Matthew Emerton,

"Moduli stacks of Galois representations"

(Noted by Yuya Murakami)

- $K/\mathbb{Q} < \infty$
- S : finite subset of finite places of K.
- $G_{K,S}$ = Galois group over K of maximal extension unramified outside S.

$$= \varprojlim_i G_i, \quad G_i : \text{finite}.$$

$\mathcal{X} \to \operatorname{Spf} \mathbb{Z}_p$: stack of d-dimensional representation of $G_{K,S}$.

A : p-adically complete ring.

$\mathcal{X}(A) = \{ \rho : G_{K,S} \to GL_d(A)$

: continuous, p-adic representation }

GL$_d$ - conjugation.

Omit quoted by conjugation.

$$\mathcal{X}^{\square}(A) = \{ \rho : G_{K,S} \longrightarrow GL_d(A) \}$$

$$= \varprojlim_{n} \varinjlim_{i} \underbrace{\{ \rho_{i,n} : G_i \longrightarrow GL_d(A)_{p^n} \}}_{\mathcal{X}_i^{\square}(A/p^n)}$$

finite group

affine scheme

$\mathcal{X}^{\square}$ is an Ind-scheme over Spf $\mathbb{Z}_p$.

$\mathcal{X} = [\mathcal{X}^{\square} / GL_d]$ is an Ind-algebraic stack / Spf $\mathbb{Z}_p$.

Carl Wang-Erickson thesis studied these:

$\mathfrak{X}$ is a formal algebraic stack.

<u>Local versions</u>:

$\mathfrak{X}_v$ for v a finite place of K.

define using $W\mathcal{O}_{K_v}$ $v \nmid p$

Emerton-Gee stack for K_v $v \nmid p$.

$$\mathrm{res} : \mathfrak{X} \longrightarrow \prod \mathfrak{X}_v$$

formal algebraic stack.

<u>Natural conjecture</u> (E-Zhu):

Xinwen Zhu

"res" is representable by

<u>algebraic stack</u> equivalent to

natural "Boston-type" conjectures.

$\mathcal{X}$: formal algebraic stack of d-dimensional $G_{K,S}$ representations.
$\downarrow$
X : formal scheme of d-dimensional pseudo-representations.

<u>Chenevier</u>:

$$X = \coprod_{\substack{\bar{\psi} \text{ reduce} \\ \text{pseudo-representation}}} X_{\bar{\psi}} = \mathrm{Spf}\, R_{\bar{\psi}}$$

$$\mathcal{X}_{\bar{\psi}} := \mathcal{X} \times_X X_{\bar{\psi}},$$

$$\mathcal{X} = \coprod_{\bar{\psi}} \mathcal{X}_{\bar{\psi}}$$

Carl Wang-Erickson :

$$\begin{array}{l} \mathcal{X}_{\bar{\psi}} \\ \downarrow \quad : \text{representable by algebraic stack.} \\ X_{\bar{\psi}} \end{array}$$

$\bar{\psi} = \bar{\rho}$ is irreduceble representation,

$G_{K,S} \longrightarrow GL_d(\mathbb{F}_p)$: irreducible, continuous.

$$\mathcal{X}_{\bar{\rho}} = [\operatorname{Spf} R_{\bar{\rho}} / \hat{\mathbb{G}}_m]$$

└ automorphisms of irreducible representations

$$\downarrow$$

$$X_{\bar{\rho}} = \operatorname{Spf} R_{\bar{\rho}}$$

<u>Ex</u> $p = 5$, $K = \mathbb{Q}$, $S = \{\infty, 5, 11\}$, $d = 2$.

$\mathcal{X}$ classifying 5-adic representation what are $\left\{\begin{array}{l} \cdot\ d = 2, \\ \cdot\ \text{cyclotomic}, \\ \cdot\ f\text{: finite flat at } 5, \\ \cdot\ \text{semistable at } 11, \\ \cdot\ \text{weight 2 modular form on} \\ \quad \Gamma_0(11). \end{array}\right.$

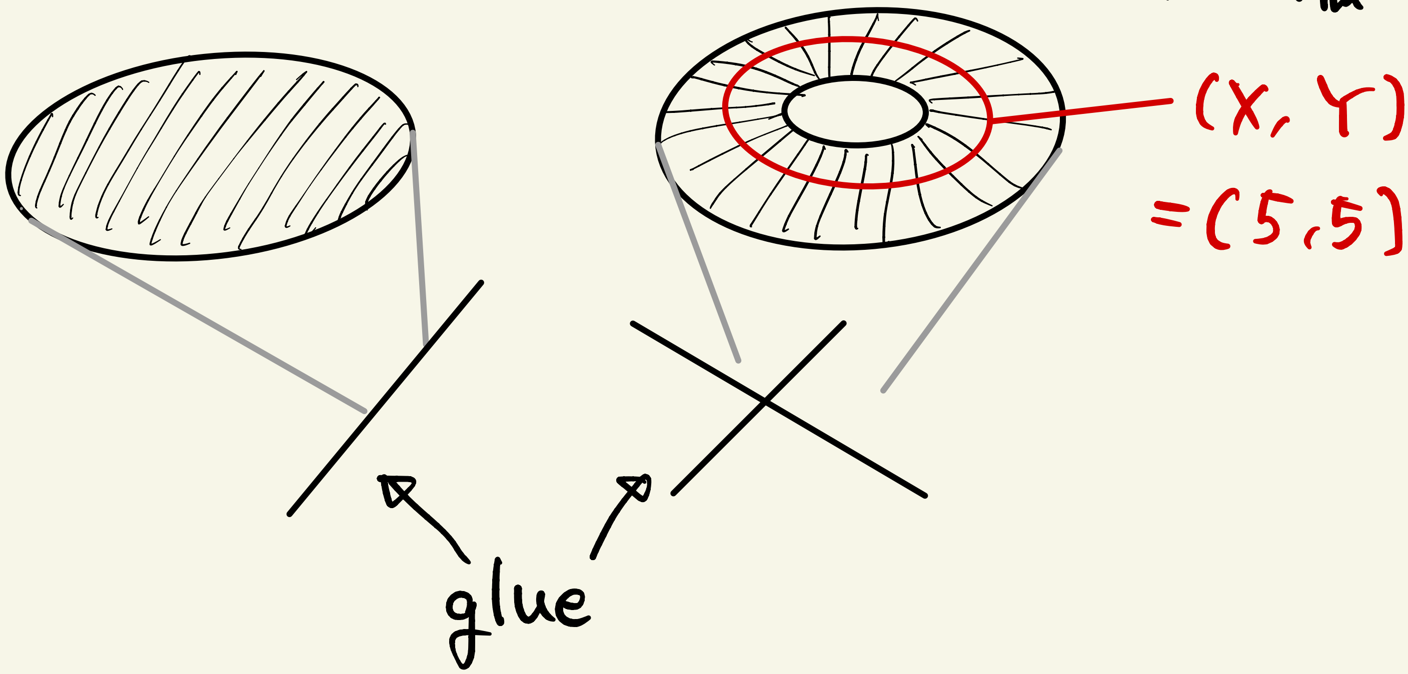
Unique $\bar{\varphi}$!! $= 1 \oplus \overline{\text{cyclo}}$.

$$\begin{array}{ll} \mathfrak{X} = \mathfrak{X}_{\bar{\varphi}} & \left[\operatorname{Spf} \mathbb{Z}_5\langle X\rangle / \mathbb{G}_m\right] \times \left[\operatorname{Spf} \mathbb{Z}_5\langle X, Y\rangle / (XY - 25) / \hat{\mathbb{G}}_m\right] \\ \downarrow & \left[\operatorname{Spec} \mathbb{F}_5[X] / \mathbb{G}_m\right] \\ X_{\bar{\varphi}} = \operatorname{Spf} \mathbb{Z}_5 \times_{\mathbb{F}_5} \mathbb{Z}_5 & \end{array}$$

$t \in \hat{\mathbb{G}}_m$ act via

$$\begin{cases} t \cdot X = t^2 X, \\ t \cdot Y = t^{-2} Y. \end{cases}$$

$[\mathrm{Spf}\, \mathbb{Z}_5\langle X\rangle / \hat{\mathbb{G}}_m]$ $\quad$ $[\mathrm{Spf}\, \mathbb{Z}_5\langle X, Y\rangle/(XY-25) \,/\, \hat{\mathbb{G}}_m]$

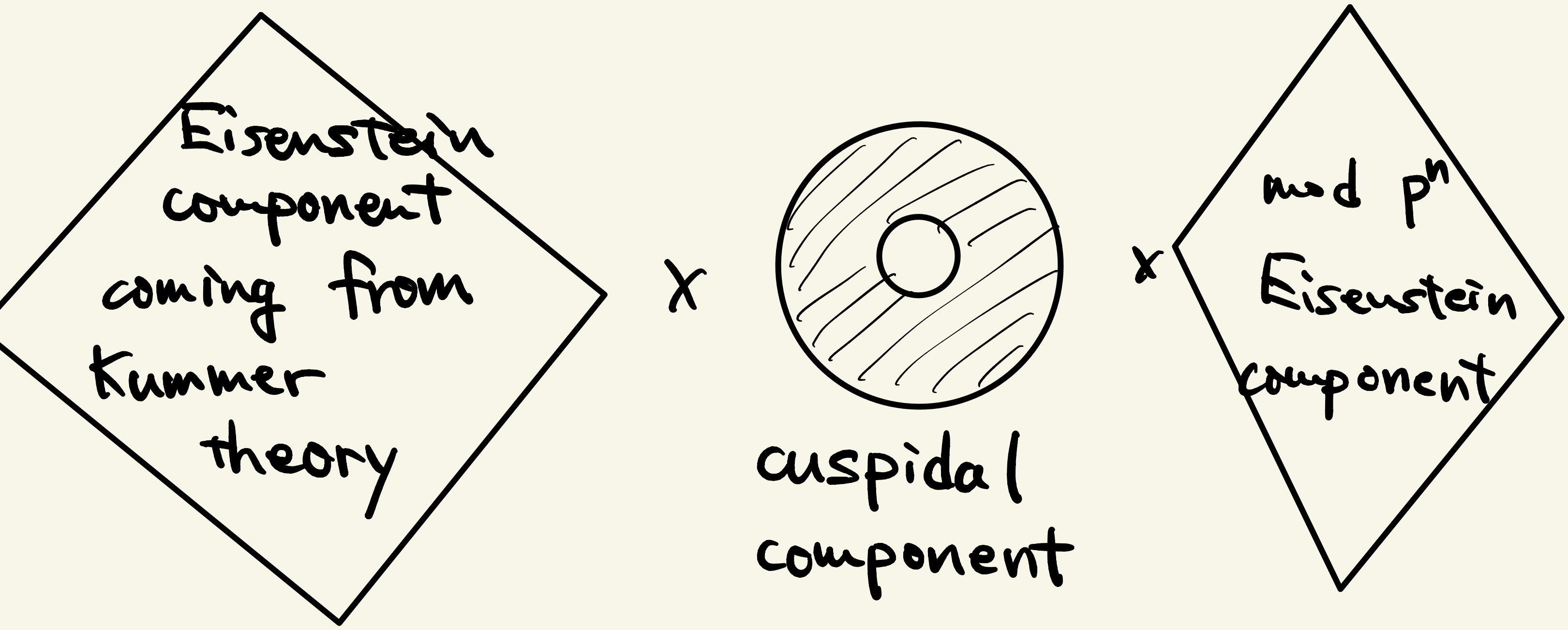


"X"

= coordinate on $\mathrm{Ext}^1_{G_{\mathbb{Q}_5}}(\mathbb{1}, \text{cyclo})$

= $\mathbb{Z}_5 \cdot$ Kummer class of 11.

On-going work of Toby Gee, Xinwen Zhu, if $K = \mathbb{Q}$, $d = 2$, $p > 2$, $\bar{\varphi}$ <u>odd</u>, then $\mathfrak{X}_{\bar{\varphi}}$ has the general structure.



dimension of this grows with #S

fixed dimension

We should, and <u>do</u>, use derived stacks (forthcoming work of E-Xinwen Zhu)

Derived point of view vanishing of Galois cohomology in degree > 2

$\Longrightarrow$ our stacks are "derived l.c.i.," i.e. locally look like

$\mathbb{A}^n / \underbrace{(f_1, \dots, f_r)}_{\text{expected \# of equ to}}$ in derived sense!!

<u>Main motivation</u> :

relation to cohomology of Shimura varieties.

If we fix a unitary Shimura variety

Hodge cocharacter μ <u>level structure</u>.

$$\begin{array}{c} \mathfrak{a} := \prod \mathfrak{a}_v \\ | \\ \mathfrak{X} \xrightarrow{\text{res.}} \prod \mathfrak{X}_v \end{array}$$

$\Gamma(\mathfrak{X}, \text{res}^! \mathfrak{a} \otimes V_\mu)$

$\overset{\text{conj.}}{=} R\Gamma_c(\text{Shimura variety}, \mathbb{Z}_p)_{\bar{\psi}}[\dim \frac{d}{-}]$

$\llcorner$ $\mathfrak{b}$-typ in middle degree.

$\bar{\psi} = \bar{\rho}$ irreducible

$\Gamma(\mathrm{Spec}\, Z_{\bar{\rho}}, \mathrm{res}^! \mathfrak{a}) \otimes V_\mu$

The End !!

$v \mid p$ Xinwen Zhu "Coherent sheaves on the stack of Langlands parameters"

$$\begin{array}{c} \mathfrak{a}_v \\ | \\ \mathfrak{X}_v \end{array}$$