Singular modular forms on quaternionic E_8

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January 2021

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Goal

This talk is about: The construction of two very nice automorphic forms on quaternionic E_8

- $E_{8,4}$: real reductive group of type E_8 with split rank four; this is quaternionic E_8
- The symmetric space $E_{8,4}/K$ does not have Hermitian structure, but still possesses automorphic forms that behave **similarly** to classical holomorphic modular forms
- **Similarly**: They have a 'robust' Fourier expansion; called 'modular' forms
- There are two modular forms on $E_{8,4}$ that can write down explicitly
- **Theorem**: These modular forms have all Fourier coefficients in $\mathbf{Q}$
- Time permitting: An application to a very interesting automorphic form on $E_{6,4}$

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Siegel modular forms

The symplectic group

- $\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}(2n) : {}^t g \left(\begin{smallmatrix} & 1_n \\ -1_n & \end{smallmatrix}\right) g = \left(\begin{smallmatrix} & 1_n \\ -1_n & \end{smallmatrix}\right)\}$
- $Sp_{2n} \supseteq U(n) \simeq \left\{\left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right) : a + ib \in U(n)\right\}$

The symmetric space

- $S_n := n \times n$ symmetric matrices
- $\mathcal{H}_n = \{Z = X + iY : X, Y \in S_n(\mathbf{R}), Y > 0\}$ the Siegel upper half-space
- $\mathcal{H}_n \simeq \mathrm{Sp}_{2n}(\mathbf{R})/U(n)$ the symmetric space

$\mathrm{Sp}_{2n}(\mathbf{R})$ acts on $Sp_{2n}(\mathbf{R})/U(n) = \mathcal{H}_n$ via

$$g \cdot Z = (aZ + b)(cZ + d)^{-1}$$

if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $n \times n$ block form.

Siegel modular forms: continued

Siegel modular form of weight $\ell > 0$:

Definition and basic properties

- $f : \mathcal{H}_n \to \mathbf{C}$ **holomorphic** such that
- $f((aZ+b)(cZ+d)^{-1}) = \det(cZ+d)^{\ell} f(Z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ some congruence subgroup of $\mathrm{Sp}_{2n}(\mathbf{Z})$
- **Fourier expansion**:

$$f(Z) = \sum_{T \in S_n(\mathbf{Q}),\, T \geq 0} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)}$$

  with $a_f(T) \in \mathbf{C}$ and $T \geq 0$ means "T is positive semi-definite".

- If $n = 1$, these are classical modular forms for SL_2

If f a Siegel modular form, can consider $f \in H^0(\Gamma\backslash\mathcal{H}_n, \mathcal{L}^{\ell})$

- a global section of a holomorphic line bundle $\mathcal{L}^{\ell}$ on $\Gamma\backslash\mathcal{H}_n$

Siegel modular forms automorphically

$$\varphi : \mathrm{Sp}_{2n}(\mathbf{Q})\backslash \mathrm{Sp}_{2n}(\mathbf{A}) \to \mathbf{C} \text{ with}$$

The definition

1. $\varphi(gk) = z(k)^{-\ell}\varphi(g)$ for all $k \in U(n)$, $z : U(n) \stackrel{\det}{\to} U(1) \subseteq \mathbf{C}^\times$
2. $\mathcal{D}_{CR,\ell}\varphi \equiv 0$: φ annihilated by linear differential operator $\mathcal{D}_{CR,\ell}$ so that f_φ on $\mathcal{H}_n$ satisfies the Cauchy-Riemann equations

The Fourier expansion

$$\begin{aligned}\varphi_f\left(\begin{pmatrix} 1 & X \\ & 1\end{pmatrix}\begin{pmatrix} Y^{1/2} & \\ & Y^{-1/2}\end{pmatrix}\right) &= \varphi_f(n(X)m) \\ &= \sum_{T \in S_n(\mathbf{Q}), T \geq 0} a_\varphi(T) e^{2\pi i \operatorname{tr}(TX)} e^{-2\pi \operatorname{tr}(TY)}\end{aligned}$$

where $iY = m \cdot i$ in $\mathcal{H}_n$ and $a_\varphi(T) \in \mathbf{C}$.

Automorphically

- $\pi = \otimes_v \pi_v$ with π_∞ a holomorphic discrete series representation

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A very nice exceptional group

$E_{7,3}$: has a symmetric space with Hermitian tube structure

- Θ: octonions with positive-definite norm form. This is an 8-dimensional, non-associative $\mathbf{R}$-algebra that comes equipped with a quadratic form $\Theta \to \mathbf{R}$ and an $\mathbf{R}$-linear conjugation $* : \Theta \to \Theta$.
- $J = H_3(\Theta)$: Hermitian 3×3 matrices with elements in Θ.

$$J = \left\{ \begin{pmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{pmatrix} : c_i \in \mathbf{R}, x_j \in \Theta \right\}.$$

$E_{7,3}$ acts on

$$\mathcal{H}_J = \{Z = X + iY : X, Y \in J, Y > 0\}$$

by "fractional linear" transformations.

For an integer $\ell > 0$, $f : \mathcal{H}_J \to \mathbf{C}$ is a holomorphic modular form of weight ℓ if

- f is holomorphic, moderate growth
- $f(\gamma Z) = j(\gamma, Z)^{\ell} f(Z)$ for all $\gamma \in \Gamma \subseteq E_{7,3}$ a congruence subgroup

These holomorphic modular forms on $E_{7,3}$ have a Fourier expansion:

$$f(Z) = \sum_{T \in J_{\mathbf{Q}}, T \geq 0} a_f(T) e^{2\pi i \operatorname{tr}(TZ)}$$

with the $a_f(T) \in \mathbf{C}$.

Kim's modular forms on $E_{7,3}$

Rank

Note that $J \supseteq S_3$ the symmetric 3×3 matrices. There is a function *rank* : $J \to \{0,1,2,3\}$ extending the rank of symmetric matrices on S_3.

Theorem 1 (H. Kim)

There exists holomorphic modular forms $\Theta_{Kim,4}$ and $\Theta_{Kim,8}$ for $E_{7,3}$ with the following properties:

1. *$\Theta_{Kim,4}$ is a weight 4, level 1 modular form with Fourier coefficients in $\mathbf{Z}$. Moreover, the Fourier coefficients $a_{\Theta_{Kim,4}}(T)$ are 0 unless $rank(T) \in \{0,1\}$.*
2. *$\Theta_{Kim,8}$ is a weight 8, level 1 modular form with Fourier coefficients in $\mathbf{Z}$. Moreover, the Fourier coefficients $a_{\Theta_{Kim,8}}(T)$ are 0 unless $rank(T) \in \{0,1,2\}$.*

The modular forms $\Theta_{Kim,4}, \Theta_{Kim,8}$ are said to be **singular**.

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Exceptional groups have 'modular forms'

The groups

$G : G_2 \subseteq D_4 \subseteq F_4 \subseteq E_{6,4} \subseteq E_{7,4} \subseteq E_{8,4}$

- $K \subseteq G$ the maximal compact. $K \twoheadrightarrow \mathrm{SU}(2)/\mu_2$.
- G/K: no Hermitian structure

Definition of modular forms on G

Let $\ell \geq 1$ be an integer. A modular form on G of weight ℓ is

- an automorphic form $\varphi : \Gamma\backslash G \to Sym^{2\ell}(\mathbf{C}^2)$
- satisfying $\varphi(gk) = k^{-1} \cdot \varphi(g)$ for all $g \in G$, $k \in K$
- and $\mathcal{D}_\ell \varphi = 0$ for a certain special linear differential operator $\mathcal{D}_\ell$

- Definition due to Gross-Wallach, Gan-Gross-Savin

These modular forms have nice properties

Theorem 2

The modular forms of weight $\ell \geq 1$ on G have a robust Fourier expansion, normalized over the integers, that is compatible with pullbacks between groups G above.

The theorem means:

- Given a modular φ form of weight ℓ, one can ask the question "Are all of φ's Fourier coefficients in some ring $R \subseteq \mathbf{C}$?"
- If $\iota : G_1 \subseteq G_2$ in the above sequence of groups, and if φ is modular form on G_2 of weight ℓ, then the pullback $\iota^*(\varphi)$ on G_1 is a modular form of weight ℓ.
- Moreover, the Fourier coefficients of $\iota^*\varphi$ are **finite sums** of the Fourier coefficients of φ

More precise Fourier expansion

Have

$$G \supseteq P = MN \supseteq [N, N] := Z$$

the Heisenberg parabolic. Set

$$\varphi_Z(g) = \int_{Z(\mathbf{Q})\backslash Z(\mathbf{A})} \varphi(zg)\, dz.$$

Can Fourier expand φ_Z along $N/[N, N]$, and via Theorem 2 this Fourier expansion is

Fourier expansion

$$\varphi_Z(nm_f m_\infty) = \sum_{\chi \in (N/Z)^\vee} \chi(n) c_{\chi,\varphi}(m_f) \mathcal{W}_\chi(m_\infty)$$

for certain completely explicit functions $\mathcal{W}_\chi : M(\mathbf{R}) \to Sym^{2\ell}(\mathbf{C}^2)$.

- The point is that the functions $\mathcal{W}_\chi$ are independent of φ
- The functions $\mathcal{W}_\chi$ are 0 unless χ satisfies a certain positivity condition

Fourier coefficients

Fourier coefficients

If $R \subseteq \mathbf{C}$ is a subring, one says that φ has Fourier coefficients in R if all the functions $c_{\chi,\varphi} : M(\mathbf{A}_f) \to \mathbf{C}$ are in fact valued in R.

- If χ is non-degenerate in a certain sense, these Fourier coefficients were defined by Gan-Gross-Savin, using a multiplicity one result of Wallach.

Motivating question

Fix G and $\ell \geq 1$. Does there exist a basis of the modular forms on G of weight ℓ, all of whose Fourier coefficients are in $\overline{\mathbf{Q}}$?

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Let $P = MN \subseteq E_{8,4}$ be the Heisenberg parabolic subgroup, $M = GE_{7,3}$.

Theorem 3 (Gan,P,Savin)

There exists square integrable automorphic forms Θ_{min} and Θ_{ntm} on $E_{8,4}$ with the following properties.

1. *Θ_{min} is a weight 4 modular form with all Fourier coefficients in $\mathbf{Z}$. Its constant term along N, $\Theta_{min,N}$ is essentially $\Theta_{Kim,4}$.*
2. *Θ_{ntm} is a weight 8 modular form with all Fourier coefficients in $\mathbf{Q}$. Its constant term along N, $\Theta_{ntm,N}$ is essentially $\Theta_{Kim,8}$.*

These modular forms are **singular** *in the sense that many of their Fourier coefficients are 0.*

The Fourier coefficients are parametrized by elements in a lattice in $W = (N/[N,N])^\vee$. There is a function $\mathrm{rank} : W \to \{0,1,2,3,4\}$.

- The Fourier coefficients $a_{\Theta_{min}}(w)$ of Θ_{min} are 0 unless $\mathrm{rank}(w) \in \{0,1\}$
- The Fourier coefficients $a_{\Theta_{ntm}}(w)$ of Θ_{ntm} are 0 unless $\mathrm{rank}(w) \in \{0,1,2\}$

Remarks

1. Gross-Wallach constructed unitary representations π_4 and π_8 of the real group $E_{8,4}$ that are small in the sense of GK dimension. The automorphic forms Θ_{min}, Θ_{ntm} should be[1] thought of as globalizations of these representations.
2. On **split** E_8 there are analogues of Θ_{min} and Θ_{ntm}. These are completely spherical automorphic forms
   - constructed by Ginzburg-Rallis-Soudry, in the case of the minimal;
   - constructed by Green-Miller-Vanhove, Ciubotaru-Trapa in the case of next-to-minimal;
   - next-to-minimal recently studied by Gourevitch-Gustafsson-Kleinschmidt-Persson-Sahi.
3. Gan constructed Θ_{min} as a special value of an Eisenstein series associated to $Ind_P^{E_{8,4}}(\delta_P^{s_{min}})$, proved it's square integrable.

[1] Proved by Gan-Savin for Θ_{min} and π_4. Should be true but not proved for Θ_{ntm} and π_8.

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Heisenberg Eisenstein series

Suppose $G = E_{8,4}$, P Heisenberg parabolic.

$$\nu : P \to \mathrm{GL}_1$$

generating the character group of P. On $G = E_{8,4}$,

$$|\nu(p)|^{29} = \delta_P(p)$$

for $p \in P$. Suppose

- $\ell \geq 1$ even
- $f(g,\ell;s) \in Ind_{P(\mathbf{A})}^{G(\mathbf{A})}(|\nu|^s)$, certain $Sym^{2\ell}(V_2)$-valued section.
- $E(g,\ell;s) = \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} f(\gamma g,\ell;s)$ absolutely convergent for $Re(s) > 29$.
- If $s = \ell+1$ in range of absolute convergence, $E(g,s=\ell+1)$ a **modular form of weight** ℓ for G

Question

Does $E(g,s=\ell+1)$ have rational Fourier coefficients?

Next to minimal

Motivated by work of Gross-Wallach on continuation of quaternionic discrete series, take $\ell = 8$ and $G = E_{8,4}$.

Proposition

The Eisenstein series $E(g, \ell = 8; s)$ is regular at $s = 9$ (even though outside the range of absolute convergence), and defines square integrable weight 8 modular form at this point.

Set

$$\theta_{ntm}(g) = E(g, \ell = 8; s = 9)$$

Theorem 4 (Savin)

The spherical constituent of the degenerate principal series $Ind_{P(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)}(|\nu|^9)$ is "small", i.e., many twisted Jacquet modules are 0. Consequently, the rank three and rank four Fourier coefficients of θ_{ntm} are 0.

More on next-to-minimal modular form

Theorem 5

The weight 8 modular form θ_{ntm} has rational Fourier coefficients.

Proof.

1. Savin's result gives vanishing of rank three and four Fourier coefficients
2. Explicit computation (outside range of abs. convergence) gives rationality of rank 1 and rank 2 Fourier coefficients
3. Constant term analyzed using work of H. Kim on weight 8 singular modular form on $GE_{7,3}$

$\square$

Explicit computation of θ_{ntm}

1. Define special $Sym^{2\ell}(V_2)$-valued Eisenstein series $E_\ell(g)$ on $\mathrm{SO}(3, 4k+3)$
2. Prove that the constant term θ_{ntm} from $E_{8,4}$ down to $\mathrm{SO}(3, 11)$ is $E_8(g)$
3. **Theorem**: the $E_\ell(g)$ have rational Fourier coefficients (in a precise sense)
4. The Fourier coefficients of $E_8(g)$ can be identified with rank 1 and rank 2 Fourier coefficients of θ_{ntm}.

To prove the $E_\ell(g)$ have rational Fourier coefficients:

Jacquet integral

Explicit computation of certain Archimedean Jacquet integral

$$\int_{V_{2,4k+2}(\mathbf{R})} e^{2\pi i (v,x)} f_\ell(wn(x))\, dx.$$

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A distinguished modular form

Let G be one of the quaternionic exceptional groups, $P = MN$ its Heisenberg parabolic, $W(\mathbf{R}) = (N/[N,N])^\vee$. Globally, there is an arithmetic invariant on the orbits of $M(\mathbf{Q})$ on $W(\mathbf{Q})$:

$q : W(\mathbf{Q})^{rk=4} \to \mathbf{Q}^\times/(\mathbf{Q}^\times)^2 = \{$ quadratic etale extensions of $\mathbf{Q}\}$.

Fact: If F a modular form on G, $\omega \in W(\mathbf{Q})$ and $q(\omega) > 0$ then $a_F(\omega) = 0$. In other words, only ω corresponding to imaginary quadratic fields can have associated nonzero Fourier coefficients

Fix an imaginary quadratic extension $E/\mathbf{Q}$. Associated to E, there is a group G_E over $\mathbf{Q}$ of type $E_{6,4}$.

Theorem 6

There is a weight 4 modular form θ_E on G_E with Fourier coefficients in $\mathbf{Z}$ such that θ_E has nonzero Fourier coefficients of all ranks and

1. *If $\omega \in W(\mathbf{Q})^{rk=4}$ and $q(\omega) \in \mathbf{Q}^\times/(\mathbf{Q}^\times)^2$ does not represent E, then the Fourier coefficient $a_{\theta_E}(\omega) = 0$*

Proof of Theorem 6:

1. Define G_E, which is simply-connected of type $E_{6,4}$
2. Carefully embed G_E in $E_{8,4}$ via $\iota_E : G_E \rightarrow E_{8,4}$
3. Define $\theta_E = \iota_E^*(\theta_{min})$, the pull-back of the modular form generating the minimal representation on $E_{8,4}$
4. The Fourier coefficients of θ_E can then be computed from those of θ_{min}
5. θ_{min} only has nonzero Fourier coefficients for the **most degenerate** ω, those ω of rank 1
6. This vanishing of $a_{\theta_{min}}(\omega)$ imposes a strong arithmetic condition on the Fourier coefficients of θ_E.

Thank you

Thank you for your attention!