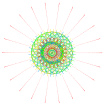
# $p$-adic Spin $L$-functions for $\mathrm{GSp}_6$

Ellen Eischen [1]



http://www.elleneischen.com

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Pair of workshops (in 2022) that might be of interest to people here: http://sites.google.com/view/automorphic2021/

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# Credits

This is joint work in progress with...

Giovanni Rosso

# Credits

This is joint work in progress with...

Giovanni Rosso

Shrenik Shah

# Credits

This is joint work in progress with...

Giovanni Rosso

Shrenik Shah

Willow Cat

# Also guest-starring and building on...

a variety of flavors of results by many different people...
especially an integral representation for the spin $L$-function on $\mathrm{GSp}_6$ for Siegel modular forms constructed by

Aaron Pollack

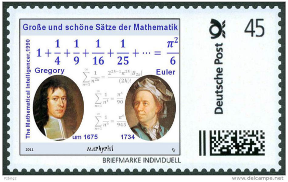
# Plan for this talk

Goal: Introduce a construction of $p$-adic Spin $L$-functions for $\mathrm{GSp}_6$.

1. Introduction to $p$-adic $L$-functions
2. Related results, especially for symplectic groups
3. Main results
4. Overview of key ingredients and steps of proofs of main results

# The Beginning: From Rational Numbers to Congruences

- Euler (1700s): For $2k$ a positive even number
  - $\zeta(2k) = 1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \cdots = \frac{-(2\pi i)^{2k}}{2\cdot(2k-1)!} \cdot \frac{B_{2k}}{2k}$
  - $\zeta(1-2k) = -\frac{B_{2k}}{2k} \in \mathbb{Q}$
- Kummer (1800s): For a prime $p$ and $2k, 2k'$ positive even integers such that $(p-1) \nmid 2k, 2k'$,
  $2k \equiv 2k' \mod \phi(p^d) \implies \zeta^{(p)}(1-2k) \equiv \zeta^{(p)}(1-2k') \mod p^d$,
  where $\zeta^{(p)}(1-2k) := (1-p^{2k-1})\zeta(1-2k)$





- Kubota-Leopoldt (1960s): Construct $p$-adic zeta-function, in fact $p$-adic Dirichlet $L$-functions
- Serre (1970s): Construct $p$-adic zeta-functions, using modular forms

# Connection with Modular Forms

Eisenstein series of weight $2k \geq 2$:

$$\tilde{G}_{2k}(z) := \sum_{(0,0)\neq(m,n)\in\mathbb{Z}\times\mathbb{Z}} \frac{1}{(mz+n)^{2k}}$$

Fourier expansion of $G_{2k}(z) := \frac{(2k-1)!}{(2\pi i)^{2k}}\tilde{G}_{2k}(z)$:

$$G_{2k}(z) = \zeta(1-2k) + 2\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n,$$

where $q = e^{2\pi iz}$ and $\sigma_{2k-1}(n) = \sum_{d\,|\,n} d^{2k-1}$.
Special case of

$$G_{k,\chi}(q) = L(1-k,\chi) + 2\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n,$$

where $\chi$ is Dirichlet character and $\sigma_{2k-1,\chi}(n) = \sum_{d\,|\,n}\chi(d)d^{2k-1}$.

# Serre's Innovation

Serre (1970s): Consider congruences between Fourier coefficients of modular forms (and construct *p-adic modular forms*...)

**Theorem (Serre,1970s)**

*Suppose have modular forms*

$$f(q) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$$
$$g(q) = b_0 + b_1 q + b_2 q^2 + b_3 q^3 + \cdots$$

*such that* $a_n \equiv b_n \mod p^d$ *for* $n \geq 1$*. Then* $a_0 \equiv b_0 \mod p^d$*.*

Apply to *p-stabilization* of $G_{k,\chi}$:

$$G_{k,\chi}^{(p)}(q) := L^{(p)}(1-k,\chi) + 2\sum_{n=1}^{\infty} \sigma_{k-1,\chi}^{(p)}(n) q^n,$$

where $L^{(p)}(1-k,\chi) := (1-\chi(p)p^{k-1})L(1-k,\chi)$ and $\sigma_{k-1,\chi}^{(p)}(n) := \sum_{d \mid n, p \nmid d} \chi(d) d^{k-1}$. Recovers Kummer, Kubota-Leopoldt!

# The origins of the theory of $p$-adic modular forms

## Theorem (Serre, 1973 Antwerp Volume 3)

*Modular forms can be put into p-adic families, and you can take p-adic limits of them. That is, if* $\{f_{k_i}(q) := \sum_{n\geq 0} a_n(k_i)\, q^n\}_i$ *is a sequence of modular forms of weight* $k_i$ *with* $k_i$ *converging in* $\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p$ *and* $\lim_{i\to\infty} a_n(k_i) = a_n$ *for* $n \geq 1$*, then* $a_0(k_i)$ *converge p-adically to some* $a_0$*.*

## Remark

Builds on work of Hecke, Klingen, and Siegel on algebraicity and of Atkin and Swinnerton-Dyer on congruences.

## Remark

This is the first instance of properties of modular forms driving the study of $p$-adic (or algebraic) properties of values of $L$-functions. (N.B. The behavior of families of Galois representations is also tied to properties of $p$-adic modular forms, as developed in work of Hida in the 1980s via systems of Hecke eigenvalues.)

# More using modular forms to study $L$-functions

In analogue with how Klingen and Siegel's work on algebraicity inspired Serre's approach to congruences and $p$-adic $L$-functions, we have:

**Theorem (Shimura, 1970s)**

*Let $f(q) = \sum_{n>0} a_n q^n$ be a cusp form of weight $k$ and $g(q) = \sum_{n\geq 0} b_n q^n$ be a modular form of weight $\ell$. Consider the Dirichlet series*

$$D(s, f, g) := \sum_{n=1}^{\infty} a_n b_n n^{-s}.$$

*Then for $\ell < k$ and $\frac{k+\ell-2}{2} < m < k$*

$$\frac{D(m, f, g)}{\pi^k \langle f, f \rangle_{\mathrm{Pet}}} \in \overline{\mathbb{Q}}$$

*(in fact, lies in $\mathbb{Q}(\{a_n, b_n\}_n)$).*

Here $\langle , \rangle_{\mathrm{Pet}}$ denotes the usual Petersson pairing on modular forms.

# From algebraic to $p$-adic properties of Rankin–Selberg convolutions

## Theorem (Shimura, 1970s)

*For each non-negative integer $r$ such that $\ell + 2r < k$,*

$$D(k-1-r, f, g) = c\pi^{\ell}\langle \tilde{f}, g\delta_{\lambda}^{(r)} E_{\lambda}\rangle_{\mathrm{Pet}},$$

*where $E_\lambda$ is a certain Eisenstein series of weight $\lambda := k - \ell - 2r$, $\delta_\lambda^{(r)}$ is a* **Maass–Shimura differential operator**, *$\tilde{f} := \sum_{n\geq 1} \overline{a_n} q^n$, and $c$ is a constant dependent on $k$, $\ell$, $r$, and the level of $f$ and $g$.*

## Theorem (Hida, 1980s)

*Reinterpret the Right Hand Side p-adically to construct p-adic L-functions (as p-adic measures).*

# General philosophy

Connection between proofs of algebraicity and constructions of $p$-adic $L$-functions

# Natural Questions, in View of Previous Examples

## Question

*For which ($\mathbb{C}$-valued) L-functions $L(s, M)$ and data $M$ does there exist a p-adic function $\mathfrak{L}_{p-\mathrm{adic}}$ such that*

$$(*)\mathfrak{L}_{p-\mathrm{adic}}(n, M) = (*')L^{(p)}(n, M)$$

*for all $(n, M)$ meeting appropriate conditions (and such that the construction exploits congruences between Fourier expansions of modular forms)?*

## Question

*What can we say about algebraicity of values of $L(n, M)$? (We need to know that values of algebraic, and furthermore p-integral, before talking about p-adic properties.)*

# Conjectures

- Coates, Perrin-Riou:
  - Conjecture existence of wide class of $p$-adic $L$-functions
  - Realized as element of an *Iwasawa algebra*, i.e. a completed group ring (where group is profinite Galois group) over a $p$-adic ring
  - Predicted form of the Euler factor at $p$
- Greenberg: Building on the Main Conjecture of Iwasawa Theory (proved by Mazur and Wiles), predicted the meaning of many $p$-adic $L$-functions through more extensive *Main Conjectures*, which concern $p$-adic variation of arithmetic data
- Deligne: Conjectures about algebraicity (again, necessary to establish before talking about $p$-adic behavior)

# Some Answers, in the Form of Examples

1. Already saw example for Dirichlet $L$-functions (Kubota-Leopoldt)
2. There exists a $p$-adic $L$-function interpolating $L^{(p)}(s, \chi)$ for appropriate $s$ and Hecke characters $\chi : \mathbb{A}_K^\times \to \mathbb{C}^\times$:
    1. Coates-Sinnott (1974): $K$ real quadratic
    2. Deligne-Ribet (1980): $K$ totally real
    3. Katz (1978): $K$ CM, with splitting condition on the prime $p$
3. There exists a $p$-adic $L$-function interpolating $L^{(p)}(s, f, \chi)$ for appropriate $s$, $\chi$, and modular forms $f$
    1. Hida (1980s, 1990s): $f$ *ordinary* (in a *Hida family*)
    2. Panchishkin (2000s): $f$ *finite slope* (in a *Coleman family*)
4. There exists a $p$-adic $L$-function interpolating $L^{(p)}(s, \pi, \chi)$ for appropriate $s$, $\chi$, and cuspidal automorphic representations $\pi$ of a unitary group
    1. Theorem of E–Harris–Li–Skinner (published in 2020): for $\pi$ *ordinary* (in a Hida family), with splitting condition on the prime $p$

# In case this all sounds incremental...

- Coates-Sinnot (case: real quadratic Hecke characters): 28 pages
- Deligne-Ribet (case: totally real Hecke characters): 60 pages
- Katz (case: CM Hecke character, with condition on $p$): 99 pages
- Hida (case: ordinary modular forms): 36 pages
  ...uses earlier work of Katz on $p$-adic families of Eisenstein series
- Panchishkin (case: finite slope modular forms): 65 pages

All of the above papers were published in *Inventiones*.

- E–Harris–Li–Skinner (case: ordinary cuspidal automorphic representations of unitary groups): 160 page paper published in *Forum of Mathematics, Pi* in 2020
  ...uses about 100 pages of my earlier work (e.g. *Crelle* '15, *Algebra Number Theory* '14) on $p$-adic families of Eisenstein series

# Strategy for constructing certain $p$-adic $L$-functions

This is the recipe employed by Hida, Panchishkin, E–Harris–Li–Skinner, and others:

1. Construct a $p$-adic family $\mathbb{E}$ of Eisenstein series (indexed by weights, like the example of $G_{2k}$ at the beginning) on some group $G$
2. Restrict ("pull back") to some (possibly smaller) subgroup $H$ inside $G$, and pair against a cusp form on $H$
3. Interpret (via an integral representation, such as the Rankin–Selberg method) as a (familiar) $L$-function

This builds on a strategy for proving algebraicity results...

# Precursor to studying $p$-adic behavior: prove algebraicity results!

Recipe for proving algebraicity results (pioneered by Shimura in his study of Rankin–Selberg convolutions):

1. Realize your $L$-function in terms of an integral representation (e.g. Rankin–Selberg convolution, doubling integral, ...), pairing an Eisenstein series against cusp form(s), so values of your $L$-function are realized as a pairing $\langle \phi, E \rangle$
2. Find an Eisenstein series with algebraic Fourier coefficients
3. Find an othonormal basis $\{\phi_i\}$ for your space of cusp forms over $\bar{\mathbb{Q}}$, and show that $\langle \phi_i, E \rangle / \langle \phi_i, \phi_i \rangle \in \bar{\mathbb{Q}}$. Use this prove similar statement for any cusp form $\phi$ in place of $\phi_i$.

It is nontrivial to carry out this recipe, and it is even nontrivial to acquire some of the ingredients (in particular, the Fourier expansion of an Eisenstein series, an integral representation of an $L$-function, and furthermore an integral representation that is suitable for studying algebraicity).

# What about symplectic groups $\mathrm{GSp}_{2n}$ and Siegel modular forms?

- In analogue with the standard Langlands $L$-functions studied by E–Harris–Li–Skinner in the setting of unitary groups, $p$-adic standard Langlands $L$-functions associated to cuspidal automorphic forms on symplectic groups were recently constructed by Zheng Liu, further extending earlier work by Böcherer–Schmidt.
- This relies on the doubling method, due to Piatetski-Shapiro–Rallis and earlier work of Garrett.
- Shimura proved corresponding algebraicity results.

**Remark**

In the setting of $\mathrm{GSp}_2 = \mathrm{GL}_2$, the standard $L$-function is $L(Sym^2 f, s)$. On the other hand, $L(f, s)$ is a spin $L$-function....

# Spin $L$-functions for symplectic groups $\mathrm{GSp}_{2n}$

We have that ${}^L\mathrm{GSp}_{2n} = \mathrm{GSpin}_{2n+1}$, which has a natural $2^n$-dimensional representation, the *spin* representation.

## Question

*What can we say about algebraicity and p-adic interpolation if we replace $L(s, \pi, \mathrm{std})$ (studied by Liu, Böcherer–Schmidt, ...) by $L(s, \pi, \mathrm{spin})$, with $\pi$ a cuspidal automorphic representation of $\mathrm{GSp}_{2n}$?*

- $n = 1$:This is $L(f, s)$ with $f$ a modular form. Using modular symbols, Manin–Shimura proved algebraicity results, and Mazur–Tate–Teitelbaum constructed $p$-adic $L$-function.
- $n = 2$:Using higher Hida theory, together with Harris's interpretation of $L$-values as cup products (useful for studying algebraicity), Loeffler–Pilloni–Skinner–Zerbes constructed $p$-adic $L$-function.
- $n \geq 3$:algebraicity and $p$-adic properties for $n = 3$ is the subject of the rest of this talk.Remain poorly understood for $n > 3$ (although Bump–Ginzburg constructed an integral representation for n=$3, 4, 5$ that appears not to be amenable to proving algebraicity results)...

# $p$-adic spin $L$-functions for symplectic groups $\mathrm{GSp}_6$

- This project focuses on analytic $p$-adic $L$-functions (obtained by $p$-adic interpolation).
- On the algebraic side, there is recent related work of Cauchi–Jacinto and Cauchi–Lemma–Jacinto.

# Main results

**Theorem (E–Rosso–Shah)**

*Let $\pi$ be a cuspidal automorphic representation of $\mathrm{GSp}_6$ associated to a Siegel modular form $\phi$ of weight $2r$. Then $\frac{L(\pi, spin, s)}{\pi^{4s-6r+6}\langle \phi|_{w_N}, \phi\rangle}$ is algebraic, for $s \in \{3r-2, \ldots, 4r-5\}$. (Here, $w_N$ is a certain involution for $G$.)*

**Theorem (E–Rosso–Shah)**

*There is a two-variable p-adic L-function $L_p(f_{(2r)}, j)$ that varies p-adically in $2r$ and $j$, interpolating the values*

$$l_\infty l_p \frac{L^{(p)}(\pi_{(2r)}, spin, s)}{\langle f_{(2r)}|_{w_N}, f_{(2r)}\rangle},$$

*where $3r-2 \le j \le 4r-5$ and $f_{(2r)}$ varies in a Hida family of ordinary Siegel modular forms. Here $l_\infty$ denotes the archimedean Euler factor (a product of gamma factors) and $l_p$ denotes the modified Euler factor at $p$.*

# A Rankin–Selberg style pairing

Let $\phi$ be a cusp form associated to a holomorphic Siegel modular form on $\mathrm{GSp}_6$ of positive, even scalar weight $2r$. Consider the pairing

$$\langle \phi, E_{2r}(g,s)\rangle = I_{2r}^*(\phi, s) = \int_{\mathrm{GSp}_6(\mathbb{Q})Z(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \phi(g)E_{2r}(g,s)dg,$$

for $E_{2r}(g,s)$ certain Eisenstein series on a group $G$ (to be defined in a few moments) restricted to $\mathrm{GSp}_6 \subset G$.

# Some things that are different in this setting

If you're familiar with the doubling method for Siegel modular forms or the Rankin–Selberg integral for modular forms, it might *feel* like you can use what you know about those settings to proceed here.
There are some key differences, though:

1. $G$ is closely related to the exceptional group $GE_7$ (and although $G$ is related to a unitary group, that's a *quaternionic* unitary group).
2. There's currently no known moduli problem for $G$, no $q$-expansion principle, etc.
3. The Rankin–Selberg style pairing here is a non-unique model.

# Preliminary setup

Fix:

- a field $F$ of characteristic 0
- a quaternion algebra $B$ over $F$

Define an $F$ vector space

$$W := F \oplus H_3(B) \oplus H_3(B) \oplus F,$$

where $H_3(B)$ denotes $3 \times 3$ hermitian matrices over $B$, i.e. matrices of the form

$$\begin{pmatrix} c_1 & a_3 & a_2^* \\ a_3^* & c_2 & a_1 \\ a_2 & a_1^* & c_3 \end{pmatrix},$$

with $c_1, c_2, c_3 \in F$ and $a_1, a_2, a_3 \in B$. Here, $a^*$ denotes the conjugate of $a$ in the quaternion algebra $B$.

The group $G$ will be defined as a certain subgroup of $\mathrm{GL}(W) \times \mathrm{GL}_1(F)$. First, though, a few conventions for operations on $H_3(B)$...

# Conventions for $H_3(B)$

Given $h = \begin{pmatrix} c_1 & a_3 & a_2^* \\ a_3^* & c_2 & a_1 \\ a_2 & a_1^* & c_3 \end{pmatrix} \in H_3(B)$, we put

$$N(h) := c_1c_2c_3 - c_1n(a_1) - c_2n(a_2) - c_3n(a_3) + \mathrm{tr}(a_1a_2a_3),$$

where $n(a) := aa^*$ for all $a \in B$. We also define

$$h^\# := \begin{pmatrix} c_2c_3 - n(a_1) & a_2^*a_1^* - c_3a_3 & a_3a_1 - c_2a_2^* \\ a_1a_2 - c_3a_3^* & c_1c_2 - n(a_2) & a_3^*a_2^* - c_1a_1 \\ a_1^*a_3^* - c_2a_2 & a_2a_3 - c_1a_1^* & c_1c_2 - n(a_3) \end{pmatrix}$$

Given an element $h' \in H_3(B)$, we define

$$\mathrm{tr}(h, h') := \mathrm{trace}(hh' + h'h).$$

# The group $G$

Define

- A symplectic form $\langle , \rangle : W \times W \to F$ by

$$\langle (a, b, c, d), (a', b', c', d') \rangle := ad' - \mathrm{tr}(b, c') + \mathrm{tr}(c, b') - da',$$

  where $\mathrm{tr}(b, c) := \frac{1}{2}(bc + cb)$ for all $b, c \in H_3(B)$.

- A quartic form $Q : W \to F$ by

$$Q((a, b, c, d)) := (ad - \mathrm{tr}(b, c))^2 + 4aN(c) + 4dN(b) - 4\mathrm{tr}(b^\#, c^\#).$$

We define $G$ to be the similitude algebraic group consisting of all $(g, \nu(g)) \in \mathrm{GL}(W) \times \mathrm{GL}_1(F)$ such that for all $u, v, w \in W$:

$$\langle ug, vg \rangle = \nu(g) \langle u, v \rangle$$
$$Q(wg) = \nu(g)^2 Q(w)$$

# About the group $G$...

- There's a group that is a double cover of $G$ and $GU_6(B)$ (quaternionic unitary group)
- $G$ is closely related to the exceptional group $GE_7$
- $G$ is a half-spin group of type $D_6$

# The Hermitian symmetric space for $G(\mathbb{R})$

The Hermitian symmetric space for $G(\mathbb{R})$ is

$\mathcal{H} := \{Z \in H_3(B \otimes_{\mathbb{R}} \mathbb{C}) | Z = X + iY, withX, Y \in H_3(B_{\mathbb{R}}), Y \text{ positive definite } \}$

How does $G$ act on $\mathcal{H}$?
We have a map $r : \mathcal{H} \to W$, $Z \mapsto (1, -Z, Z^{\#}, -N(Z))$.
The action of $G$ on $\mathcal{H}$ and the factor of automorphy $j_{G(\mathbb{R})}(g, Z)$ are defined simultaneously by

$$r(Z)g^{-1} = j_{G(\mathbb{R})}(g, Z)r(gZ).$$

There is an embedding $\mathrm{GSp}_6 \hookrightarrow G$ and corresponding embedding of Siegel upper half space into $\mathcal{H}$.

# Key input: Eisenstein series

1. We have an Eisenstein series $E_{2r}(g,s)$ that is holomorphic at $s = r$ and whose Fourier expansion when $s = r$ has coefficients of the form

$$N(h)^{r-5}\left(\prod_{\ell \mid N(h)} P_\ell(\ell^{2r})\right),$$

with $P_\ell$ a polynomial with rational coefficients and $h$ is a $3 \times 3$ hermitian quaternionic matrix.

2. We easily see that these Fourier coefficients satisfy congruences...which would be nice for constructing an Eisenstein measure...BUT we don't have a $q$-expansion principle here.
3. Pulls back nicely to the space for $\mathrm{GSp}_6$, where we DO have a $q$-expansion principle and can construct a $p$-adic family of (pullbacks of) Eisenstein series
4. There is a differential operator $\mathcal{D}$ such that $\mathcal{D}E_{2r}(g,s) = E_{2r+2}(g,s)$. So it suffices to study $\mathcal{D}$ and $E_{2r}(g,r)$. Has nice action in $q$-expansions...

# Example: Calculating the action of the differential operator

- We can show that the differential operator acting on a weight $k$ modular form $f(Z)$ on $\mathcal{H}$ has the form
$$N(Y)^{-k}\Delta\left(N(Y)^k f\right),$$
where $\Delta$ is the determinant of a matrix of partial derivatives.
- Note the similarity with the familiar Maass–Shimura operators in the setting of classical automorphic forms.
- Given the way the action of $G$ and the automorphy factor are defined, how can we see that this operator raises the weight of a modular form on $G$ by 2?
- The answer follows from two key equations (whose proofs are a nice exercise in exploring how to think about the action of $G$ and the embedding $r : \mathcal{H} \hookrightarrow W$):For any $\alpha \in G^+(\mathbb{R})$ and $z, w \in \mathcal{H}$, we have:

$$\begin{aligned} N(\alpha z - \alpha w) &= \nu(\alpha)^{-1} j(\alpha, z)^{-1} j(\alpha, w)^{-1} N(z - w) && (1) \\ N(\mathrm{Im}(\alpha z)) &= |j(\alpha, Z)|^{-2} \nu(\alpha)^{-1} N(\mathrm{Im} z) && (2) \end{aligned}$$

(These rely on the observation that $\langle r(z), r(w)\rangle = N(z - w)$.)

# Key realizations

- Key realization that enables algebraicity theorem: Write $\frac{L(\pi, spin, s)}{\pi^{4s-6r+6}\langle\phi|_{w_N}, \phi\rangle} = \frac{\langle\phi|_{w_N}, E_{2r}(z, s+5-3r)\rangle}{\langle\phi|_{w_N}, \phi\rangle}$ with Eisenstein series $E_{2r}$ and a Rankin–Selberg style pairing as above, and recall that $\mathcal{D}E_{2r}(z, s) = E_{2r+2}(z, s)$ ...
- Following Hida's approach to constructing $p$-adic $L$-functions associated to Rankin–Selberg convolutions of modular forms, we write $\ell_{f_{2r}}(E_{2r}(s)) = \frac{\langle f_{2r}|_{w_N}, E_{2r}(z, s+5-3r)\rangle}{\langle\phi|_{w_N}, f_{2r}\rangle}$, and we obtain a $p$-adic $L$-function associated to a Hida family of (ordinary) Siegel modular forms $f_{(2r)}$.

# Thank you!

For more information:

- Visit `http://www.elleneischen.com`
- Email eeischen@uoregon.edu
- Also, check out the pair of workshops here: `http://sites.google.com/view/automorphic2021/`

# About the image on the first slide

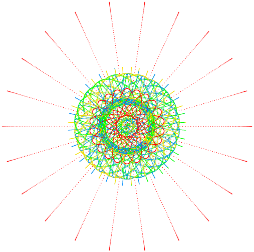


Figure: The Gaussian periods in the 255, 255-th cyclotomic extension of $\mathbb{Q}$ fixed by the action of the automorphism $\zeta \mapsto \zeta^{254}$, where $\zeta$ is a primitive 255, 255-th root of unity. Image credit: E. Eischen, based on earlier work by W. Duke, S. R. Garcia, T. Hyde, and R. Lutz.