On the Kodaira dimension of unitary Shimura varieties

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Introduction

Today, I will talk about the *singularities* and *Kodaira dimension* of unitary Shimura varieties.

- Canonical singularities on unitary Shimura varieties for $\mathrm{U}(1,n)$ $(n > 4)$
- Unitary Shimura varieties of general type for $\mathrm{U}(1,n)$ $(9 \leqslant n \leqslant 12,\ F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3}))$
- Uniruled unitary Shimura varieties for $\mathrm{U}(1,n)$ $(n = 3,4,5,\ F = \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3}))$

We show the canonical singularities by the Reid-Shepherd-Barron-Tai criterion.

To prove the results on the Kodaira dimension, we follow the strategy of Gritsenko-Hulek-Sankaran (Invent Math, 2007) and Gritsenko-Hulek (J Alg Geom, 2014).

We use reflective modular forms on orthogonal groups (Borcherds lifts and Gritsenko lifts), and their restriction to unitary groups (Hofmann (Math Ann, 2014)).

Introduction

- Normal variety X over $\mathbb{C}$ has canonical singularities if $\exists r \geqslant 1$, $\exists \epsilon \colon Y \to X$: resolution of singularities s.t. $\epsilon_\star(\omega_Y^{\otimes r}) = \mathscr{O}_X(rK_X)$.
- For a smooth variety X over $\mathbb{C}$, let $P_d := \dim_{\mathbb{C}} H^0(X, K_X^d)$ $(d \geqslant 0)$. The Kodaira dimension $\kappa(X)$ is defined to be $-\infty$ if $P_d = 0$ for all d; otherwise, $\kappa(X) := \min\{k \mid P_d/d^k : \text{bounded}\}$. This is a birational invariant.
  If X is singular, we take a resolution to singularities.
- An irreducible variety X over $\mathbb{C}$ is called uniruled if there exists a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ where Y is an irreducible variety over $\mathbb{C}$ with $\dim Y = \dim X - 1$.

Remark

- We call X is of general type if $\kappa(X) = \dim(X)$.
- Uniruled varieties have $\kappa(X) = -\infty$. The converse is conjectured, but it is not known in dimension > 3.

References

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Introduction

Unitary Shimura varieties

$F := \mathbb{Q}(\sqrt{d})$ $(d < 0)$,
$\langle\ ,\ \rangle \colon L \times L \to F$: Hermitian lattice of sign $(1, n)$ over $\mathscr{O}_F$ $(n > 0)$.
$L^\vee := \{v \in L \mid \mathrm{Tr}_{F/\mathbb{Q}}\langle v, w\rangle \in \mathbb{Z} \text{ for any } w \in L\}$: dual latttice of L
$\mathrm{U}(L)$: unitary group of $(L, \langle\ ,\ \rangle)$
$\widetilde{\mathrm{U}}(L) := \{g \in \mathrm{U}(L) \mid g|_{L^\vee/L} = 1\}$: discriminant kernel

$$\begin{aligned} D_L &:= \{w \in \mathbb{P}(L \otimes_{\mathscr{O}_F} \mathbb{C}) \mid \langle w, w\rangle > 0\} \\ &\cong \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\} \end{aligned}$$

: Hermitian symmetric domain associated with $\mathrm{U}(L)(\mathbb{R}) \cong \mathrm{U}(1, n)$.
For a finite index subgroup $\Gamma \subset \mathrm{U}(L)$, we define

$$\mathscr{F}_L(\Gamma) := \Gamma \backslash D_L \text{ (unitary Shimura variety).}$$

This is a quasi-projective variety of dimension n over $\mathbb{C}$.

Unitary Shimura varieties

For a quadratic lattice M over $\mathbb{Z}$, we define

$$\begin{aligned}\widetilde{\mathrm{O}}(M) &:= \{g \in \mathrm{O}(M)(\mathbb{Z}) \mid g|_{M^\vee/M} = \mathrm{id}\}\\ \widetilde{\mathrm{O}}^+(M) &:= \widetilde{\mathrm{O}}(M) \cap \mathrm{O}^+(M)(\mathbb{Z})\\ \mathscr{D}_M &:= \{w \in \mathbb{P}(M \otimes_{\mathbb{Z}} \mathbb{C}) \mid (w,w) = 0,\ (w,\overline{w}) > 0\}^+.\end{aligned}$$

$(L, \langle\ ,\ \rangle)$:Hermitian lattice of sign $(1,n)$ over $\mathscr{O}_F$
$(L_Q, (\ ,\ ))$: quadratic lattice of signature $(2,2n)$ associated with L over $\mathbb{Z}$.
Here L_Q is L considered as a free $\mathbb{Z}$-module and $(\ ,\ ) := \mathrm{Tr}_{F/\mathbb{Q}}\langle\ ,\ \rangle$
We have an embedding

$$\mathrm{U}(L)(\mathbb{R}) \cong \mathrm{U}(1,n) \hookrightarrow \mathrm{O}^+(L_Q)(\mathbb{R}) \cong \mathrm{O}^+(2,2n)$$
$$\iota\colon D_L \hookrightarrow \mathscr{D}_{L_Q}.$$

For $r \in L_Q$, we define the reflection with respect to r

$$\sigma_r(\ell) := \ell - \frac{2(\ell,r)}{(r,r)} r \in \mathrm{O}(L_Q)(\mathbb{Q})\ (\ell \in L_Q).$$

Unitary Shimura varieties

For $\lambda \in L$ with $\lambda \neq 0$, we define the special divisors

$$H(\lambda) := \{w \in D_L \mid \langle w, \lambda \rangle = 0\} \subset D_L,$$
$$\mathscr{H}(\lambda) := \{w \in \mathscr{D}_{L_Q} \mid (w, \lambda) = 0\} \subset \mathscr{D}_{L_Q}.$$

Then, we have

$$\iota(H(\lambda)) = \iota(D_L) \cap \mathscr{H}(\lambda) \subset \mathscr{D}_{L_Q}.$$

We say a quadratic lattice M over $\mathbb{Z}$ is 2-elementary if

$$M^\vee/M \cong (\mathbb{Z}/2\mathbb{Z})^{\ell(M)}.$$

We also define

$$\delta(M) := \begin{cases} 0 & ((v,v) \in \mathbb{Z} \text{ for any } v \in M^\vee) \\ 1 & ((v,v) \notin \mathbb{Z} \text{ for some } v \in M^\vee). \end{cases}$$

Main results

Let $\mathscr{F}_L(\Gamma) := \Gamma \backslash D_L$ be the unitary Shimura variety ($\Gamma \subset \mathrm{U}(L)$).

Theorem (Canonical singularities [M1, arXiv:2008.08095])

1. If $n > 4$, then $\mathscr{F}_L(\Gamma)$ has canonical singularities at all points.
2. There exists a toroidal compactification $\overline{\mathscr{F}_L(\Gamma)}$ of $\mathscr{F}_L(\Gamma)$ which has canonical singularities over cusps.

Remark

1. (Gritsenko-Hulek-Sankaran, 2007) Similar results for Shimura varieties of $\mathrm{SO}(2, n)$ $(n > 8)$.
2. Our results are generalization of the results of Behrens (2012).

Condition $(\star)$

The above toroidal compactification $\overline{\mathscr{F}_L(\Gamma)}$ has no ramification divisors through cusps.

Main results

Theorem (General type [M1, arXiv:2008.08095])

Let $F = \mathbb{Q}(\sqrt{-1})$ or $F = \mathbb{Q}(\sqrt{-3})$. $(n = \dim D_L)$
Assume that $\exists L_Q \hookrightarrow II_{2,26} := \mathbb{U}^{\oplus 2} \oplus \mathbb{E}_8(-1)^{\oplus 3}$ such that

- $r((L_Q)^{\perp}) := \#\{v \in (L_Q)^{\perp} \mid (v, v) = -2\} > 0$.
- $r((L_Q)^{\perp}) < r((L_Q)_r^{\perp})$ for any $r \in L_Q$ with $-\sigma_r \in \widetilde{\mathrm{O}}^+(L_Q)$, where $(L_Q)_r := \{v \in L_Q \mid (v, r) = 0\}$.
- $n > (24 + r((L_Q)^{\perp}))/(\#\mathscr{O}_F^{\times})$.

We assume $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ satisfies the condition $(\star)$. Then $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ is of general type.

Remark

Kondo (1993,1999), Gritsenko-Hulek-Sankaran (2007) and Ma (2018) showed the Kodaira dimension of certain orthogonal Shimura varieties are non-negative, more generally, some of them are general type.

Main results

Example

$F = \mathbb{Q}(\sqrt{-1})$

$M_{II_{2,26}}$: Hermitian lattice of sign (1,13) defined by $A \oplus B^{\oplus 3}$ where

$$A := \begin{pmatrix} 0 & -\frac{\sqrt{-1}}{2} \\ \frac{\sqrt{-1}}{2} & 0 \end{pmatrix}, \quad B := \begin{pmatrix} -1 & \frac{\sqrt{-1}}{2} & \frac{\sqrt{-1}}{2} & -\frac{1}{2} \\ -\frac{\sqrt{-1}}{2} & -1 & -\frac{1}{2} & -\frac{\sqrt{-1}}{2} \\ -\frac{\sqrt{-1}}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{-1}}{2} & -\frac{1}{2} & -1 \end{pmatrix}$$

(B : Iyanaga's matrix)

$L_{\mathbb{A}_1(-1)^{\oplus 2}}$: Hermitian lattice of sign (0,1) defined by (-1).

L : orthogonal complement of $L_{\mathbb{A}_1(-1)^{\oplus 2}}$ in $M_{II_{2,26}}$.

Then L satisfies the top three conditions in the Theorem. (The condition $(\star)$ has not been proved for $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ yet.)

Main results

Example

$F = \mathbb{Q}(\sqrt{-3})$

$N_{II_{2,26}}$: Hermitian lattice of sign (1,13) defined by $C \oplus D^{\oplus 3}$ where

$$C := \begin{pmatrix} 0 & -\frac{\sqrt{-3}}{3} \\ \frac{\sqrt{-3}}{3} & 0 \end{pmatrix}, \quad D := \begin{pmatrix} -1 & 0 & -\frac{\sqrt{-3}}{3} & -\frac{\sqrt{-3}}{3} \\ 0 & -1 & -\frac{\sqrt{-3}}{3} & \frac{\sqrt{-3}}{3} \\ \frac{\sqrt{-3}}{3} & \frac{\sqrt{-3}}{3} & -1 & 0 \\ \frac{\sqrt{-3}}{3} & -\frac{\sqrt{-3}}{3} & 0 & -1 \end{pmatrix}$$

$L_{\mathbb{A}_2(-1)^{\oplus k}}$: Hermitian lattice defined by $\begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}^{\oplus k}$ $(1 \leqslant k \leqslant 4)$.

$L := (L_{\mathbb{A}_2(-1)^{\oplus k}})^{\perp}$ in $M_{II_{2,26}}$. Then L satisfies the top three conditions in the Theorem. (The condition $(\star)$ has not been proved for $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ yet.)

Main results

Theorem ([M2, arXiv:2008.13106])

Let $(L, \langle\ ,\ \rangle)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1,5)$ and let $(L_Q, (\ ,\ ))$ be the associated quadratic lattice over $\mathbb{Z}$ of signature $(2,10)$. Assume that

1. L_Q is even 2-elementary, $\delta(L_Q) = 0$ and $\ell(L_Q) \leqslant 8$. Moreover, $\ell(L_Q) \leqslant 6$ if $F = \mathbb{Q}(\sqrt{-3})$.
2. $2\langle \ell, r\rangle \in \mathscr{O}_F$ for any $\ell, r \in L$ with $\langle r, r\rangle = -1$.

Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled.

Remark

To prove this Theorem, we use reflective modular forms constructed by Yoshikawa. Using reflective modular forms constructed by Gritsenko-Hulek, we can give 3 more sufficient conditions for uniruledness in terms of Hermitian lattices.

Main results

Theorem (Uniruledness [M2, arXiv:2008.13106])

1. For $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, there exist Hermitian lattices L over $\mathscr{O}_F$ of signature $(1,5)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.
2. For $F = \mathbb{Q}(\sqrt{-1})$, there exist Hermitian lattices L over $\mathscr{O}_F$ of signature $(1,4)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.
3. For $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-2})$, there exist Hermitian lattices L over $\mathscr{O}_F$ of signature $(1,3)$ such that $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ are uniruled.

Remark

Gritsenko-Hulek (2014) proved certain orthogonal Shimura varieties are uniruled.

Uniruled unitary Shimura varieties

Example

$F = \mathbb{Q}(\sqrt{-2})$

$L_{\mathbb{U}\oplus\mathbb{U}(2)}$:Hermitian lattice of sign (1,1) defined by $\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$.

$L_{\mathbb{D}_4}$:Hermitian lattice of sign (0,2) defined by $\begin{pmatrix} -1 & -\frac{1+\sqrt{-2}}{2} \\ -\frac{1-\sqrt{-2}}{2} & -1 \end{pmatrix}$.

$(L_{\mathbb{U}\oplus\mathbb{U}(2)} \oplus L_{\mathbb{D}_4} \oplus L_{\mathbb{D}_4})_Q \cong \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4(-1) \oplus \mathbb{D}_4(-1)$.
Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled for $L := L_{\mathbb{U}\oplus\mathbb{U}(2)} \oplus L_{\mathbb{D}_4} \oplus L_{\mathbb{D}_4}$.

Canonical singularities on unitary Shimura varieties

$V := \mathbb{C}^n$, $G \subset \mathrm{GL}(V)$: finite group, $g \in G$: order$=m$
$\xi^{a_1}, \ldots, \xi^{a_n}$: eigenvalues of g acting on V where $\xi = \exp(2\pi\sqrt{-1}/m)$ and $0 \leqslant a_i < m$.
g is called quasi-reflection if $a_1 = \cdots = a_{n-1} = 0$ for some order, and reflection if the remaining a_n is $m/2$. The age of g is defined by

$$\mathrm{A}(g) := \sum_{i=1}^{n} \frac{a_i}{m}.$$

Theorem (The Reid-Shepherd-Barron-Tai criterion)

Assume every $g \in G$ with $g \neq 1$ does not act on V as a quasi-reflection. Then V/G has canonical singularities if and only if $\mathrm{A}(g) \geqslant 1$ for any $g \neq 1$.

Canonical singularities on unitary Shimura varieties

$D_L = \{w \in \mathbb{P}(L \otimes_{\mathscr{O}_F} \mathbb{C}) \mid \langle w, w \rangle > 0\}$
$[w] \in D_L$, $W := \mathbb{C}w \subset L \otimes_{\mathscr{O}_F} \mathbb{C}$
$S := (W + \overline{W})^{\perp} \cap L$ and $T := S^{\perp} \subset L$, then $S_{\mathbb{C}} \cap T_{\mathbb{C}} = \{0\}$
We put

$$G := \mathrm{Stab}_{\Gamma}([w]) := \{g \in \Gamma \mid g[w] = [w]\}.$$

To investigate the singularities of $\mathscr{F}_L(\Gamma)$, we shall consider the tangent space of D_L at $[w] \in D_L$:

$$V := T_{[w]} D_L \cong \mathrm{Hom}_{\mathbb{C}}(W, W^{\perp}).$$

Since $L_{\mathbb{C}} = S_{\mathbb{C}} \oplus T_{\mathbb{C}}$, we calculate eigenvalues of G on $W^{\perp} \cap S_{\mathbb{C}}$ and $W^{\perp} \cap T_{\mathbb{C}}$.

$$G_0 := \{g \in G \mid gw = w\}$$

$G/G_0 \cong (\mathbb{Z}/r\mathbb{Z})^{\times}$ acts on $T_{\mathbb{C}}$ (for $\exists r$).

Canonical singularities on unitary Shimura

Lemma

Let $g \in G$. Assume that g does not act on V as a quasi-reflection and $n > 3$. Then $\mathrm{A}(g) \geqslant 1$.

Sketch of Proof.

When $\varphi(r) > 2$, eigenvalues arising from $W^{\perp} \cap T_{\mathbb{C}}$ gives $\mathrm{A}(g) \geqslant 1$ by Bertrand's Postulate.
When $\varphi(r) \leqslant 2$, there remains a finite number of cases and Lemma follows from the direct calculation. To show the claim for this case, we use $n > 3$.

Corollary ([M1, arXiv:2008.08095])

1. If $n > 3$, then $\mathscr{F}_L(\Gamma)$ has canonical singularities away from the ramification divisors of $\pi_\Gamma \colon D_L \to \mathscr{F}_L(\Gamma)$.
2. If $n > 4$, then $\mathscr{F}_L(\Gamma)$ has canonical singularities at all points.

Canonical singularities on unitary Shimura varieties

$\overline{\mathscr{F}_L(\Gamma)}$: a toroidal compactification (Ash-Mumford-Rapoport-Tai)
There exist only 0-dim cusps on $\mathscr{F}_L(\Gamma)$. $\overline{\mathscr{F}_L(\Gamma)}$ is locally isomorphic to X/G where X is a toric variety and G is a finite group.
Hence the problem is reduced to toric varieties.

Lemma

Let X be an n-dimensional smooth toric variety over $\mathbb{C}$ and M be the character group of X. Assume that a finite group $G \subset \mathrm{GL}_{\mathbb{Z}}(M)$ acts on X. Then X/G has canonical singularities.

Theorem

There is a toroidal compactification $\overline{\mathscr{F}_L(\Gamma)}$ of $\mathscr{F}_L(\Gamma)$ such that $\overline{\mathscr{F}_L(\Gamma)}$ has canonical singularities on the boundary $\overline{\mathscr{F}_L(\Gamma)} \backslash \mathscr{F}_L(\Gamma)$.

Unitary Shimura varieties of general type

Theorem (Low weight cusp form trick [GHS])

Let $n > 1$. Assume that $\mathscr{F}_L(\Gamma)$ satisfies the condition $(\star)$. If there exists a non-zero cusp form $F_0 \in S_k(\Gamma, \chi)$ of weight $k < n$ which vanishes on ramification divisors, then $\mathscr{F}_L(\Gamma)$ is of general type.

Proof.

$\overline{\mathscr{F}_L(\Gamma)}$: toroidal compactification of $\mathscr{F}_L(\Gamma)$ with canonical singularities and no ramification divisors in the boundary components.
Then we have an injection

$$M_{(n-k)m}(\Gamma, 1) \hookrightarrow H^0(\overline{\mathscr{F}_L(\Gamma)}, (K_{\overline{\mathscr{F}_L(\Gamma)}})^{\otimes m}), \quad F \mapsto F_0^m F.$$

On the other hand, $\dim_{\mathbb{C}} M_{(n-k)m}(\Gamma, 1)$ grows like m^n when $m \to \infty$.
(The Hirzebruch-Mumford proportionality principle)
Therefore $\mathscr{F}_L(\Gamma)$ is of general type. □

Unitary Shimura varieties of general type

For $r \in L$ with $\langle r, r\rangle \neq 0$ and $\xi \in \mathscr{O}_F^\times \backslash \{1\}$, we define the reflection

$$\tau_{r,\xi}(\ell) := \ell - (1-\xi)\frac{\langle \ell, r\rangle}{\langle r, r\rangle} r \in \mathrm{U}(L)(\mathbb{Q}) \quad (\ell \in L).$$

Assume $n > 3$. The ramification divisors of $D_L \to \mathscr{F}_L(\Gamma)$ and $\mathscr{D}_{L_Q} \to \Gamma' \backslash \mathscr{D}_{L_Q}$ are defined by

$$\bigcup_{\substack{r \in L/\{\pm 1\}:\text{primitive} \\ \exists \xi,\ \tau_{r,\xi} \in \Gamma \ \text{or} \ -\tau_{r,\xi} \in \Gamma}} H(r), \qquad \bigcup_{\substack{r \in L_Q/\{\pm 1\}:\text{primitive} \\ \sigma_r \in \Gamma' \ \text{or} \ -\sigma_r \in \Gamma'}} \mathscr{H}(r).$$

Proposition

For $F = \mathbb{Q}(\sqrt{d})$, assume $d \equiv 2, 3 \pmod 4$ or $d = -3$. Then

$$\iota\Big(\bigcup_{\substack{r \in L/\{\pm 1\}:\text{primitive} \\ \exists \xi,\ \tau_{r,\xi} \in \widetilde{\mathrm{U}}(L) \ \text{or} \ -\tau_{r,\xi} \in \widetilde{\mathrm{U}}(L)}} H(r)\Big) \subset \bigcup_{\substack{r \in L_Q/\{\pm 1\}:\text{primitive} \\ \sigma_r \in \widetilde{\mathrm{O}}^+(L_Q) \ \text{or} \ -\sigma_r \in \widetilde{\mathrm{O}}^+(L_Q)}} \mathscr{H}(r) \cap \iota(D_L).$$

Unitary Shimura varieties of general type

$\Phi_{12} \in M_{12}(\mathrm{O}^+(II_{2,26}), \det)$: the Borcherds lift of Δ^{-1}.
S : primitive quadratic sublattice of $II_{2,26}$ of signature $(2, s)$ for $s \geqslant 1$.

$$R_S := \{r \in II_{2,26} \mid r^2 = -2,\ (r, S) = 0\},\ r(S^\perp) = \#R_S.$$

Let $S_r \subset S$ be the orthogonal complement of r in S. Now we define the quasi-pullback of Φ_{12} as

$$\Phi_{12}|_S := \left(\frac{\Phi_{12}(Z)}{\prod_{r \in R_S/\{\pm 1\}} (Z, r)} \right) \Bigg|_{\mathscr{D}_S}.$$

Lemma ([GHS])

Assume $r(S^\perp) < r(S_r^\perp)$ for any $r \in S$ satisfying $-\sigma_r \in \widetilde{\mathrm{O}}^+(S)$. Then $\Phi_{12}|_S$ is a modular form of weight $12 + r(S^\perp)/2$ vanishing on the ramification divisors of $\mathscr{D}_S \to \widetilde{\mathrm{O}}^+(S)\backslash\mathscr{D}_S$. Moreover, if $r(S^\perp) \neq 0$, it is a cusp form.

Unitary Shimura varieties of general type

Theorem (General type [M1, arXiv:2008.08095])

Let $F = \mathbb{Q}(\sqrt{-1})$ or $F = \mathbb{Q}(\sqrt{-3})$. Assume that $\exists L_Q \hookrightarrow II_{2,26}$ such that

- $r((L_Q)^{\perp}) := \#\{v \in (L_Q)^{\perp} \mid (v,v) = -2\} > 0$.
- $r((L_Q)^{\perp}) < r((L_Q)^{\perp}_r)$ for any $r \in L_Q$ with $-\sigma_r \in \widetilde{\mathrm{O}}^+(L_Q)$, where $(L_Q)_r := \{v \in L_Q \mid (v,r) = 0\}$.
- $n > (24 + r((L_Q)^{\perp}))/(\#\mathscr{O}_F^{\times})$.

We assume $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ satisfies the condition $(\star)$. Then $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ is of general type.

Proof.

We put $w := \#\mathscr{O}_F^{\times}$. The quasi-pullback $\Phi_{12}|_{L_Q}$ is a cusp form of weight $12 + r((L_Q)^{\perp})/2$ vanishing on the ramification divisors. A $(w/2)$-th root of $\iota^{\star}(\Phi_{12}|_{L_Q})$ is a modular form of weight $(24 + r((L_Q)^{\perp}))/w < n$ vanishing on the ramification divisors. By the low weight cusp form trick, $\mathscr{F}_L(\widetilde{\mathrm{U}}(L))$ is of general type. □

Unitary Shimura varieties of general type

F	$\mathbb{Q}(\sqrt{-1})$	
signature of L	(1,12)	(1,11)
$(L_Q)^{\perp}$	$\mathbb{A}_1(-1)^{\oplus 2}$	$\mathbb{A}_2(-1)^{\oplus 2}$
weight of a square root of $\iota^{\star}(\Phi_{12}\|_{L_Q})$	7	9

F	$\mathbb{Q}(\sqrt{-3})$			
signature of L	(1,12)	(1,11)	(1,10)	(1,9)
$(L_Q)^{\perp}$	$\mathbb{A}_2(-1)$	$\mathbb{A}_2(-1)^{\oplus 2}$	$\mathbb{A}_2(-1)^{\oplus 3}$	$\mathbb{A}_2(-1)^{\oplus 4}$
weight of a third root of $\iota^{\star}(\Phi_{12}\|_{L_Q})$	5	6	7	8

F	$\mathbb{Q}(\sqrt{-3})$	
signature of L	(1,11)	(1,10)
$(L_Q)^{\perp}$	$\mathbb{D}_4(-1)$	$\mathbb{A}_2(-1) \oplus \mathbb{D}_4(-1)$
weight of a third root of $\iota^{\star}(\Phi_{12}\|_{L_Q})$	8	9

Uniruled unitary Shimura varieties

A modular form $F_k \in M_k(\Gamma, \chi)$ on D_L is called reflective if

$$\mathrm{Supp}(\mathrm{div}\, F_k) \subset \bigcup_{\substack{r \in L/\{\pm 1\},\ r \text{ is primitive} \\ \sigma_r \in \Gamma \text{ or } -\sigma_r \in \Gamma}} H(r).$$

A reflective modular form F_k is called strongly reflective if the multiplicity of each irreducible component of $\mathrm{div}(F_k)$ is 1.
For modular forms on $\mathscr{D}_{L_Q}$, we define the notions similarly.

Theorem (Uniruledness criterion [GH])

Let $n > 1$. Let $a, k > 0$ be positive integers satisfying $k > an$. If there exists a non-zero reflective modular form $F_{a,k} \in M_k(\Gamma, \chi)$ of weight k for which the multiplicity of every irreducible component of $\mathrm{div}(F_{a,k})$ is less than or equal to a, then $\mathscr{F}_L(\Gamma)$ is uniruled.

Proof.

Use the numerical criterion of uniruledness due to Miyaoka and Mori. $\square$

Uniruled unitary Shimura varieties

Strongly reflective modular forms are very rare. In some special cases, we can construct strongly reflective modular forms by Borcherds lifts and Gritsenko lifts.

Theorem

1. (Yoshikawa, 2013) Let M_1 be an even 2-elementary quadratic lattice over $\mathbb{Z}$ of signature $(2, 10)$ and $\delta(M_1) = 0$. There exists a strongly reflective modular form Ψ_{M_1} of weight $w(M_1) = 2^{(16-\ell(M_1))/2} - 4$ on $\mathscr{D}_{M_1}$ for $\mathrm{O}^+(M_1)$.
2. (Yoshikawa, 2013) Let $M_2 := \mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_6(-1)$ be a quadratic lattice over $\mathbb{Z}$ of signature $(2, 8)$. There exists a strongly reflective modular form Ψ_{M_2} of weight $w(M_2) = 102$ on $\mathscr{D}_{M_2}$ for $\mathrm{O}^+(M_2)$.
3. (Gritsenko-Hulek, 2016) Let $N := \mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(-2)$. There exists a strongly reflective cusp form Φ_{124} of weight 124.

Uniruled unitary Shimura varieties

Theorem ([M2, arXiv:2008.13106])

Let $(L, \langle\ ,\ \rangle)$ be a Hermitian lattice over $\mathscr{O}_F$ of signature $(1,5)$ and let $(L_Q, (\ ,\ ))$ be the associated quadratic lattice over $\mathbb{Z}$ of signature $(2,10)$. Assume that

1. L_Q is even 2-elementary, $\delta(L_Q) = 0$ and $\ell(L_Q) \leqslant 8$. Moreover, $\ell(L_Q) \leqslant 6$ if $F = \mathbb{Q}(\sqrt{-3})$.
2. $2\langle \ell, r\rangle \in \mathscr{O}_F$ for any $\ell, r \in L$ with $\langle r, r\rangle = -1$.

Then $\mathscr{F}_L(\mathrm{U}(L)(\mathbb{Z}))$ is uniruled.

Proof.

Since L_Q is 2-elementary with $\ell(L_Q) \leqslant 8$ and $\delta(L_Q) = 0$, we have a strongly reflective modular form Ψ_{L_Q} on $\mathscr{D}_{L_Q}$ constructed by Yoshikawa.

$$\mathrm{div}(\Psi_{L_Q}) = \bigcup_{r \in \Delta_{L_Q}/\{\pm 1\}} \mathscr{H}(r), \quad \Delta(L_Q) := \{v \in L_Q \mid (v,v) = -2\}$$

Put $\Xi := \{r \in L \mid \langle r, r \rangle = -1\}$. When $F \neq \mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, we have

$$\mathrm{div}(\iota^\star \Psi_{L_Q}) = \bigcup_{r \in \Xi/\{\pm 1\}} H(r).$$

For any $r \in L$ with $\langle r, r \rangle = -1$, we have $2\langle \ell, r \rangle \in \mathscr{O}_F$ for any $\ell \in L$. Hence for such r, we have

$$\frac{2\langle \ell, r \rangle}{\langle r, r \rangle} \in \mathscr{O}_F,$$

so $H(r)$ is contained in the ramification divisors of $\pi_{\mathrm{U}(L)} \colon D_L \to \mathrm{U}(L)\backslash D_L$. Therefore $\iota^\star \Psi_{L_Q}$ is a strongly reflective modular form of weight $\geqslant 12$ on D_L. Hence $\mathscr{F}_L(\mathrm{U}(L))$ is uniruled. (When $F = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, Theorem is proved in the same way.) $\square$

Uniruled unitary Shimura varieties

Quadratic lattices of sign (2,10)	$\ell(L_Q)$	$\delta(L_Q)$	F
$\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{E}_8(-2)$	10	0	$\mathbb{Q}(\sqrt{-1})$
$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(-2)$	8	0	$\mathbb{Q}(\sqrt{-1})$
$\mathbb{U} \oplus \mathbb{U}(2) \oplus \mathbb{D}_4(-1) \oplus \mathbb{D}_4(-1)$	6	0	$\mathbb{Q}(\sqrt{-2})$
$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_4(-1) \oplus \mathbb{D}_4(-1)$	4	0	$\mathbb{Q}(\sqrt{-1})$
$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_8(-1)$	2	0	$\mathbb{Q}(\sqrt{-1})$
$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(-1)$	0	0	$\mathbb{Q}(\sqrt{-1})$

Quadratic lattices of sign (2,8)	F
$\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{D}_6(-1)$	$\mathbb{Q}(\sqrt{-1})$

Summary

- Canonical singularities on unitary Shimura varieties for $\mathrm{U}(1,n)$ $(n>4)$
- Unitary Shimura varieties of general type for $\mathrm{U}(1,n)$
  $(9 \leqslant n \leqslant 12,\ F=\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3}))$
- Uniruled unitary Shimura varieties for $\mathrm{U}(1,n)$
  $(n=3,4,5,\ F=\mathbb{Q}(\sqrt{-1}),$ or $\mathbb{Q}(\sqrt{-2}))$

Problem.

- The Kodaira dimension of unitary Shimura varieties over
  $F \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-2})$ and $\mathbb{Q}(\sqrt{-3})$.
- Construction of reflective modular forms on D_L, not using reflective modular forms on $\mathscr{D}_{L_Q}$
- Unitary analogue of the "Arithmetic Mirror Symmetry Conjecture" (Gritsenko, Ma)