On the formal degree conjecture for inner forms of Sp_4 and GSp_4

Hirotaka KAKUHAMA

Kyoto University

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Today's talk

Let

- F be a p-adic field,
- G be a reductive group over F,
- A be the maxmal F-split torus of the center of G,
- π be an irreducible square integrable representation of $G(F)$.

Then, the formal degree $\deg \pi$ is a positive real number satisfying

$$\int_{G(F)/A(F)} (\pi(g)v_1, v_2)\overline{(\pi(g)v_3, v_4)}\, dg = \frac{1}{\deg \pi}(v_1, v_3)\overline{(v_2, v_4)}.$$

Here, $(\ ,\ )$ is a non-zero $G(F)$-invariant Hermitian pairing of the space of π.

I will talk about the following topics.

- The formal degree conjecture of Hiraga-Ichino-Ikeda, which describes $\deg \pi$ in terms of the Langlands parameter (ϕ, η).
- The local Langlands correspondence for the non-split inner forms of Sp_4, GSp_4, which is established by using the local theta correspondence.
- The formal degree conjecture for them.
- The behavior of the formal degrees under the local theta correspondence.

Contents

§1 Quaternionic unitary groups and similitude groups

Let

- F be a non-Archimedean local field of characteristic 0,
- D be the division quaternion algebra over F,
- $V_{1,1} = D^2$ be the two-dimensional right Hermitian space over D on which the Hermitian form

$$(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix})_{1,1} = x_1^* y_2 + x_2^* y_1$$

  is defined,
- $G_{1,1}$ be the unitary group of $V_{1,1}$,
- $\widetilde{G}_{1,1}$ be the similitude group of $V_{1,1}$.

Then, $G_{1,1}, \widetilde{G}_{1,1}$ are the (unique) non-split inner form of $\mathrm{Sp}_4, \mathrm{GSp}_4$ respectively.

Moreover, we consider

- the skew-Hermitian space $W_{1,1} = D^2$ on which the skew-Hermitian form

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{1,1} = x_1^* y_2 - x_2^* y_1$$

  is defined,
- the skew-Hermitian space $W_{3,0} = D^3$ on which the skew-Hermitian form

$$\langle (x_1, y_1, z_1), (x_2, y_2, z_2) \rangle_{3,0} = x_1^* \alpha x_2 + y_1^* \beta y_2 + z_1^* \alpha\beta z_2$$

  is defined, (Here, α, β are elements of $D^\times$ such that $\alpha^* = -\alpha$, $\beta^* = -\beta$, and $\beta\alpha = -\alpha\beta$.)
- the unitary groups $H_{1,1}$, $H_{3,0}$ of $W_{1,1}$, $W_{3,0}$ respectively,
- and the similitude groups $\widetilde{H}_{1,1}$, $\widetilde{H}_{3,0}$ of $W_{1,1}$, $W_{3,0}$ respectively.

Then,

- $H_{1,1}$ is the inner form of $\mathrm{SO}_{2,2}$,
- $H_{3,0}$ is the inner form of $\mathrm{SO}_{3,3}$.

Moreover, there are accidental isomorphisms

- $\widetilde{H}_{1,1} \cong D^{\times} \times \mathrm{GL}_2(F)/\{(a, a^{-1}) \mid a \in F^{\times}\}$,
- $\widetilde{H}_{3,0} \cong D_4^{\times} \times F^{\times}/\{(a, a^{-2}) \mid a \in F^{\times}\}$.
  Here, D_4 denotes a central division algebra over F with $[D_4 : F] = 16$.

Note that the local Langlands correspondence and the formal degree conjecture has been established for $F^{\times}$, $D^{\times}$, $\mathrm{GL}_2(F)$, $D_4^{\times}$. Thus, they are also available for $\widetilde{H}_{1,1}$ and $\widetilde{H}_{3,0}$.

§2 Local theta correspondence

Setup

- $\epsilon = \pm 1$: fixed sign,
- D: division quaternion algebra over F with the canonical involution $*$,
- V: m-dimensional right ϵ-Hermitian space over D equipped with the non-degenerate ϵ-Hermitian form $(\ ,\ )$, i.e.
    - V is an m-dimensional right D-vector space, and
    - $(\ ,\ ) : V \times V \to D$ is a non-degenerate pairing satisfying

$$(xa, yb + zc) = a^*(x, y)b + a^*(x, z)c$$
$$(y, x) = \epsilon(x, y)^*$$

  for $x, y, z \in V$ and $a, b, c \in D$,
- $G(V)$: the unitary group of V,
- W: n-dimensional left $(-\epsilon)$-Hermitian space over D equipped with the $(-\epsilon)$-Hermitian form $\langle\ ,\ \rangle$,
- $G(W)$: the unitary group of W.

- Then, $(G(V), G(W))$ consists a reductive dual pair,
- $l = 2n - 2m - \epsilon$,
- $\mathbb{W} := V \otimes_D W$ with Symplectic form $\langle\langle\ ,\ \rangle\rangle$ given by $T(\frac{1}{2}(\ ,\ ) \cdot \langle\ ,\ \rangle^*)$ where T is the reduced trace of D,
- $\psi : F \to \mathbb{C}^\times$: a non-trivial additive character of F,
- ω_ψ: the Weil representation realized on the space $\mathcal{S}(\mathbb{X})$ where $\mathbb{W} = \mathbb{X} \oplus \mathbb{Y}$ is a polar decomposition,
- For an irreducible representation π of $G(W)$, denote

$$\Theta_\psi(\pi, V) := (\omega_\psi \otimes \pi^\vee)_{G(W)}$$

  the $G(W)$-coinvariant space.
- denote by $\theta_\psi(\pi, V)$ the maximal semisimple quotient of $\Theta_\psi(\pi, V)$ if it is non-zero (put $\theta_\psi(\pi, V) = 0$ if $\Theta_\psi(\pi, V) = 0$). It is known to be irreducible if it is non-zero ((a part of) Howe duality: (Waldspurger'90, Gan-Sun'17)).

Local theta correspondence for similitude groups

For an irreducible representation $\widetilde{\pi}$ of $\widetilde{G}_{1,1}$, we can define irreducible representations

$$\theta_\psi(\widetilde{\pi}, V_{1,1}), \text{ and } \theta_\psi(\widetilde{\pi}, V_{3,0})$$

of $\widetilde{H}_{1,1}$ and $\widetilde{H}_{3,0}$. They satisfy

- for a irreducible constituent π of $\widetilde{\pi}|_{G_{1,1}}$,

$$\theta_\psi(\pi.V) \neq 0 \Leftrightarrow \theta_\psi(\widetilde{\pi}, V) \neq 0,$$

- if $\widetilde{\pi}|_{G_{1,1}}$ is of the form $(\oplus_{i=1}^t \pi_i)^k$, then, we have

$$\theta_\psi(\widetilde{\pi}, V)|_H = (\oplus_{i=1}^t \theta_\psi(\pi_i, V))^k.$$

§3 Local Langlands correspondence

Notations:

- F: a non-Archimedean local field with $\mathrm{ch}(F) = 0$,
- G: a connected reductive group over F,
  (However, we consider orthogonal groups as exceptions.)
- $\Pi(G(F))$: the set of the irreducible smooth admissible representations of $G(F)$.

The local Langlands correspondence (= LLC) is a classification theory of the irreducible representations $\Pi(G(F))$ of $G(F)$, which is still conjecture in general. Let $\pi \in \Pi(G(F))$. Roughly speaking, LLC asserts that π is characterized by its L-parameter ϕ_π and an irreducible representation η_π of a certain finite group $\widetilde{\mathcal{S}}(\widehat{G})$.

Classification by L-parameters

An L-parameter for G is a conjugacy class by $\widehat{G}$ of homomorphisms

$$\phi : L_F \to {}^LG$$

satisfying some properties. Here,

- W_F denotes the Weil group of F,
- $L_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ is the local Langlands group,
- ${}^LG$ denotes the L-group of G,
- $\widehat{G}$ denotes the Langlands dual group of G.

We denote by $\Phi_F(G)$ the set of all L-parameters for G. Then, LLC asserts that there is a finite-to-one map

$$\mathcal{L} : \Pi(G(F)) \to \Phi_F(G).$$

For $\phi \in \Phi_F(G)$, we put $\Pi_\phi(G(F)) := \mathcal{L}^{-1}(\phi)$. It is called the L-packet for G associated with ϕ.

Classification of an L-packet

Let $\phi \in \Phi_F(G)$. Then, there is a bijection

$$\Pi_\phi(G(F)) \to \mathrm{Irr}(\widetilde{\mathcal{S}}_\phi(\widehat{G}), \zeta'_G).$$

Here,

- $C_\phi(\widehat{G})$ is the centralizer of $\mathrm{Im}(\phi)$ in $\widehat{G}$,
- $\widehat{G}_{sc}$ the simply connected cover of $\widehat{G}/Z_{\widehat{G}}$,
- $\widetilde{S}_\phi(\widehat{G})$ is the preimage of $C_\phi(\widehat{G})Z_{\widehat{G}}/Z_{\widehat{G}} \subset \widehat{G}/Z_{\widehat{G}}$ in $\widehat{G}_{\mathrm{sc}}$,
- $\widetilde{\mathcal{S}}_\phi(\widehat{G})$ is the component group $\pi_0(\widetilde{S}_\phi(\widehat{G}))$ of $\widetilde{S}_\phi(\widehat{G})$,
- ζ'_G is a character of $Z(\widehat{G}_{\mathrm{sc}})$ associated with G, (It is not necessarily canonical.)
- $\mathrm{Irr}(\widetilde{\mathcal{S}}_\phi(\widehat{G}), \zeta'_G)$ is the set of irreducible representations of $\widetilde{\mathcal{S}}_\phi(\widehat{G})$ whose restriction to $Z(\widehat{G}_{\mathrm{sc}})$ is the scalar multiplication by ζ'_G.

Known cases:

- the general linear group GL_n by Harris-Taylor'01, by Henniart'00, and by Scholze'13,
- the quasi-split special orthogonal groups $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n,\chi)$ by Arthur'13,
- the symplectic group $\mathrm{Sp}(2n)$ by Arthur'13,
- quasi-split unitary groups $\mathrm{U}(n)$ by Mok'15,
- (non-quasi-split) unitary groups $\mathrm{U}(W)$ by Kaletha-Minguez-Shin-White'14.

Langlands parameters for $\widetilde{G}_{1,1}$

Theorem (Gan-Tantono'14)

Let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}_{1,1}$, then the exactly one of $\theta_\psi(\widetilde{\pi}, V_{1,1}) \neq 0$ or $\theta_\psi(\widetilde{\pi}, V_{3,0}) \neq 0$ occurs. Moreover,

- *if $0 \neq \theta_\psi(\widetilde{\pi}, V_{1,1}) \cong \rho \boxtimes \tau$, then*

$$\phi_{\widetilde{\pi}} = \phi_\rho \oplus \phi_\tau,$$

- *if $0 \neq \theta_\psi(\widetilde{\pi}, V_{3,0}) = \Pi \boxtimes \mu$, then*

$$\phi_{\widetilde{\pi}} = \phi_\Pi.$$

Put $\widetilde{\phi} = \phi_{\widetilde{\pi}}$. Then, the irreducible representation $\eta_{\widetilde{\pi}}$ is also given case by case. We note that the component group $S_{\widetilde{\phi}}$ is Abelian.

Langlands parameters for $G_{1,1}$

Denote by $\mathfrak{p} : \widehat{\widetilde{G}_{1,1}} \to \widehat{G_{1,1}}$ the standard projection. We call an irreducible representation $\widetilde{\pi}$ of $\widetilde{G}_{1,1}$ the case S if $\theta_\psi(\widetilde{\pi}, V_{1,1}) \neq 0$ and $\phi_{\widetilde{\pi}} = \phi_1 \oplus (\phi_1 \otimes \chi)$ for some quadratic character ϕ.

Theorem (Choiy'17)

Let π be an irreducible representation of $G_{1,1}$, and let $\widetilde{\pi}$ be an irreducible representation of $\widehat{G}_{1,1}$ so that $\widetilde{\pi}|_{G_{1,1}} \supset \pi$. Then,

$$\phi_\pi = \mathfrak{p} \circ \phi_{\widetilde{\pi}}.$$

Moreover, if k denotes the multiplicity of π in $\widetilde{\pi}$, then we have

$$k = \begin{cases} \frac{1}{2}\dim \eta_\pi & (\ \textit{Case S}\ ) \\ \dim \eta_\pi & (\ \textit{otherwise}\ ) \end{cases}.$$

We do not write down the definition of η_π.

§4 Formal degree conjecture

Notations

- F: non-Archimedean local field of $\mathrm{ch}(F) = 0$, $\mathcal{O}_F/\varpi_F = \mathbb{F}_q$,
- G: connected reductive group $/F$,
- A: maximal F split torus of the center of G,
- π: square integrable irreducible representation of G,
- $(\ ,\ )$: non-zero $G(F)$-invariant Hermitian form on π.

The formal degree of π is a non-zero constant $\deg \pi$ satisfying

$$\int_{G/A} (\pi(g)x_1, x_2)\overline{(\pi(g)x_3, x_4)}\, dg = \frac{1}{\deg \pi}(x_1, x_3)\overline{(x_2, x_4)}$$

for all $x_1, \ldots, x_4 \in \pi$.

Haar measures

Fix a non-trivial additive character ψ of F. We choose the Haar measure dg on G/A by using the motive of G (Gan-Gross'99). It depend only on G and ψ. For example:

- Suppose that G is unramified and $A = 1$. Then the Haar measure dg is normalized so that

$$|G(\mathcal{O}_F)| = q^{-\dim G} \# G(\mathbb{F}_q).$$

- The Haar measure dg on $G_{1,1}$ is given by

$$|G_{1,1} \cap \mathrm{GL}_2(\mathcal{O}_D)| = q^{-3}(1 - q^{-1})(1 + q^{-1})^2.$$

- The Haar measure dg on $H_{3,0}$ is given by

$$|H_{3,0}| = 2q^{-6}(1 + q^{-1})(1 + q^{-2}).$$

Formal degree conjecture

Let

- A be the maximal split torus of the center of G,
- $C'_\phi = C_\phi \cap (\widehat{G/A})$,
- $Z'(\widehat{G}) = Z(\widehat{G}) \cap (\widehat{G/A})$.

Put $S_\phi := C_\phi / Z_(\widehat{G})^\Gamma$. Here Γ is the absolute Galois group of F.
Then, we have $S_\phi = C'_\phi / Z'(\widehat{G})^\Gamma$.

Conjecture (Hiraga-Ichino-Ikeda'08)

Let π be a square-integrable irreducible representation of G and let (ϕ, η) be its Langlands parameter. Then $\# C'_\phi < \infty$, and

$$\deg \pi = \frac{\dim \eta}{\# C'_\phi} \cdot |\gamma(0, \mathrm{Ad} \circ \phi, \psi)|.$$

Here,

- $\mathrm{Ad} : {}^L G \to \mathrm{GL}(\mathrm{Lie}({}^L G))$ is the Adjoint representation of ${}^L G$,
- $\gamma(s, \mathrm{Ad} \circ \phi, \psi)$ is the Adjoint γ-factor. It has an expression

$$\gamma(s, \mathrm{Ad} \circ \phi, \psi) := \frac{L(1 - s, \mathrm{Ad} \circ \phi^\vee)}{L(s, \mathrm{Ad} \circ \phi)} \cdot \epsilon(s, \mathrm{Ad} \circ \phi, \psi).$$

Remark

The conjecture is refined by Gross-Reeder'10.

Known cases

- Archimedean cases (Hiraga-Ikeda-Ichino'08),
- Inner forms of $\mathrm{GL}_n(F)$ (Hiraga-Ikeda-Ichino'08),
- $\mathrm{SO}_{2n+1}(F)$ (Ichino-Lapid-Mao'16),
- $\mathrm{Mp}_{2n}(F)$ (Ichino-Lapid-Mao'16),
- $\mathrm{U}_n(E)$ (Beuzart-Plessis'18),
- $\mathrm{Sp}_4(F)$ (Gan-Ichino'14).

Moreover, we have

Theorem 1 (K.)

The formal degree conjecture holds for $G_{1,1}$ and $\widetilde{G}_{1,1}$.

§5 Outline of the proof

For a while, we discuss in the general setting. Let V be an m-dimensional ϵ-Hermitian space, and let W be an n-dimensional $(-\epsilon)$-Hermitian space. Suppose that $l = 1$.

Theorem 2 (K.)

Let π be a square integrable irreducible representation of $G(W)$, and let $\sigma = \theta_\psi(\pi, V)$. Suppose that $\sigma \neq 0$. Then σ is also square-integrable. Moreover, we have

$$\frac{\deg \pi}{\deg \sigma} = \alpha(V, W) \cdot \sigma(-1) \cdot \gamma^V(0, \sigma \boxtimes \chi_W, \psi)$$

where $\gamma^V(s, \sigma \boxtimes \chi_W, \psi)$ is the standard γ-factor defined by the doubling method, and

$$\alpha(V, W) = \begin{cases} (-1)^n \chi_V(-1) \epsilon(\frac{1}{2}, \chi_V, \psi) & (-\epsilon = 1), \\ \frac{1}{2} \chi_W(-1) \epsilon(\frac{1}{2}, \chi_W, \psi) & (-\epsilon = -1). \end{cases}$$

Remarks

- The factor $\gamma^V(s, \sigma \boxtimes \chi, \psi)$ is defined as a factor appearing in the local functional equation

$$Z^V(M^*(s, \chi, \psi) f_s, \xi) = \gamma^V(s, \sigma \boxtimes \chi) Z^V(f_s, \xi)$$

where ξ is a matrix coefficient of σ, $f_s \in \mathrm{Ind}_P^{G(V^\square)}(\chi_s \circ \Delta)$, $Z^V(f_s, \xi)$ is a zeta integral

$$Z^V(f_s, \xi) = \int_{G(V)} f_s(g) \xi(g)\, dg,$$

$M^*(s, \chi, \psi)$ is a normalized intertwining operator. Then, we can prove that $\gamma^V(s, \sigma \boxtimes \chi, \psi)$ satisfies some notable properties which is expected to characterize the standard γ-factor (Yamana'14, K.'20). In particular, the above theorem is stated unconditionally.

- A similar result has been established by Gan-Ichino for the following reductive dual pairs:
  - $(\mathrm{Sp}_{2m}(F), \mathrm{O}_{2m+2}(F))$,
  - $(\mathrm{O}_{2m}(F), \mathrm{Sp}_{2m}(F))$,
  - $(\mathrm{Sp}_{2m}(F), \mathrm{O}_{2m+1}(F))$,
  - $(\mathrm{O}_{2m+1}(F), \mathrm{Sp}_{2m}(F))$,
  - $(\mathrm{U}_{m}(F), \mathrm{U}_{m}(F))$,
  - $(\mathrm{U}_{m}(F), \mathrm{U}_{m+1}(F))$.
- The proof of Theorem 2 differs from that of the result of Gan-Ichino'14.
  - First, we deduce Theorem 2 to the case where either V or W is anisotropic.
  - Then, we compute the constant by using some explicit local zeta values.

- For non-quaternionic classical groups, the behavior of the Langlands parameter under the theta correspondence of equal or almost equal rank has been established.
  - Unitary dual pairs (Gan-Ichino'16),
  - Symplectic-orthogonal dual pairs (Atobe-Gan'17)
- For quaternionic unitary groups, this has not been formulated yet. However, Theorem 2 tells us an information of the dimensions of η_π, η_σ. We assume $\phi_\pi = (\phi_\sigma \otimes \chi_V^{-1}\chi_W) \oplus \chi_W$. Then, Theorem 2 indicates that

$$\frac{\dim \eta_\pi}{\dim \eta_\sigma} = \frac{\# C_{\phi_\pi}}{\# C_{\phi_\sigma}} \times \begin{cases} 1 & (-\epsilon = 1), \\ \frac{1}{2} & (-\epsilon = -1). \end{cases}$$

Now, we prove Theorem 1. Let G be one of $G_{1,1}$, $H_{1,1}$, $H_{3,0}$, and let $\widetilde{G}$ be one of $\widetilde{G}_{1,1}$, $\widetilde{H}_{1,1}$, $\widetilde{H}_{3,0}$.

Theorem (Choiy17)

Let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$. Then, we have a decomposition

$$\widetilde{\pi}|_G = (\bigoplus_{i=1}^{t} \pi_i)^{\oplus k}$$

where $\pi_1, \ldots, \pi_t$ are irreducible representations of G and

$$k = \begin{cases} \frac{1}{2} \dim \eta_i & G = G_{1,1} \text{ and } \widetilde{\pi} \text{ has the L-parameter of Case S}, \\ \dim \eta_i & \text{otherwise.} \end{cases}$$

Then, we have

Proposition (Gan-Ichino'14)

$$\deg \pi = \frac{\# Z'(\widehat{\widetilde{G}})}{\# Z(\widehat{G})} \cdot \frac{k}{\# X(\widetilde{\pi})} \cdot \deg \widetilde{\pi}.$$

where λ is the similitude norm, and

$$X(\widetilde{\pi}) = \{\chi \in \mathrm{Hom}(F^{\times}, \mathbb{C}^{\times}) \mid (\chi \circ \lambda)\widetilde{\pi} \cong \widetilde{\pi}\}.$$

Moreover, if we denote by $X(\widetilde{\phi})$ the stabilizer

$$\{a \in H^1(W_F, \widehat{\mathrm{GL}_1}) \mid a\widetilde{\phi} = \widetilde{\phi} \text{ as } L\text{-parameters } \},$$

we have

Proposition (Chao-Li'14)

$$\mathcal{S}_{\widetilde{\phi}}(\widetilde{G}) \to \mathcal{S}_{\phi}(G) \to X(\widetilde{\phi}) \to 1$$

Finally, the reciprocity map of the local class field theory induces an embedding $X(\widetilde{\pi}) \to X(\widetilde{\phi})$. Moreover, we have

$$[X(\widetilde{\phi}) : X(\widetilde{\pi})] = \begin{cases} 2 & G = G_{1,1} \text{ and } \widetilde{\pi} \text{ has the } L\text{-parameter of Case I-(b),} \\ 1 & \text{otherwise.} \end{cases}$$

Hence, we have

Lemma

Let π be a square integrable irreducible representation of G, let (ϕ, η) be its Langlands parameter, let $\widetilde{\pi}$ be an irreducible representation of $\widetilde{G}$ so that its restriction $\widetilde{\pi}|_G$ to G contains π, and let $(\widetilde{\phi}, \widetilde{\eta})$ be the Langlands parameter of $\widetilde{\pi}$. Then, we have

$$\deg \widetilde{\pi} = \frac{\dim \widetilde{\eta}}{\dim \eta} \cdot \frac{\# C_\phi(G)}{\# C'_{\widetilde{\phi}}(\widetilde{G})} \cdot \deg \pi, \quad \textit{and } \mathrm{Ad} \circ \widetilde{\phi} = \mathrm{Ad} \circ \phi.$$

Then, the proof of Theorem 1 goes as follows:

$$\begin{tikzcd}
\text{FDC for } H_{1,1}, H_{3,0} \arrow[r, Rightarrow, "Theorem2"] & \text{FDC for } G_{1,1} \arrow[d, Rightarrow, "Lemma"] \\
\text{FDC for } \widetilde{H}_{1,1}, \widetilde{H}_{3,0} \arrow[u, Rightarrow, "Lemma"] & \text{FDC for } \widetilde{G}_{1,1}
\end{tikzcd}$$

Now, we prove Theorem 2. Consider

- X, X': right vector space over D with $\dim X = \dim X'$,
- $V' = X + V + X'$: an ϵ-Hermitian space containing V,
- Y, Y': left vector space over D with $\dim Y = \dim Y' = \dim X$,
- $W' = Y + W + Y'$: a $(-\epsilon)$-Hermitian space containing W.

Suppose that either V or W is anisotropic. Then, as in Gan-Ichino'14, we have

Proposition

$$\alpha(V', W') = \alpha(V, W)$$

- The proof of this proposition uses a result of Plancherel formula of Hiereman'04.

Minimal case

- It remains to determine either $\alpha(V, W)$ in the following types of cases
  - Minimal case (I): $\epsilon = 1$, and V is a (unique) isotropic 2 dimensional ϵ-Hermitian space, W is a (unique) anisotropic 3 dimensional $(-\epsilon)$-Hermitian space;
  - Minimal case (II): the case where V is anisotropic.
- Let $\beta(V, W)$ be the proportional constant appearing in the local Siegel-Weil formula. Then, it is known that $\alpha(V, W)$ is expressed by $\beta(V, W)$. In our study, we relate $\beta(V, W)$ to local zeta values in the "minimal case", which we can compute in enough cases. In this way, we have a formula of $\alpha(V, W)$.

Remark: the local Siegel-Weil formula

- The Weil representation $\omega_\psi^\square$ of the reductive dual pair $(G(W^\square), G(V))$ is realized on $\mathcal{S}(V \otimes W^\triangle)$. It is known that there is a $G(W^\square)$-invariant map
$$\mathcal{S}(V \otimes W^\triangle) \to I^W(-\frac{1}{2}, \chi_V) : \phi \mapsto F_\phi$$
given by $F_\phi(g) = [\omega_\psi^\square(g)\phi](0)$.
- Consider the map
$$\mathcal{I} : \mathcal{S}(V \otimes W^\triangle) \otimes \overline{\mathcal{S}(V \otimes W^\triangle)} \to \mathbb{C}$$
by
$$\mathcal{I}(\phi, \phi') = \int_{G(V)} (\omega_\psi^\square(h)\phi, \phi')\, dh$$
where $(\ ,\ )$ is the L^2-inner product of $\mathcal{S}(V \otimes W^\triangle)$.

- For $\phi \in \mathcal{S}(V \otimes W^{\triangle})$, we can take $F_\phi^\dagger \in I^W(\frac{1}{2}, \chi_V)$ so that $M^{V^*}(s, \chi, A_0, \psi)F_\phi^\dagger = F_\phi$ (Yamana'11).
- Consider the map

$$\mathcal{E} : \mathcal{S}(V \otimes W^{\triangle}) \otimes \overline{\mathcal{S}(V \otimes W^{\triangle})} \to \mathbb{C}$$

given by

$$\mathcal{E}(\phi, \phi') = \int_G F_\phi^\dagger((g,1)) \cdot F_{\phi'}((g,1))\, dg.$$

- Denoting $\Delta G(W)$ the diagonal subgroup of $G(W) \times G(W) \subset G(W^\square)$, both $\mathcal{I}$ and $\mathcal{E}$ are $\Delta G(W) \times G(V)^2$-invariant maps.

Theorem (Gan-Ichino'14)

There is a constant $\beta(V, W)$ such that $\mathcal{I} = \beta(V, W) \cdot \mathcal{E}$.

A key lemma

Lemma

Let τ be a Weyl element of $G(W^{\square})$ so that $\tau(W^{\triangle}) = W^{\triangledown}$, and $\tau(W^{\triangledown}) = W^{\triangle}$. Then, for $f \in I(\rho, 1)$, we have

$$\int_{G(W)} f((g,1))\, dg = C(\tau) \cdot \int_{U(W^{\triangle})} f(\tau u)\, du$$

where $C(\tau)$ is a constant depending only on the Weyl element τ.

- We can compute $C(\tau)$ by taking f as a special test function.
- Thus, we deform $\mathcal{E}$ to an integral over $U(W^{\triangledown})$.