H^i_{\mathrm{LT}} := \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$: rep. of $\mathrm{GL}_n(\mathbb{Q}_p) \times D^\times \times W_{\mathbb{Q}_p}$

Theorem (Deligne, Carayol ($n = 2$), Harris-Taylor ($n \geq 3$))

Let π be a sc (=supercuspidal) rep. of $\mathrm{GL}_n(\mathbb{Q}_p)$. Put $\rho = \mathrm{JL}(\pi)$ and $\sigma = \mathrm{LLC}(\pi)$. Then

$$\mathrm{Hom}_{D^\times}(H^i_{\mathrm{LT}}, \rho)^{\mathrm{sm}} = \begin{cases} \pi \boxtimes \sigma(\frac{n-1}{2}) & i = n-1, \\ 0 & i \neq n-1. \end{cases}$$

Key of the proof in the case $n = 2$:

- Relation between $\{M_K\}_{K \subset \mathrm{GL}_2(\mathbb{Z}_p)}$ and the modular curve.

Lubin-Tate tower ($n = 2$) vs modular curve

Fix a sufficiently small compact open subgroup $K^p \subset \mathrm{GL}_2(\mathbb{A}^{\infty,p})$.

- $\mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p}$: integral modular curve over $\widehat{\mathbb{Z}}_p^{\mathrm{ur}}$ with level $\mathrm{GL}_2(\mathbb{Z}_p)K^p$
- $\mathrm{Sh}^{\mathrm{ss}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p} \subset \mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}$:
  the supersingular locus (0-dimensional)
- $\mathrm{Sh}^{\mathrm{ss.red.}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} := \mathrm{sp}^{-1}(\mathrm{Sh}^{\mathrm{ss}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\overline{\mathbb{F}}_p}) \subset \mathrm{Sh}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$:
  the supersingular reduction locus (rigid analytic open)
- $\mathrm{Sh}^{\mathrm{ss.red.}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ = inverse image of $\mathrm{Sh}^{\mathrm{ss.red.}}_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$ in $\mathrm{Sh}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}}$.

$$\rightsquigarrow \mathrm{Sh}^{\mathrm{ss.red.}}_{KK^p,\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} = \widetilde{D}^\times \backslash (M_K \times \mathrm{GL}_2(\mathbb{A}^{\infty,p})/K^p),$$

where $\widetilde{D}$ is a quaternion division algebra over $\mathbb{Q}$ with $\mathrm{ram}(\widetilde{D}) = \{\infty, p\}$.

Summary:
The supersingular reduction locus of the modular curve is uniformized by the Lubin-Tate tower.

Rapoport-Zink tower for GSp_4

$\exists$ GSp_4-version of the Lubin-Tate tower

$= $ Rapoport-Zink tower $\{M_K\}_{K\subset \mathrm{GSp}_4(\mathbb{Z}_p)}$:

For simplicity we put $G := \mathrm{GSp}_4$.

- Each M_K is a 3-dimensional smooth rigid space over $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$.
- $M_K/M_{G(\mathbb{Z}_p)}$ is a finite étale covering.
  If $K \lhd G(\mathbb{Z}_p)$, it is Galois with Galois group $G(\mathbb{Z}_p)/K$.
- $G(\mathbb{Q}_p)$ acts on $\{M_K\}_{K\subset G(\mathbb{Z}_p)}$ (Hecke action).
- $J(\mathbb{Q}_p)$ acts on $\{M_K\}_{K\subset G(\mathbb{Z}_p)}$, where J is a non-trivial inner form of G.

$H^i_{\mathrm{RZ}} := \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$.
This is a representation of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$.

Aim of this talk

$H^i_{\mathrm{RZ}} := \varinjlim_K H^i_c(M_K \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)$: rep. of $G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$

Aim of this talk:
Describe the $G(\mathbb{Q}_p)$-sc part of H^i_{RZ} by means of LLC for G and J.

Remark

- LLC for G and J are established by Gan-Takeda and Gan-Tantono, respectively. No geometry is needed (use LLC for GL_2 and GL_4).
- LLC for G and J are more difficult (=interesting?) than that for GL_n.
  - $\exists$ L-packet which is not a singleton.
  - $\exists$ L-packet containing both sc rep. and non-sc rep.
  - $\exists$ non-trivial theory of local A-packets.

$\rightsquigarrow$ I expect that description of H^i_{RZ} is more interesting than that of H^i_{LT}.

LLC for G and J

- LLC for G is due to Gan-Takeda.
- LLC for J is due to Gan-Tantono.

For an L-parameter $\phi\colon W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G} = \mathrm{GSp}_4(\mathbb{C})$, let Π_ϕ^G (resp. Π_ϕ^J) denote the L-packet for G (resp. J) attached to ϕ.

Assume that Π_ϕ^G contains a sc rep. $\to \exists 4$ cases:

	$\#\Pi_\phi^G = \#\Pi_\phi^J$	$\#$sc in Π_ϕ^G	$\#$sc in Π_ϕ^J
I (stable)	1	1	1
II	2	2	2
III	2	1	1
IV	2	1	0

We focus on the case III (I and II are easier):
$r\colon \widehat{G} = \mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C}) \rightsquigarrow r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$
($\exists \phi_0$: 2-dim. irr. rep. of $W_{\mathbb{Q}_p}$, $\exists \chi$: a character of $W_{\mathbb{Q}_p}$)

LLC for $G(\mathbb{Q}_p)$ and $J(\mathbb{Q}_p)$

The case III: $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$.

- $\Pi^G_\phi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{disc}}\}$; π_{sc} = non-generic sc, π_{disc} = generic non-sc.
- $\Pi^J_\phi = \{\rho_{\mathrm{sc}}, \rho_{\mathrm{disc}}\}$; ρ_{sc} = sc, ρ_{disc} = non-sc.

Need technical assumption: $\det \phi_0 = 1$, $\chi^2 = 1$ ($\Rightarrow \mathrm{Im}\,\phi \subset \mathrm{Sp}_4(\mathbb{C})$).

We also consider the following A-parameter ψ related to ϕ:

$$\begin{tikzcd} W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\psi"] \arrow[d, "\text{swap } \mathrm{SL}_2 \text{ factors}"'] & \widehat{G}(\mathbb{C}) \\ W_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \arrow[ur, "\phi"'] & \end{tikzcd}$$

Π^G_ψ, Π^J_ψ: local A-packets of G and J, respectively.

- $\Pi^G_\psi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{nt}}\}$; π_{nt} = non-tempered.
- $\Pi^J_\psi = \{\rho'_{\mathrm{sc}}, \rho_{\mathrm{nt}}\}$; ρ'_{sc} = sc (expected to equal ρ_{sc}), ρ_{nt} = non-temp.

Main Theorem

For an irred. smooth rep. ρ of $J(\mathbb{Q}_p)$, put $H^{i,j}_{\mathrm{RZ}}[\rho] := \mathrm{Ext}^j_{J(\mathbb{Q}_p)}(H^i_{\mathrm{RZ}}, \rho)^{\mathcal{D}_c\text{-sm}}_{\mathrm{sc}}$.
Here $(-)_{\mathrm{sc}}$ is the $G(\mathbb{Q}_p)$-supercuspidal part.
Split semisimple rank of $J(\mathbb{Q}_p) = 1 \implies H^{i,j}_{\mathrm{RZ}}[\rho] = 0$ if $j \neq 0, 1$.

Theorem (Ito-M.)

(i) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \phi_0(\frac{3}{2}) & i = 3, \\ 0 & i \neq 3, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] = 0$. Similar for ρ'_{sc}.

(ii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(1) & i = 3, \\ 0 & i \neq 3, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(2) & i = 4, \\ 0 & i \neq 4. \end{cases}$

(iii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(2) & i = 4, \\ 0 & i \neq 4, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(1) & i = 3, \\ 0 & i \neq 3. \end{cases}$

Recall: $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$.
$\Pi^G_\phi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{disc}}\}$, $\Pi^J_\phi = \{\rho_{\mathrm{sc}}, \rho_{\mathrm{disc}}\}$, $\Pi^G_\psi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{nt}}\}$, $\Pi^J_\psi = \{\rho'_{\mathrm{sc}}, \rho_{\mathrm{nt}}\}$.

Main Theorem

Theorem (the case $j = 0$)

(i) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \phi_0(\frac{3}{2}) & i = 3, \\ 0 & i \neq 3. \end{cases}$

(ii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(1) & i = 3, \\ 0 & i \neq 3. \end{cases}$

(iii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(2) & i = 4, \\ 0 & i \neq 4. \end{cases}$

<u>Very rough summary</u>:

- A piece of LLC for G and J appear in H^3_{RZ}.
- Non-tempered local A-packet for J contributes to H^4_{RZ}.
- $\exists$ sc rep. appearing outside the middle degree.
  (it happens only when its L-parameter has non-trivial $\mathrm{SL}_2(\mathbb{C})$-part)

Main Theorem

Remark

By working in a suitable derived category, we may also consider the derived version $H^*_{\mathrm{RZ}}[\rho] := \mathsf{Ext}^*_{J(\mathbb{Q}_p)}(R\Gamma_{\mathrm{RZ}}, \rho)^{\mathcal{D}_c\text{-sm}}_{\mathrm{sc}}$ of $H^{i,j}_{\mathrm{RZ}}[\rho]$.
We can recover ϕ and ψ from the $W_{\mathbb{Q}_p}$-action and the Lefschetz operator on $H^*_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]$ and $H^*_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$, respectively. (cf. Dat's work in the GL_n case)

Rough ideas of proof

- Inputs from local geometry.
- Use of global method:
  - Relation between RZ tower and Siegel threefold.
  - Suitable globalization of elements of Π^G_ϕ, Π^G_ψ, Π^J_ϕ, Π^J_ψ.
- $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$ is easier than $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]$.

Inputs from local geometry

Theorem (Ito-M.)

$H^i_{\mathrm{RZ,sc}} = 0$ if $i \neq 2, 3, 4$.

- $2 = 3 - 1 = \dim M_{\mathrm{GSp}_4(\mathbb{Z}_p)} - \dim \mathcal{M}_{\mathrm{red}}$, where $\mathcal{M}$ is the natural formal model of $M_{\mathrm{GSp}_4(\mathbb{Z}_p)}$. (cf. the supersingular locus of the Siegel threefold is 1-dimensional.)
- Method is similar to M.'s proof of $H^i_{\mathrm{LT,sc}} = 0$ for $i \neq n-1$, but much more complicated (mainly because connected components of $\mathcal{M}$ are not affine).

Theorem (M.)

$H^2_{\mathrm{RZ,sc}}$ doesn't contain $J(\mathbb{Q}_p)$-non-sc subquotient.

- Use $H^2_{\mathrm{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}} \xleftrightarrow{\text{Zelevinsky involution}} H^5_{\mathrm{RZ},G(\mathbb{Q}_p)\text{-sc},J(\mathbb{Q}_p)\text{-non-sc}}$ and the theorem above.

Inputs from local geometry

Theorem (M.)

$\varinjlim_K H_c^i((M_K/p^{\mathbb{Z}}) \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)[\pi_{\mathrm{sc}}^\vee]$ has finite length as a representation of $J(\mathbb{Q}_p)$.

- Use the duality isomorphism between RZ tower for G and RZ tower for J due to Kaletha-Weinstein and Chen-Fargues-Shen.

RZ tower vs Siegel threefold

- For $K' \subset G(\mathbb{A}^\infty)$, $\mathrm{Sh}_{K'} :=$ Siegel threefold/$\mathbb{Q}$ with level K'.
- $H^i_c(\mathrm{Sh}) := \varinjlim_{K'} H^i_c(\mathrm{Sh}_{K'} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$: rep. of $G(\mathbb{A}^\infty) \times \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.
  It is rather understood by using GLC for G.
- $\mathrm{Sh}^{\mathrm{ss.red.}}_{K', \widehat{\mathbb{Q}}^{\mathrm{ur}}_p} \subset \mathrm{Sh}_{K', \widehat{\mathbb{Q}}^{\mathrm{ur}}_p}$: supersingular reduction locus.
- For $K^p \subset G(\mathbb{A}^{\infty,p})$, $\mathrm{Sh}^{\mathrm{ss.red.}}_{KK^p, \widehat{\mathbb{Q}}^{\mathrm{ur}}_p}$ is uniformized by M_K.
  $\rightsquigarrow \exists$ Hochschild-Serre spectral sequence (due to Fargues):

$$E_2^{r,s} = \mathrm{Ext}^r_{J(\mathbb{Q}_p)}(H^{6-s}_{\mathrm{RZ}}(3), \mathcal{A}(J)_{\mathbf{1}})_{\mathrm{sc}} \Rightarrow H^{r+s}(\mathrm{Sh}^{\mathrm{ss.red.}}_{\widehat{\mathbb{Q}}^{\mathrm{ur}}_p})_{\mathrm{sc}}.$$

  Here $\mathcal{A}(J)_{\mathbf{1}}$ is the space of automorphic forms on $J(\mathbb{A})$, trivial at ∞.
  ($\exists$ suitable globalization of $J/\mathbb{Q}_p$ to $\mathbb{Q}$.)
- $H^{r+s}(\mathrm{Sh}^{\mathrm{ss.red.}}_{\widehat{\mathbb{Q}}^{\mathrm{ur}}_p})_{\mathrm{sc}} \cong H^{r+s}_c(\mathrm{Sh})_{\mathrm{sc}}$.
  (Boyer's trick and Imai-M. or Lan-Stroh (stronger))

Main Theorem again

Now we are ready to prove:

Theorem

(i) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \phi_0(\frac{3}{2}) & i=3, \\ 0 & i \neq 3, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] = 0$. Similar for ρ'_{sc}.

(ii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(1) & i=3, \\ 0 & i \neq 3, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(2) & i=4, \\ 0 & i \neq 4. \end{cases}$

(iii) $H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(2) & i=4, \\ 0 & i \neq 4, \end{cases}$ $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \begin{cases} \pi_{\mathrm{sc}} \boxtimes \chi(1) & i=3, \\ 0 & i \neq 3. \end{cases}$

Recall: $r \circ \phi = (\phi_0 \boxtimes \mathbf{1}) \oplus (\chi \boxtimes \mathbf{Std})$.
$\Pi^G_\phi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{disc}}\}$, $\Pi^J_\phi = \{\rho_{\mathrm{sc}}, \rho_{\mathrm{disc}}\}$, $\Pi^G_\psi = \{\pi_{\mathrm{sc}}, \pi_{\mathrm{nt}}\}$, $\Pi^J_\psi = \{\rho'_{\mathrm{sc}}, \rho_{\mathrm{nt}}\}$.

Determination of $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$

By using Gan's result ("The Saito-Kurokawa space ..."), choose

- Π: cuspidal automorphic representation of $G(\mathbb{A})$
- Σ: cuspidal automorphic representation of $J(\mathbb{A})$

such that

- $\Pi_p \cong \pi_{\mathrm{sc}}$, Π^∞ contributes to $H^2_c(\mathrm{Sh})$ and $H^4_c(\mathrm{Sh})$.
- if Π' is an autom. rep. of $G(\mathbb{A})$ such that $\Pi'_v \cong \Pi_v$ $(\forall v \neq p, \infty)$ and Π'_p is sc, then $\Pi = \Pi'$ (a kind of strong multiplicity one).
- $\Sigma_p \cong \rho_{\mathrm{nt}}$, $\Sigma_\infty \cong \mathbf{1}$.
- if Σ' is an autom. rep. of $J(\mathbb{A})$ such that $\Sigma'_v \cong \Sigma_v$ $(\forall v \neq p)$, then $\Sigma = \Sigma'$ (a kind of strong multiplicity one).
- $\Pi^{\infty,p} = \Sigma^{\infty,p}$ (note: $G(\mathbb{A}^{\infty,p}) = J(\mathbb{A}^{\infty,p})$).

Take $\Pi^{\infty,p}$-isotypic part of the spectral sequence

$$E_2^{r,s} = \mathrm{Ext}^r_{J(\mathbb{Q}_p)}(H^{6-s}_{\mathrm{RZ}}(3), \mathcal{A}(J)_{\mathbf{1}})_{\mathrm{sc}} \Rightarrow H^{r+s}_c(\mathrm{Sh})_{\mathrm{sc}}.$$

Determination of $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}]$

By taking $\Pi^{\infty,p}$-isotypic part of the spectral sequence

$$E_2^{r,s} = \mathrm{Ext}^r_{J(\mathbb{Q}_p)}(H^{6-s}_{\mathrm{RZ}}(3), \mathcal{A}(J)_{\mathbf{1}})_{\mathrm{sc}} \Rightarrow H^{r+s}_c(\mathrm{Sh})_{\mathrm{sc}},$$

we get a short exact sequence

$$0 \to H^{i+1,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \to \pi_{\mathrm{sc}} \boxtimes H^{6-i}_c(\mathrm{Sh})[\Pi^\infty](3) \to H^{i,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \to 0.$$

By assumption, $H^{6-i}_c(\mathrm{Sh})[\Pi^\infty](3) \neq 0$ only if $i = 2, 4$.

On the other hand, by input from local geometry, $H^{5,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] = H^{2,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] = 0$. Hence

$$H^{4,0}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H^2_c(\mathrm{Sh})[\Pi^\infty](3), \ \ H^{3,1}_{\mathrm{RZ}}[\rho_{\mathrm{nt}}] \cong \pi_{\mathrm{sc}} \boxtimes H^4_c(\mathrm{Sh})[\Pi^\infty](3).$$

Determination of $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}]$

Choose Π and Σ similarly, but Π^∞ contributes to $H^3_c(\mathrm{Sh})$.
Then get a short exact sequence

$$0 \to H^{4,1}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \to \pi_{\mathrm{sc}} \boxtimes H^3_c(\mathrm{Sh})[\Pi^\infty](3) \to H^{3,0}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}] \to 0.$$

Since $H^3_c(\mathrm{Sh})[\Pi^\infty](3)$ is 2-dim. indecomposable as a $W_{\mathbb{Q}_p}$-rep.,
it suffices to determine $\dim H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{disc}}][\pi_{\mathrm{sc}}]$. This is done by

- $[\rho_{\mathrm{nt}}] + [\rho_{\mathrm{disc}}] = [\text{induced}]$ in the Grothendieck group.
- $\sum_{i=0}^\infty (-1)^i \dim \mathrm{Ext}^i_{J(\mathbb{Q}_p)/p^{\mathbb{Z}}}(V, \text{induced}) = 0$ for V of finite length (Schneider-Stuhler), and the finiteness result explained before:

Theorem

$\varinjlim_K H^i_c((M_K/p^{\mathbb{Z}}) \otimes_{\widehat{\mathbb{Q}}_p^{\mathrm{ur}}} \mathbb{C}_p, \overline{\mathbb{Q}}_\ell)[\pi_{\mathrm{sc}}^\vee]$ has finite length as a representation of $J(\mathbb{Q}_p)$.

Determination of $H^{i,j}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}]$ and $H^{i,j}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}]$

It is the simplest case because $H^{i,1}_{\mathrm{RZ}}[\rho_{\mathrm{sc}}] = H^{i,1}_{\mathrm{RZ}}[\rho'_{\mathrm{sc}}] = 0$.