

A proof of Ibukiyama's conjecture on Siegel modular forms of half-integral weight and of degree 2

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Introduction

Main theorem (Ibukiyama's conjecture)

For any integer $k \geq 3$ and even integer $j \geq 0$, there exists an isomorphism

$$\rho : S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) \xrightarrow{\simeq} S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$$

such that if $F \in S^+_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$ is a Hecke eigenform, then so is $\rho(F) \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$, and they satisfy

$$L(s, F) = L(s, \rho(F), \mathrm{spin}).$$

Tools

- Arthur's multiplicity formula for SO_5
- Gan-Ichino's multiplicity formula for Mp_4
- Jacobi forms and Jacobi groups

Sketch

$$\begin{array}{ccc}
 S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L_{\mathrm{disc}}^2(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow & & \parallel \\
 S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) & & L_{\mathrm{disc}}^2(\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}})) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow \wr & & \updownarrow \\
 \{ \text{Jacobi cusp form} \} & \rightsquigarrow & L_{\mathrm{disc}}^2(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))
 \end{array}$$

Remarks:

- We have an accidental isomorphism $\mathrm{PGSp}_4 \cong \mathrm{SO}_5$;
- The A -parameters for Mp_4 are the same as those for SO_5 .

Introduction

The detailed statement

Multiplicity formulae

$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$ side

$S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$ side

At the Last

Integral weight Siegel cusp forms 1

- $\mathfrak{H}_n = \{ Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \mid X = {}^tX, \quad Y = {}^tY > 0 \}$
- $\mathrm{GSp}_{2n} = \{ g \in \mathrm{GL}_{2n} \mid {}^tgJ_ng = \nu(g)J_n, \nu(g) \in \mathrm{GL}_1 \}, \mathrm{Sp}_{2n} = \ker(\nu)$
- $\mathrm{GSp}_{2n}^+(\mathbb{R}) = \{ g \in \mathrm{GSp}_{2n}(\mathbb{R}) \mid \nu(g) > 0 \} \curvearrowright \mathfrak{H}_n, J(g, Z) = CZ + D$
 ◀ $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$
- (Sym_j, V_j) : the symmetric tensor rep. of degree j of $\mathrm{GL}_2(\mathbb{C})$
- $[f|_{k,j}g](Z) = \nu(g)^{k+\frac{j}{2}} \det(J(g, Z))^{-k} \mathrm{Sym}_j(J(g, Z))^{-1} f(gZ)$
 ◀ $f : \mathfrak{H}_2 \rightarrow V_j, g \in \mathrm{GSp}_4^+(\mathbb{R})$
 ▶ (k, j) means the weight $\det^k \mathrm{Sym}_j$.

$$S_{k,j}(\mathrm{Sp}_4(\mathbb{Z})) = \{ f : \mathfrak{H}_2 \rightarrow V_j \mid f|_{k,j}\gamma = f, \forall \gamma \in \mathrm{Sp}_4(\mathbb{Z}), \text{ holom.}, \text{ cusp} \}$$

Integral weight Siegel cusp forms 2

$f \in S_{k,j}(\mathrm{Sp}_4(\mathbb{Z})), m \in \mathbb{Z}_{>0}$

- $X(m) = \{x \in \mathrm{Mat}_4(\mathbb{Z}) \cap \mathrm{GSp}_4^+(\mathbb{R}) \mid \nu(x) = m\}$
- Hecke operators

$$f|_{k,j}T(m) = m^{k+\frac{j}{2}-3} \sum_{g \in \mathrm{Sp}_4(\mathbb{Z}) \backslash X(m)} f|_{k,j}g$$

f : Hecke eigenform

$\rightsquigarrow$ the spinor L -function

$$L(s, f, \mathrm{spin}) = \prod_p L(s, f, \mathrm{spin})_p$$

Half-integral weight Siegel cusp forms 1

- $\widetilde{\mathrm{GSp}}_4^+(\mathbb{R})$: the 4-fold covering group of $\mathrm{GSp}_4^+(\mathbb{R})$:
 - $\left\{ (g, \phi(Z)) \in \mathrm{GSp}_4^+(\mathbb{R}) \times \mathrm{Hol}(\mathfrak{H}_2, \mathbb{C}) \mid \phi(Z)^4 = \frac{\det J(g, Z)^2}{\det g} \right\}$
 - $(g_1, \phi_1(Z))(g_2, \phi_2(Z)) = (g_1 g_2, \phi_1(g_2 Z) \phi_2(Z))$
- $\theta(Z) = \sum_{x \in \mathbb{Z}^2} e({}^t x Z x)$, $Z \in \mathfrak{H}_2$, $e(z) = \exp(2\pi i z)$
 - an embedding $\Gamma_0(4) \hookrightarrow \widetilde{\mathrm{GSp}}_4^+(\mathbb{R})$, $\gamma \mapsto \left(\gamma, \frac{\theta(\gamma Z)}{\theta(Z)} \right)$
- $\left(\frac{-1}{\gamma} \right) = \left(\frac{-1}{\det D} \right)$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(4)$
- $\left[F|_{k-\frac{1}{2}, j}(g, \phi(Z)) \right] (Z) = \nu(g)^{\frac{j}{2}} \phi(Z)^{-2k+1} \mathrm{Sym}_j(J(g, Z))^{-1} F(gZ)$
 - $F : \mathfrak{H}_2 \rightarrow V_j$, $(g, \phi(Z)) \in \widetilde{\mathrm{GSp}}_4^+(\mathbb{R})$

$$S_{k-\frac{1}{2}, j}(\Gamma_0(4), \left(\frac{-1}{\cdot} \right)) =$$

$$\left\{ F : \mathfrak{H}_2 \rightarrow V_j \mid F|_{k-\frac{1}{2}, j} \gamma = \left(\frac{-1}{\gamma} \right) F, \forall \gamma \in \Gamma_0(4), \quad \text{holom.}, \text{ cusp} \right\}$$

Half-integral weight Siegel cusp forms 2

$$F \in S_{k-\frac{1}{2}, j}(\Gamma_0(4), \begin{pmatrix} -1 \\ \cdot \end{pmatrix})$$

- Fourier series expansion $F(Z) = \sum_T A(T) e(\mathrm{tr}(TZ))$
 - T runs over positive definite half-integral symmetric matrices of degree 2.

$S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), \begin{pmatrix} -1 \\ \cdot \end{pmatrix})$: the set of $F \in S_{k-\frac{1}{2}, j}(\Gamma_0(4), \begin{pmatrix} -1 \\ \cdot \end{pmatrix})$ such that $A(T) = 0$ unless $T \equiv (-1)^k r^t r \pmod{4}$ for some $r \in \mathbb{Z}^2$

Half-integral weight Siegel cusp forms 3

Hecke operators at odd prime $p \neq 2$

- $$K_s(p^2) = \begin{pmatrix} 1_{2-s} & & & \\ & p1_s & & \\ & & p^2 1_{2-s} & \\ & & & p1_s \end{pmatrix}, \quad s = 0, 1, 2.$$

- $\Gamma_0(4)(K_s(p^2), p^{1-\frac{s}{2}})\Gamma_0(4) = \bigsqcup_t \Gamma_0(4)\widetilde{g}_{s,t}$ in $\widetilde{\mathrm{GSp}}_4^+(\mathbb{R})$

- $F|_{k-\frac{1}{2}, j} T_s(p) = \sum_t \begin{pmatrix} -1 \\ g_{s,t} \end{pmatrix} F|_{k-\frac{1}{2}, j} \widetilde{g}_{s,t}, \quad \widetilde{g}_{s,t} = (g_{s,t}, *)$

$\rightsquigarrow$ the Euler p -factor $L(s, F)_p$

2 is a bad prime for $\Gamma_0(4)$.

We use Jacobi forms to define $T_s(2)$ and $L(s, F)_2$.

Holomorphic and skew-holomorphic Jacobi cusp forms 1

- the Jacobi group $\mathrm{Sp}_4^J = \mathrm{Sp}_4 \ltimes \mathcal{H}_2 \subset \mathrm{Sp}_6$

◀ an embedding $\mathrm{Sp}_4 \hookrightarrow \mathrm{Sp}_6$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & \\ & A & & B \\ & & 1 & \\ & C & & D \end{pmatrix}$

- ◀ the Heisenberg group

$$\mathcal{H}_2 = \left\{ ([\lambda, \mu], \kappa) = \begin{pmatrix} 1 & {}^t\lambda & \kappa & {}^t\mu \\ & 1_2 & \mu & \\ & & 1 & \\ & & -\lambda & 1_2 \end{pmatrix} \in \mathrm{Sp}_6 \right\}$$

$$J_{(k,j)}^{\mathrm{hol}, \mathrm{cusp}} =$$

$$\left\{ F : \mathfrak{H}_2 \times \mathbb{C}^2 \rightarrow V_j \mid F|_{(k,j)}^{\mathrm{hol}} \gamma = F, \forall \gamma \in \mathrm{Sp}_4^J(\mathbb{Z}), \quad \text{holom.}, \text{ cusp} \right\}$$

$$J_{(k,j)}^{\mathrm{skew}, \mathrm{cusp}} =$$

$$\left\{ F : \mathfrak{H}_2 \times \mathbb{C}^2 \rightarrow V_j \mid F|_{(k,j)}^{\mathrm{skew}} \gamma = F, \forall \gamma \in \mathrm{Sp}_4^J(\mathbb{Z}), \quad \text{skew-holom.}, \text{ cusp} \right\}$$

Holomorphic and skew-holomorphic Jacobi cusp forms 2

Slash operators

- For $([\lambda, \mu], \kappa) \in \mathcal{H}_2(\mathbb{R})$ and $F : \mathfrak{H}_2 \times \mathbb{C}^2 \rightarrow V_j$,

$$\begin{aligned} \left[F|_{(k,j)}^{\mathrm{hol}}([\lambda, \mu], \kappa) \right] (Z, w) &= \left[F|_{(k,j)}^{\mathrm{skew}}([\lambda, \mu], \kappa) \right] (Z, w) \\ &= e({}^t\lambda Z\lambda + 2{}^t\lambda w + {}^t\lambda\mu + \kappa) F(Z, w + Z\lambda + \mu) \end{aligned}$$

- For $g \in \mathrm{GSp}_4^+(\mathbb{R})$ and $F : \mathfrak{H}_2 \times \mathbb{C}^2 \rightarrow V_j$,

$$\begin{aligned} \left[F|_{(k,j)}^{\mathrm{hol}} g \right] (Z, w) &= \nu(g)^{k+\frac{j}{2}} e(-{}^t w J(g, Z)^{-1} C w) \\ &\times \det(J(g, Z))^{-k} \mathrm{Sym}_j(J(g, Z))^{-1} F(gZ, \nu(g)^{\frac{1}{2}} {}^t J(g, Z)^{-1} w) \\ \left[F|_{(k,j)}^{\mathrm{skew}} g \right] (Z, w) &= \nu(g)^{k+\frac{j}{2}} e(-{}^t w J(g, Z)^{-1} C w) \\ &\times \frac{|\det J(g, Z)|}{\det J(g, Z)} \overline{\det J(g, Z)^{-k} \mathrm{Sym}_j(J(g, Z))^{-1} F(gZ, \nu(g)^{\frac{1}{2}} {}^t J(g, Z)^{-1} w)} \end{aligned}$$

Holomorphic and skew-holomorphic Jacobi cusp forms 3

$F : \mathfrak{H}_2 \times \mathbb{C}^2 \rightarrow V_j$ is skew-holomorphic.

$\stackrel{\text{def}}{\Leftrightarrow} F(Z, w)$ is holomorphic in $w \in \mathbb{C}^2$ and real analytic in the real part X and imaginary part Y of $Z \in \mathfrak{H}_2$.

Cusp condition

- $F \in J_{(k,j)}^{\text{hol}, \text{cusp}}$ has a Fourier expansion

$$F(Z, w) = \sum_{\substack{(N,r) \in L_2^* \times \mathbb{Z}^2 \\ 4N - r^{\text{t}}r > 0}} A(N, r) e(\text{tr}(NZ) + {}^{\text{t}}rw)$$

- $F \in J_{(k,j)}^{\text{skew}, \text{cusp}}$ has a Fourier expansion

$$F(Z, w) = \sum_{\substack{(N,r) \in L_2^* \times \mathbb{Z}^2 \\ 4N - r^{\text{t}}r < 0}} A(N, r) e(\text{tr}(NZ - \frac{1}{2}i(4N - r^{\text{t}}r)Y)) e({}^{\text{t}}rw)$$

◀ L_2^* : the set of all half-integral symmetric matrices

Holomorphic and skew-holomorphic Jacobi cusp forms 4

Hecke operators at any prime p including 2

- $K_s(p^2) = \begin{pmatrix} 1_{2-s} & & & \\ & p1_s & & \\ & & p^2 1_{2-s} & \\ & & & p1_s \end{pmatrix}, (s = 0, 1, 2) : \text{as before}$
- $\mathrm{Sp}_4(\mathbb{Z}) K_s(p^2) \mathrm{Sp}_4(\mathbb{Z}) = \bigsqcup_t \mathrm{Sp}_4(\mathbb{Z}) g_{s,t}$
- $F|_{(k,j)}^* T_s^J(p) = \sum_{\lambda, \mu \in (\mathbb{Z}/p\mathbb{Z})^2} \sum_t F|_{(k,j)}^* g_{s,t} |_{(k,j)}^* ([\lambda, \mu], 0)$
 - ◀ $F \in J_{(k,j)}^{\star, \mathrm{cusp}}, \star = \text{hol, skew}$

Half-integral weight Siegel cusp forms 4

Theorem (Ibukiyama et al.)

There exists a linear isomorphism

$$\sigma : J_{(k,j)}^{\star, \text{cusp}} \xrightarrow{\simeq} S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)), \quad \star = \begin{cases} \text{hol} & k \text{ odd} \\ \text{skew} & k \text{ even} \end{cases}$$

such that

$$F|_{(k,j)}^{\star} T_s^J(p) = p^{3+\frac{s}{2}} \left(\frac{-1}{p}\right)^s \sigma(F)|_{k-\frac{1}{2}, j} T_s(p),$$

for any odd prime p .

↪ $T_s(2)$ on $S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$

↪ Euler 2-factor $L(s, F)_2$ and L -function $L(s, F) = \prod_p L(s, F)_p$

Main theorem (again)

Main theorem (Ibukiyama's conjecture)

For any integer $k \geq 3$ and even integer $j \geq 0$, there exists an isomorphism

$$\rho : S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) \xrightarrow{\cong} S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$$

such that if $F \in S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$ is a Hecke eigenform, then so is $\rho(F) \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$, and they satisfy

$$L(s, F) = L(s, \rho(F), \mathrm{spin}).$$

Multiplicity formulae

A-parameters

The notion of elliptic A -parameters for SO_5 and for Mp_4 are the same.
An elliptic A -parameter for SO_5 or Mp_4 is a formal direct sum

$$\phi = \bigoplus_i \phi_i \boxtimes S_{d_i},$$

where

- ϕ_i : an irr. self-dual cuspidal automorphic rep. of $\mathrm{GL}_{n_i}(\mathbb{A}_{\mathbb{Q}})$;
- S_d : the irr. d -dimensional rep. of $\mathrm{SL}_2(\mathbb{C})$;
- d_i : odd $\Rightarrow \phi_i$: symplectic;
- d_i : even $\Rightarrow \phi_i$: orthogonal;
- $i \neq j \Rightarrow (\phi_i, d_i) \neq (\phi_j, d_j)$;
- $\sum_i n_i d_i = 4$.

The localization at a place v (of $\mathbb{Q}$)

$$\phi_v = \bigoplus_i \phi_{i,v} \boxtimes S_{d_i} : L_{\mathbb{Q}_v} \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \mathrm{Sp}_4(\mathbb{C})$$

- ◀ $\phi_{i,v}$ is identified with its L -parameter, via the LLC for GL_n .
- ◀ $L_{\mathbb{Q}_v}$: the Langlands group of $\mathbb{Q}_v$

A-parameters (explicit)

The A -parameter ϕ for SO_5 or Mp_4 is one of the following forms.

- (1) $\phi = \chi \boxtimes S_4$
- (2) $\phi = \chi \boxtimes S_2 \oplus \chi' \boxtimes S_2$
- (3) $\phi = \sigma \boxtimes S_2$
- (4) $\phi = \chi \boxtimes S_2 \oplus \mu \boxtimes S_1$
- (5) $\phi = \mu \boxtimes S_1 \oplus \mu' \boxtimes S_1$
- (6) $\phi = \tau \boxtimes S_1$

- ◀ $\chi, \chi' : \text{quadratic characters of } \mathbb{Q}^\times \backslash \mathbb{A}^\times$
- ◀ $\sigma : \text{an irr. cuspidal orthogonal autom. rep. of } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$
- ◀ $\mu, \mu' : \text{irr. cuspidal symplectic autom. reps of } \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$
- ◀ $\tau : \text{an irr. cuspidal symplectic autom. rep. of } \mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$

Multiplicity formula for SO_5

For an elliptic A -parameter $\phi = \bigoplus_i \phi_i \boxtimes S_{d_i}$,

- The global component group $S_\phi = \bigoplus_i (\mathbb{Z}/2\mathbb{Z}) a_i$
- The local component group $S_{\phi_v} = \pi_0[\mathrm{Cent}(\mathrm{im}(\phi_v), \mathrm{Sp}_4(\mathbb{C}))]$

$\rightsquigarrow S_\phi \rightarrow S_{\phi_v}$, and $\Delta : S_\phi \rightarrow S_{\phi, \mathbb{A}} = \prod_v S_{\phi_v}$

Multiplicity formula for SO_5 (Arthur)

For the split SO_5 , Arthur gave a character $\epsilon_\phi \in \widehat{S}_\phi$ and finite sets $\Pi_{\phi_v}(\mathrm{SO}_5) = \left\{ \sigma_{\eta_v} \mid \eta_v \in \widehat{S}_{\phi_v} \right\}$ of semisimple representations of $\mathrm{SO}_5(\mathbb{Q}_v)$ of finite length indexed by characters of S_{ϕ_v} such that

$$L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) = \bigoplus_{\phi} \bigoplus_{\eta \in \widehat{S}_{\phi, \mathbb{A}}} n_{\eta} \sigma_{\eta}, \quad \sigma_{\eta} = \bigotimes_v \sigma_{\eta_v},$$

where $\eta = \bigotimes_v \eta_v$ and $n_{\eta} = \begin{cases} 1, & \text{if } \eta \circ \Delta = \epsilon_{\phi}, \\ 0, & \text{otherwise.} \end{cases}$

The metaplectic group Mp_4

The metaplectic group Mp_4 is the nontrivial 2-fold covering group of Sp_4 . (It is not an algebraic group.)

- $1 \rightarrow \{\pm 1\} \rightarrow \mathrm{Mp}_4(\mathbb{Q}_v) \rightarrow \mathrm{Sp}_4(\mathbb{Q}_v) \rightarrow 1$
 - ▶ a canonical splitting $\mathrm{Sp}_4(\mathbb{Z}_p) \hookrightarrow \mathrm{Mp}_4(\mathbb{Q}_p)$ for $p \neq 2, \infty$
 - ↪ the notion of spherical representations for $p \neq 2$
- $1 \rightarrow \{\pm 1\} \rightarrow \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathrm{Sp}_4(\mathbb{A}_{\mathbb{Q}}) \rightarrow 1$
 - ▶ a canonical splitting $\mathrm{Sp}_4(\mathbb{Q}) \hookrightarrow \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}})$
 - ↪ the notion of automorphic forms $\varphi : \mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C}$
- A representation of Mp_4 is said to be genuine if it does not factor through the covering group $\mathrm{Mp}_4 \rightarrow \mathrm{Sp}_4$.
- $L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$: the genuine discrete spectrum of $L^2(\mathrm{Sp}_4(\mathbb{Q}) \backslash \mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}}))$

Multiplicity formula for Mp_4

- Recall S_ϕ , S_{ϕ_v} , and $\Delta : S_\phi \rightarrow S_{\phi, \mathbb{A}} = \prod_v S_{\phi_v}$.
- Fix an additive character $\psi : \mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}} \rightarrow \mathbb{C}^1$ s.t. $\psi_\infty = \mathrm{e}$.

Multiplicity formula for Mp_4 (Gan-Ichino)

Gan-Ichino gave a character $\tilde{\epsilon}_\phi \in \widehat{S}_\phi$ and finite sets (dependent on ψ_v) $\Pi_{\phi_v, \psi_v}(\mathrm{Mp}_4) = \left\{ \pi_{\eta_v, \psi_v} \mid \eta_v \in \widehat{S}_{\phi_v} \right\}$ of semisimple representations of $\mathrm{Mp}_4(\mathbb{Q}_v)$ of finite length indexed by characters of S_{ϕ_v} such that

$$L^2_{\mathrm{disc}}(\mathrm{Mp}_4) = \bigoplus_{\phi} \bigoplus_{\eta \in \widehat{S}_{\phi, \mathbb{A}}} m_{\eta} \pi_{\eta, \psi}, \quad \pi_{\eta, \psi} = \bigotimes_v \pi_{\eta_v, \psi_v},$$

where $m_{\eta} = \begin{cases} 1, & \text{if } \eta \circ \Delta = \tilde{\epsilon}_\phi, \\ 0, & \text{otherwise.} \end{cases}$

Sketch (again)

$$\begin{array}{ccc}
 S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) & \longleftrightarrow (A\text{-parameters}) \\
 \downarrow & & \parallel \\
 S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) & & L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow | \mathfrak{z} & & \updownarrow \\
 J_{(k,j)}^{\star, \mathrm{cusp}} & \rightsquigarrow L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}})) &
 \end{array}$$

$$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \rightsquigarrow L_{\mathrm{disc}}^2(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}}))$$

- $f \in S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))$: a Hecke eigenform
- ↪ $\Phi_f(g) = [f|_{j+3,2k-6} g_{\infty}](i1_2)$, for $g = \gamma g_{\infty} \kappa \in \mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$
 - ◀ strong approximation $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GSp}_4(\mathbb{Q}) \mathrm{GSp}_4^+(\mathbb{R}) \mathrm{GSp}_4(\hat{\mathbb{Z}})$
 - ◀ level $\mathrm{Sp}_4(\mathbb{Z})$
- ↪ $\varphi_{f,v}(g) = \langle v, \Phi_f(g) \rangle$, for $v \in V_{2k-6}^{\vee}$
- ↪ π_f : an irr. cuspidal autom. rep. of $\mathrm{PGSp}_4(\mathbb{A}_{\mathbb{Q}}) \cong \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})$
 - ▶ $\pi_{f,\infty}$: holom. d.s. w/ lowest K -type $(k-3, k+j)$
 - ▶ $\pi_{f,p}$: spherical (unramified) w/ same Satake parameter as f , for all p
- ↪ ϕ_f : the A -parameter of π_f (Arthur's multiplicity formula)

$$S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) \longleftrightarrow (A\text{-parameters})$$

Lemma (essentially by Chenevier-Lannes)

We have $\phi_f = \tau_f \boxtimes S_1$,

$\exists \tau_f$: an irr. symplectic cuspidal autom. rep. of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ s.t.

$$L(s, f, \mathrm{spin}) = L(s - j - k + \tfrac{3}{2}, \tau_f) \text{ and } \tau_{f, \infty} = \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}.$$

Moreover, $\mathbb{C}f \mapsto \tau_f$ is a bijection between

$\{1\text{-dim'l Hecke eigenspace in } S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z}))\}$ and

$\{\text{irr. symp. cusp. autom. rep. of } \mathrm{GL}_4 \text{ that is unramified at any } p \text{ and } \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}} \text{ at } \infty\}.$

$$\blacktriangleleft \mathcal{D}_a = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(re^{i\theta} \mapsto e^{2ia\theta})$$

∴) Rule out other possibilities by using the conditions:

$\pi_{f, \infty}$: holom. d.s. w/ lowest K -type $(k-3, k+j)$

$\pi_{f, p}$: unramified w/ same Satake parameter as f , for all p

j : even

For the last assertion, do the converse. □

Sketch (again)

$$\begin{array}{ccc}
 S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow & & \parallel \\
 S_{k-\frac{1}{2},j}^+(\Gamma_0(4), (\frac{-1}{\cdot})) & & L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow \wr & & \updownarrow \\
 J_{(k,j)}^{\star, \mathrm{cusp}} & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))
 \end{array}$$

$$S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), (\frac{-1}{\cdot})) \rightsquigarrow L_{\mathrm{disc}}^2(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))$$

- $F \in J_{(k, j)}^{\star, \mathrm{cusp}}$: a Hecke eigenform, $F' = \sigma(F) \in S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), (\frac{-1}{\cdot}))$
- ↪ $\Phi_F(g) = [F|_{k, j}^{\star} g_{\infty}](i1_2, 0)$, for $g = \gamma g_{\infty} \kappa \in \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}})$
 - ◀ strong approximation $\mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}) = \mathrm{Sp}_4^J(\mathbb{Q}) \mathrm{Sp}_4^J(\mathbb{R}) \mathrm{Sp}_4^J(\hat{\mathbb{Z}})$
 - ◀ level $\mathrm{Sp}_4^J(\mathbb{Z})$
- ↪ $\varphi_{F, v}(g) = \langle v, \Phi_F(g) \rangle$, for $v \in V_j^{\vee}$
- ↪ π_F : an irr. cuspidal autom. rep. of $\mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}})$
 - ▶ $\pi_{F, \infty}$: d.s. w/ a specific lowest K -type
 - ▶ $\pi_{F, p}$: spherical (unramified) w/ same Satake parameter as F , $\forall p$
 - ▶ The center of Sp_4^J is $\mathcal{Z} = \{([0, 0], z) \in \mathcal{H}_2\} \cong \mathbb{G}_a$.
 π_F has the central character $\psi : \mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}} \rightarrow \mathbb{C}^1$ w/ $\psi_{\infty} = \mathrm{e}$.
- $L_{\mathrm{disc}}^2(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi}$: the max. subspace on which $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ acts by ψ .

$$L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi} \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \quad 1$$

- $\pi_{S,\psi}$: the Schrödinger rep. of $\mathcal{H}_2(\mathbb{A}_{\mathbb{Q}})$ w/ cent. char. ψ
- ω_{ψ} : the Weil rep. of $\mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}})$ rel. to ψ
- $\rightsquigarrow \pi_{SW,\psi}$: the Schrödinger-Weil rep. of $\mathrm{Mp}_4(\mathbb{A}_{\mathbb{Q}}) \ltimes \mathcal{H}_2(\mathbb{A}_{\mathbb{Q}})$
- The Stone-von Neumann theorem gives the following fact.

Proposition (essentially by Berndt-Schmidt)

For any irr. subrep. $\pi' \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$,
 we have an irr. subrep. $\pi = \pi' \otimes \pi_{SW,\psi} \subset L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi}$.
 This is a bijective correspondence.

- Similar bijective correspondences exist over local fields.
- ... compatible with the global correspondence:

$$\pi' = \bigotimes_v \pi'_v \Rightarrow \pi = \pi' \otimes \pi_{SW,\psi} = \bigotimes_v (\pi'_v \otimes \pi_{SW,\psi_v})$$

$$L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi} \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4)_2$$

- $\pi_F \subset L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi} \rightsquigarrow \pi'_F \subset L^2_{\mathrm{disc}}(\mathrm{Mp}_4)$
 - ▶ $\pi'_{F, \infty} : \text{d.s., w/ lowest } K\text{-type} \begin{cases} (k+j-\frac{1}{2}, k-\frac{1}{2}) & \star = \text{hol} \\ (-k+\frac{1}{2}, -k-j+\frac{1}{2}) & \star = \text{skew} \end{cases}$
 - ▶ $\pi'_{F, p}$: The followings can be seen. Note that $\pi_{F, p} = \pi'_{F, p} \otimes \pi_{SW, \psi_p}$

Proposition

$\pi'_{F, p}$ is spherical, and hence unramified for $p \neq 2$.

$\therefore$) If $p \neq 2$, then π_{SW, ψ_p} is spherical. □

$$L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi} \longleftrightarrow L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \quad 2$$

Proposition

$\pi'_{F,p}$ is unramified for any prime p including 2.

$\therefore$) Intertwining operators for $\mathrm{Sp}_4^J(\mathbb{Q}_p)$ and $\mathrm{Mp}_4(\mathbb{Q}_p)$ are compatible with the correspondence $\pi'_{F,p} \mapsto \pi_{F,p} = \pi'_{F,p} \otimes \pi_{SW, \psi_p}$.

$$\begin{array}{ccc}
 (\text{a std. module of } \mathrm{Sp}_4^J(\mathbb{Q}_p)) & \longleftrightarrow & (\text{a std. module of } \mathrm{Mp}_4(\mathbb{Q}_p)) \\
 \downarrow \text{intertwining op.} & & \downarrow \text{intertwining op.} \\
 (\text{an unram. rep. of } \mathrm{Sp}_4^J(\mathbb{Q}_p)) & \longleftrightarrow & (\text{an unram. rep. of } \mathrm{Mp}_4(\mathbb{Q}_p))
 \end{array}$$

□

$\rightsquigarrow \phi_F$: the A -parameter of π'_F (Gan-Ichino's multiplicity formula)

$$S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), (\frac{-1}{\cdot})) \longleftrightarrow (A\text{-parameters})$$

$$F' \in S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), (\frac{-1}{\cdot})) \longleftrightarrow F \in J_{(k, j)}^{\star, \text{cusp}} \\ \rightsquigarrow \pi_F \subset L_{\text{disc}}^2(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi} \longleftrightarrow \pi'_F \subset L_{\text{disc}}^2(\mathrm{Mp}_4) \rightsquigarrow \phi_F$$

Lemma

We have $\phi_F = \tau_F \boxtimes S_1$,

$\exists \tau_F$: an irr. symplectic cuspidal autom. rep. of $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$ s.t.
 $L(s, F') = L(s - j - k + \frac{3}{2}, \tau_F)$ and $\tau_{F, \infty} = \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}}$.

Moreover, $\mathbb{C}F' \mapsto \tau_F$ is a bijection between
 $\{1\text{-dim'l Hecke eigenspace in } S_{k-\frac{1}{2}, j}^+(\Gamma_0(4), (\frac{-1}{\cdot}))\}$ and
 $\{\text{irr. symp. cusp. autom. rep. of } \mathrm{GL}_4 \text{ that is unramified at any } p \text{ and } \mathcal{D}_{k+j-\frac{3}{2}} \oplus \mathcal{D}_{k-\frac{5}{2}} \text{ at } \infty\}$.

$\therefore$) Rule out other possibilities by using the local information on $\pi'_{F, v}$
 (and that j is even).

For the last assertion, do the converse.



Sketch (again)

$$\begin{array}{ccc}
 S_{j+3,2k-6}(\mathrm{Sp}_4(\mathbb{Z})) & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{SO}_5(\mathbb{Q}) \backslash \mathrm{SO}_5(\mathbb{A}_{\mathbb{Q}})) \longleftrightarrow (A\text{-parameters}) \\
 \vdots & & \parallel \\
 S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) & & L^2_{\mathrm{disc}}(\mathrm{Mp}_4) \longleftrightarrow (A\text{-parameters}) \\
 \downarrow \wr & & \updownarrow \\
 J_{(k,j)}^{\star,\mathrm{cusp}} & \rightsquigarrow & L^2_{\mathrm{disc}}(\mathrm{Sp}_4^J(\mathbb{Q}) \backslash \mathrm{Sp}_4^J(\mathbb{A}_{\mathbb{Q}}))_{\psi}
 \end{array}$$

At the last

$$F \in S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right)) \rightsquigarrow \varphi_F \in \pi_F \subset L_{\mathrm{disc}}^2(\mathrm{Mp}_4)$$

- level $\Gamma_0(4) \rightsquigarrow$ NOT $\mathrm{Sp}_4(\mathbb{Z}_2)$ -invariant
- There does not exist the notion of spherical representations of $\mathrm{Mp}_4(\mathbb{Q}_2)$.

$$\pi \subset L_{\mathrm{disc}}^2(\mathrm{Mp}_4) \rightsquigarrow F_\pi \in S_{k-\frac{1}{2},j}(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$$

- difficult to see $F_\pi \in S_{k-\frac{1}{2},j}^+(\Gamma_0(4), \left(\frac{-1}{\cdot}\right))$