A cohomological interpretation of archimedean zeta integrals for $\mathrm{GL}_3 \times \mathrm{GL}_2$

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Contents

We want to generalize

$$\sum_{0\le j\le k-2}\binom{k-2}{j}\sqrt{-1}^{-j-1}\frac{\Gamma_{\mathbb{C}}(j+1)L(j+1,f)}{\Omega_f^{\pm}}X^{k-2-j}Y^j\in K_f[X,Y],$$

where

- $f=\sum_{n=1}^{\infty}a(n,f)q^n$: elliptic newform of weight k.
- $L(s,f)=\sum_{n=1}^{\infty}\frac{a(n,f)}{n^s}$.
- $K_f=\mathbb{Q}(\{a(n,f)\mid n\in\mathbb{N}\})$: Hecke field of f. $K_f/\mathbb{Q}$: finite.
- $\pm=(-1)^j, 0\le j\le k-2$.
- $\Omega_f^{\pm}\in\mathbb{C}^{\times}/K_f^{\times}$: canonical periods for f.

Remark

1. The polynomial can be understood as the Mellin transform of the image $\delta(f) = f(z)(X - zY)^{k-2}\mathrm{d}z$ of f via the Eichler-Shimura map over periods.
2. Assuming p is odd, we can refines the definitions of periods $\Omega_f^{\pm}$, depending only on f, p and p-adic units, so that we can discuss the integrality on L-values.
3. The binomial coefficients in the formula is important for the proof of Kummer congruences of L-values, i. e. a construction of p-adic L-function (Mazur-Tate-Teitelbaum).
4. $\exists$ generalizations to $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ due to
   - **(Algebraicity)** Mazur-Kazhdan-Schmidt, Kasten-Schmidt, Raghuram-Shahidi, Raghuram
   - **(p-adic L-functions)** Mazur-Kazhdan-Schmidt, Januszewski

p : prime number. Fix $\mathbb{C} \cong \mathbb{C}_p$.
$\mathcal{M}$: pure motives over $\mathbb{Q}$ of good ordinary at p.
Suppose that Tate motive is not a direct summand of $\mathcal{M}$.
K : sufficiently large finite extension of $\mathbb{Q}_p$, $\mathcal{O}$: the ring of integer of K.
$\mu_{p^\infty} \subset \mathbb{C}$: the group of p-power roots of unity.

$\phi : \mathbb{Q}^\times \backslash \mathbb{Q}_{\mathbf{A}}^\times \to \mathbb{C}^\times$: algebraic Hecke character satisfying

1. the conductor $c(\phi)$ of ϕ is p-power;
2. $L(0, \mathcal{M}(\phi))$ is a critical value ($\mathcal{M}(\phi)$ is the twist of $\mathcal{M}$ by ϕ).

$\widehat{\phi} : \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})) \to \mathbb{C}_p^\times$: the p-adic avatar of ϕ (e.g. $\varepsilon_{\mathrm{cyc}} = \widehat{|\cdot|_{\mathbf{A}}}$).

Conjecture (Coates-Perrin-Riou "Existence of p-adic L-functions")

There should exist $\mathscr{L}_p(\mathcal{M}) \in \Lambda := \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})]]$ such that for each algebraic Hecke character $\phi : \mathbb{Q}^\times \backslash \mathbb{Q}_{\mathbf{A}}^\times \to \mathbb{C}^\times$ satisfying the proceeding conditions, we have

$$\widehat{\phi}(\mathscr{L}_p(\mathcal{M})) = \mathcal{E}_\infty(\mathcal{M}(\phi))\mathcal{E}_p(\mathcal{M}(\phi))\frac{L(0, \mathcal{M}(\phi))}{\Omega(\mathcal{M})},$$

where

- $\mathcal{E}_*(\mathcal{M}(\phi))$ is the modified Euler factor at $*$.
- $\Omega(\mathcal{M})$ is the period of $\mathcal{M}$, which is a product of Deligne's $c^+(\mathcal{M})$ and a power of π.

Remark

We sometimes use a substitute of Deligne's periods for constructions of p-adic L-functions. (e.g. canonical periods.)

$\pi^{(n)}$: coh. cusp. autom. rep. of $\mathrm{GL}_n(\mathbb{Q}_{\mathbf{A}})$.
$W_{\mathbb{R}} = \mathbb{C}^\times \sqcup (\mathbb{C}^\times j)$: Weil group of $\mathbb{R}$.
Define 1 (resp. 2)-dim. rep ϕ_ν^δ (resp. $\phi_{\nu,l}$) of $W_{\mathbb{R}}$ to be
$\bullet \phi_\nu^\delta(z) = (zz^c)^\nu, \quad \phi_\nu^\delta(j) = (-1)^\delta,$
$\bullet \phi_{\nu,l}(z) = (zz^c)^\nu \begin{pmatrix} (z^c/z)^{\frac{l}{2}} & 0 \\ 0 & (z/z^c)^{\frac{l}{2}} \end{pmatrix}, \quad \phi_{\nu,l}(j) = \begin{pmatrix} 0 & (-1)^l \\ 1 & 0 \end{pmatrix}.$

Then the Langlands parameter of $\pi_\infty^{(n)}$ is given by

$$\begin{cases} \bigoplus_{i=1}^m \phi_{\nu^{(n)}, l_i^{(n)}} & (n = 2m : \text{even}) \\ \phi_{\nu^{(n)}}^\delta \oplus \bigoplus_{i=1}^m \phi_{\nu^{(n)}, l_i^{(n)}} & (n = 2m+1 : \text{odd}) \end{cases} \quad \text{with } l_1^{(n)} > \cdots > l_m^{(n)}$$

Normalize $\nu^{(n)} = (-l_1^{(n)} + n - 1)/2$.
Suppose that $(\pi^{(n+1)}, \pi^{(n)})$ satisfies the following **interlace condition**:
$l_1^{(n+1)} > l_1^{(n)} > l_2^{(n+1)} > l_2^{(n)} > \cdots > l_n^{(n+1)} > l_n^{(n)} > l_{n+1}^{(n+1)}$.
$\mathcal{M}[\pi^{(n)}]$: (conjectural) motive attached to $\pi^{(n)}$ (Clozel).
$L(s, \mathcal{M}[\pi^{(n)}]) = L(s - \frac{n-1}{2}, \pi^{(n)})$.
$\mathcal{M} = \mathcal{M}[\pi^{(n+1)}] \times \mathcal{M}[\pi^{(n)}]$.

$\phi : \mathbb{Q}^\times \backslash \mathbb{Q}_{\mathbf{A}}^\times \to \mathbb{C}^\times$: algebraic Hecke character as above.
Write the infty type of ϕ as $z_\infty^m \quad (m \in \mathbb{Z})$.

Theorem (Januszewski "p-adic L-functions for $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$")

For $m \in \mathbb{Z}$ as above, there exists $\mathscr{L}_p^{(m)}(\mathcal{M}) \in \mathcal{O}[[\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})]]$ and a "period" $\Omega(\mathcal{M}, m)$ such that, for each $\phi \neq |\cdot|_{\mathbf{A}}^m$, we have

$$\widehat{\phi}(\mathscr{L}_p^{(m)}(\mathcal{M})) = \mathcal{E}_p(\mathcal{M}(\phi)) \frac{L^{(\infty)}(0, \mathcal{M}(\phi))}{\Omega(\mathcal{M}, m)},$$

Remark

1. The periods $\Omega(\mathcal{M}, m)$ depend on m. It is an inverse of a product of a certain weighted sum of archimedean local integrals and Raghuram-Shahidi's Whittaker periods.
2. It seems difficult to compare $\mathscr{L}_p^{(m)}(\mathcal{M})$'s for different m's. (Kummer congruence. Januszewski (arXiv:1708.02616))
3. If $\phi = |\cdot|_{\mathbf{A}}^m$, the interpolation formula is not yet known.

$\pi_\infty^{(3)} \cong \mathrm{Ind}_{P_{2,1}(\mathbb{R})}^{\mathrm{GL}_3(\mathbb{R})}(D_{\nu,l_3} \boxtimes \chi_{\nu,\delta}), \quad \pi_\infty^{(2)} \cong D_{\nu,l_2}.$

$(\chi_{\nu,\delta}(u) = \mathrm{sgn}(u)^\delta |u|^\nu, D_{\nu,l}(t1_2) = t^{2\nu}(t \in \mathbb{R}_{>0}), D_{\nu,l}|_{\mathrm{SL}_2(\mathbb{R})} = D_l^+ \oplus D_l^-.)$

(Interlace condition) $0 < l_2 < l_3$.

We normalize $\nu_2 = -\frac{l_2}{2} + \frac{1}{2}, \nu_3 = -\frac{l_3}{2} + 1$.

$\mathcal{M} = \mathcal{M}[\pi^{(3)}] \times \mathcal{M}[\pi^{(2)}]$ has the pure weight $l_2 + l_3$.

$L(m, \mathcal{M})$ is critical if and only if $\begin{cases} \frac{l_3}{2} + 1 \leq m \leq \frac{l_3}{2} + l_2, & (l_2 \leq \frac{l_3}{2}), \\ l_2 + 1 \leq m \leq l_3, & (\frac{l_3}{2} < l_2 < l_3). \end{cases}$

Corollary

Suppose that $n = 2$. The $\exists$ a period $\Omega^\pm(\pi^{(3)} \times \pi^{(2)}) \in \mathbb{C}^\times$ so that for each $\phi \neq |\cdot|_{\mathbf{A}}^m$ as above,

$$\widehat{\phi}(\mathscr{L}_p^{(m)}(\mathcal{M})) = \mathcal{E}_\infty(\mathcal{M}(\phi))\mathcal{E}_p(\mathcal{M}(\phi))\frac{L(0, \mathcal{M}(\phi))}{\Omega^\pm(\pi^{(3)} \times \pi^{(2)})},$$

where $\pm 1 = (-1)^{m+\delta+\frac{l_3}{2}}$

2nd consequence of Main Theorem

The period of Rankin-Selberg L-function is a product of (a refinement of) Raghuram-Shahidi's Whittaker periods:
$\Omega^{\pm}(\pi^{(3)} \times \pi^{(2)}) = \Omega_{\pi^{(3)}} \times \Omega^{\pm}_{\pi^{(2)}}$.
Here $\Omega^{\pm}_{\pi^{(2)}}$ is the canonical periods of $\pi^{(2)}$ from its definition.
A priori $\Omega_{\pi^{(3)}}$ has no relation with the motives.
2nd corollary gives a motivic back ground on $\Omega_{\pi^{(3)}}$:

Corollary

Suppose that Deligne's conjecture for $\pi^{(3)}$ (existence of motives, algebraicity of critical values).
Then we have $\Omega_{\pi^{(3)}} = (2\pi\sqrt{-1})^{\frac{l_3}{2}} c^{+}(\mathcal{M})c^{-}(\mathcal{M})$.

The proofs are done by a PRECISE formula for L-values.

Field of rationality

(Field of rationality)

$\mathbb{Q}(\pi^{(n)})$: field of rationality of $\pi^{(n)}$, i. e., the fixed subfield of $\mathbb{C}$ under $\{\sigma \in \mathrm{Aut}(\mathbb{C}) \mid {}^{\sigma}\pi^{(n)} \cong \pi^{(n)}\}$. ($\mathbb{Q}(\pi^{(n)})/\mathbb{Q}$: finite).

$E := \mathbb{Q}(\pi^{(n+1)}, \pi^{(n)})$.

(Rationality on $(\mathfrak{g}, K)$ cohomology) ($\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}), K_n = \mathbb{R}^{\times}\mathrm{SO}_n(\mathbb{R})$)

$Y^{(n)}_{\mathcal{K}_n} = \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{Q}_{\mathbf{A}})/\mathbb{R}^{\times}\mathrm{SO}_n(\mathbb{R})\mathcal{K}_n$

$V^{\vee}_{\xi}$: contragredient of rep. of $\mathrm{GL}_n(\mathbb{R})$ of highest wt ξ.

$\mathcal{L}^{\vee}_{\xi}$: loc. sys. on $Y^{(n)}_{\mathcal{K}_n}$ assoc. with $V^{\vee}_{\xi}$.

$\pi^{(n)}$: coh. $\iff \exists \xi$ s.t. $H^*(\mathfrak{g}, K; \pi^{(n)} \otimes V^{\vee}_{\xi}) \neq 0$.

Note that • $H^*(\mathfrak{g}, K; \pi^{(n)} \otimes V^{\vee}_{\xi}) \subset H^*(Y^{(n)}_{\mathcal{K}_n}, \mathcal{L}^{\vee}_{\xi})$.

• $b_n = [\frac{n^2}{4}] \leq * \leq t_n = b_n + [\frac{n-1}{2}]$. $(b_3 = 2, t_3 = 3)$.

$V^{\vee}_{\xi}$: defined over $\mathbb{Q}(\pi^{(n)}) \rightsquigarrow H^*(Y^{(n)}_{\mathcal{K}_n}, \mathcal{L}^{\vee}_{\xi}) \cong H^*(Y^{(n)}_{\mathcal{K}_n}, \mathcal{L}^{\vee}_{\xi})_{\mathbb{Q}(\pi^{(n)})} \otimes \mathbb{C}$.

Define $H^{b_n}(\mathfrak{g}, K; \pi^{(n)} \otimes V^{\vee}_{\xi})_{\mathbb{Q}(\pi^{(n)})}$ to be

$$H^{b_n}(\mathfrak{g}, K; \pi^{(n)} \otimes V^{\vee}_{\xi}) \cap H^*(Y^{(n)}_{\mathcal{K}_n}, \mathcal{L}^{\vee}_{\xi})_{\mathbb{Q}(\pi^{(n)})}.$$

(Whittaker functions)

$\psi : \mathbb{Q}\backslash\mathbb{Q}_{\mathbf{A}} \longrightarrow \mathbb{C}^\times$: add. char. $\psi_\infty(z) = \exp(2\pi\sqrt{-1}z)$.

$\mathcal{W}(\pi^{(n)}, \psi) = \mathcal{W}(\pi^{(n)}_\infty, \psi_\infty) \otimes \mathcal{W}(\pi^{(n)}_{\mathrm{fin}}, \psi_{\mathrm{fin}})$.

$\sigma \in \mathrm{Aut}(\mathbb{C}) \longleftrightarrow u_\sigma \in \widehat{\mathbb{Z}}^\times$ s.t. $\sigma(\psi(x)) = \psi(u_\sigma x)$ for $\forall x \in \mathbb{Q}^\times_{\mathbf{A},\mathrm{fin}}$.

Define $T_\sigma : \mathcal{W}(\pi^{(n)}_{\mathrm{fin}}, \psi_{\mathrm{fin}}) \to \mathcal{W}({}^\sigma\pi^{(n)}_{\mathrm{fin}}, \psi_{\mathrm{fin}})$ by

$$T_\sigma W(g) = \sigma\left(W(\mathrm{diag}(u_\sigma^{-(n-1)}, \ldots, u_\sigma^{-1}, 1)g)\right).$$

(Raghuram-Shahidi's Whittaker periods)

Fix $\mathbf{w}^{(n)}_\infty = \mathbf{w}(\pi^{(n)}_\infty, \pm) \in H^{b_n}(\mathfrak{g}, K; H_{\pi^{(n)},K} \otimes V_\xi^\vee)[\pm]$ (1-dim.).

$$\rightsquigarrow \mathcal{W}(\pi^{(n)}_{\mathrm{fin}}, \psi_{\mathrm{fin}}) \to H^{b_n}(\mathfrak{g}, K; H_{\pi^{(n)},K} \otimes V_\xi^\vee)[\pm] \otimes \pi^{(n)}_{\mathrm{fin}} \to H^{b_n}(Y^{(n)}_{K_n}\mathcal{L}_\xi^\vee).$$

Define **Whittaker periods** $p^{b_n}(\pi^{(n)}_{\mathrm{fin}}, \mathbf{w}^{(n)}_\infty, \pm) \in \mathbb{C}^\times$ to be

$$\mathrm{Image}\left(\mathcal{W}(\pi^{(n)}_{\mathrm{fin}}, \psi_{\mathrm{fin}})^{\mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\pi^{(n)}))}\right) = p^{b_n}(\pi^{(n)}_{\mathrm{fin}}, \mathbf{w}^{(n)}_\infty, \pm)\left(H^{b_n}(\mathfrak{g}, K; \pi^{(n)} \otimes V_\xi^\vee)_{\mathbb{Q}(\pi^{(n)})}[\pm]\right).$$

Raghuram-Shahidi's Whittaker periods depend on the choice of $\mathbf{w}^{(n)}_\infty$.

Cup product pairing

$$\mathcal{Y}^{(n)}_{\mathcal{K}_n} = \mathrm{GL}_n(\mathbb{Q})\backslash \mathrm{GL}_n(\mathbb{Q}_{\mathbf{A}})/\mathrm{SO}_n(\mathbb{R})\mathcal{K}_n \xrightarrow{\mathrm{p}_n} Y^{(n)}_{\mathcal{K}_n}$$

$$\iota : \mathrm{GL}_n \longrightarrow \mathrm{GL}_{n+1}; g \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

(Branching rule)

$$H^{b_{n+1}}(\mathcal{Y}^{(n)}_{\mathcal{K}_n}, \iota^*\mathcal{L}^\vee_{\xi^{(n+1)}}) \ni \iota^*\mathrm{p}^*_{n+1}\eta^{(n+1)}_{\pi^{(n+1)}} \overset{\nabla^m}{\longmapsto} \nabla^m \iota^*\mathrm{p}^*_{n+1}\eta^{(n+1)}_{\pi^{(n+1)}} \in H^{b_{n+1}}(\mathcal{Y}^{(n)}_{\mathcal{K}_n}, \iota^*\mathcal{L}_{\xi^{(n)}}(\det{}^m))$$

(Cup product)

$$\cup : H^{b_{n+1}}(\mathcal{Y}^{(n)}_{\mathcal{K}_n}, \iota^*\mathcal{L}_{\xi^{(n)}}(\det{}^m)) \times H^{b_n}(\mathcal{Y}^{(n)}_{\mathcal{K}_n}, \iota^*\mathcal{L}^\vee_{\xi^{(n)}}) \longrightarrow H^{b_{n+1}+b_n}(\mathcal{Y}^{(n)}_{\mathcal{K}_n}, \det{}^m)$$

(Numerical coincidence) $b_{n+1} + b_n = \dim \mathcal{Y}^{(n)}_{\mathcal{K}_n}$

$\Longrightarrow I(m, \pi^{(n+1)}, \pi^{(n)}) := \nabla^m \iota^*\mathrm{p}^*_{n+1}\eta^{(n+1)}_{\pi^{(n+1)}} \cup \mathrm{p}^*_n\eta^{(n)}_{\pi^{(n)}} \in E$ is "essentially" given by the zeta-integral due to Jacquet-Piatetski-Shapiro-Shalika.

("essentially" = arch. local integral includes the information of local systems.)

Theorem (Raghuram)

$$I(m, \pi^{(n+1)}, \pi^{(n)}) = \underbrace{I_\infty(m, \pi_\infty^{(n+1)}, \pi_\infty^{(n)})}_{\text{sum of arch. loc. int.}} \times \frac{L^{\mathrm{fin}}(m, \mathcal{M}(\pi^{(n+1)} \times \pi^{(n)}))}{p^{b_{n+1}}(\pi_{\mathrm{fin}}^{(n+1)}, \mathbf{w}_\infty^{(n+1)}, \pm) p^{b_n}(\pi_{\mathrm{fin}}^{(n)}, \mathbf{w}_\infty^{(n)}, \pm)} \in E$$

Remark

1. The study of the algebraicity is reduced to the study of I_∞. Sun (JAMS 2017) proved the non-vanishing of I_∞.
2. Combining with Deligne's conjecture,
$$\frac{I_\infty(m, \pi_\infty^{(n+1)}, \pi_\infty^{(n)})}{p^{b_{n+1}}(\pi_{\mathrm{fin}}^{(n+1)}, \mathbf{w}_\infty^{(n+1)}, \pm) p^{b_n}(\pi_{\mathrm{fin}}^{(n)}, \mathbf{w}_\infty^{(n)}, \pm)} \sim_{\overline{\mathbb{Q}}^\times} \frac{\Gamma(m, \mathcal{M}(\pi^{(n+1)} \times \pi^{(n)}))}{c^+(\mathcal{M}(\pi^{(n+1)} \times \pi^{(n)}))}$$
However, $\nexists$ motivic explanation of each I_∞ and $p^{b_n}(\pi_{\mathrm{fin}}^{(n)}, \mathbf{w}_\infty^{(n)}, \pm)$.
3. It's difficult to study the relation between $I_\infty(m, \pi_\infty^{(n+1)}, \pi_\infty^{(n)})$'s for different m's. This is one of difficulties for the Kummer congruences for p-adic Rankin-Selberg L-functions.

Statement

Write $\Omega^{\pm}_{\pi^{(n)}} = p^{b_n}(\pi^{(n)}_{\mathrm{fin}}, \mathbf{w}^{(n)}_{\infty}, \pm)$ for a "suitable" $\mathbf{w}^{(n)}_{\infty}$.

Main Theorem (Hara-N. (arXiv:2012.13213))

Suppose that $(-1)^{m+\delta+\frac{l_3}{2}} = \pm 1$. Then

$$I(m, \pi^{(3)}, \pi^{(2)}) = (-1)^{\delta}\sqrt{-1}^{\frac{l_3}{2}-m+1}\binom{\frac{l_3}{2}-1}{m-\frac{l_3}{2}-1}\binom{\frac{l_3}{2}-1}{\frac{l_3}{2}+l_2-m}\frac{L(m,\mathcal{M})}{\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}}.$$

If $(-1)^{m+\delta+\frac{l_3}{2}} \neq \pm 1$ holds, we have $I(m, \pi^{(3)}, \pi^{(2)}) = 0$.

Remark

S. Y. Chen (arXiv:2012.00625) independently proved

$$I(m, \pi^{(3)}, \pi^{(2)}) \sim_{\mathbb{Q}^{\times}} \sqrt{-1}^{\frac{l_3}{2}-m+1}\frac{L(m,\mathcal{M})}{\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}}.$$

Cohomological cusp. autom. forms on GL_3

$\phi^{\delta}_{\nu_3} \oplus \phi_{\nu_3, l_3}$: Langlands parameter of $\pi^{(3)}_{\infty}$.
$\Lambda_3 = \{\boldsymbol{\lambda} = (\lambda, \delta) \mid \lambda \in \mathbb{Z},\ \lambda \geq 0,\ \delta \in \mathbb{Z}/2\mathbb{Z}\}$
Define an action $\tau^{(3)}_{\boldsymbol{\lambda}}$ of $u \in \mathrm{O}_3(\mathbb{R})$ on $P \in \mathbb{C}[z_1, z_2, z_3]_{\lambda}$ by

$$\tau^{(3)}_{\boldsymbol{\lambda}}(u) P(z_1, z_2, z_3) = (\det u)^{\delta} P((z_1, z_2, z_3)u).$$

$$\mathcal{V}_{\boldsymbol{\lambda}} := \begin{cases} (z_1^2 + z_2^2 + z_3^2)\mathbb{C}[z_1, z_2, z_3]_{\lambda-2}, & (\lambda \geq 2), \\ 0, & \text{(otherwise)}. \end{cases}$$

$V^{(3)}_{\boldsymbol{\lambda}} := \mathbb{C}[z_1, z_2, z_3]_{\lambda} / \mathcal{V}_{\boldsymbol{\lambda}}$
$\left\{ v^{(3), \boldsymbol{\lambda}}_{\pm\mu} := (\pm z_1 + \sqrt{-1} z_2)^{\mu} z_3^{\lambda-\mu} \right\}_{0 \leq \mu \leq \lambda}$: basis of $V^{(3)}_{\boldsymbol{\lambda}}$.
The minimal $\mathrm{O}_3(\mathbb{R})$-type is $\tau_{(l_3+1, \delta)}$. ($\boldsymbol{\lambda} = l_3 + 1$.)
$V^{(3)}_{\boldsymbol{\lambda}} \hookrightarrow \pi; v^{(3), \boldsymbol{\lambda}}_{\pm\mu} \longmapsto f^{\boldsymbol{\lambda}}_{\pm\mu} \in \pi (0 \leq \mu \leq l_3 + 1)$.
$\boldsymbol{f} = \begin{pmatrix} f^{\boldsymbol{\lambda}}_{l_3+1} & f^{\boldsymbol{\lambda}}_{l_3} & \ldots & f^{\boldsymbol{\lambda}}_{-l_3-1} \end{pmatrix}$: cusp form on $\mathrm{GL}_3(\mathbb{Q}_{\mathbf{A}})$

(Normalization of f)

$W_\mu^{\boldsymbol{\lambda}}$: Whittaker function attached to $f_\mu^{\boldsymbol{\lambda}}$ $(-l_3-1 \le \mu \le l_3+1)$

$\varphi_+^{(3)} : V_{\boldsymbol{\lambda}}^{(3)} \to \mathcal{W}(\pi_\infty^{(3)}, \psi_\infty)$: $\mathbb{R}^\times \mathrm{SO}_3(\mathbb{R})$-homomorphism satisfying the following explicit formula on the radial parts of Whittaker functions:

$(z_{\boldsymbol{a}} := z_1^{a_1} z_2^{a_2} z_3^{a_3} \in V_{\boldsymbol{\lambda}}^{(3)})$

$$\begin{aligned}
&\varphi_+^{(3)}(z_{\boldsymbol{a}})(\mathrm{diag}(y_1y_2y_3, y_2y_3, y_3)) \\
&= (-1)^{a_1}\sqrt{-1}^{a_2} \frac{y_1y_2(y_2y_3)^{3\nu_3}}{(4\pi\sqrt{-1})^2} \times \int_{t_2}\int_{t_1} \frac{\Gamma_{\mathbb{C}}(t_1+\nu_3+\frac{\lambda_3-1}{2})\Gamma_{\mathbb{R}}(t_2+\nu_3+a_1)}{\Gamma_{\mathbb{R}}(t_1+t_2+a_1+a_3)} \\
&\qquad \times \Gamma_{\mathbb{C}}\left(t_2-\nu_3+\frac{\lambda_3-1}{2}\right)\Gamma_{\mathbb{R}}(t_2-\nu_3+a_3) y_1^{-t_1} y_2^{-t_2}\, \mathrm{d}t_1 \mathrm{d}t_2.
\end{aligned}$$

(Miyazaki (Manus. Math., 2009))

We normalize $f_\mu^{\boldsymbol{\lambda}}$ so that the radial part of $W_\mu^{\boldsymbol{\lambda}}$ is described as above.

Description of $(\mathfrak{g}, K)$ cohomology

$H_{\pi^{(3)},K_3}$: $(\mathfrak{gl}_3(\mathbb{R}), K_3)$-module attached to $\pi_\infty^{(3)}$. ($K_3 = \mathbb{R}^\times \mathrm{SO}_3(\mathbb{R})$.)

$H^i(\mathfrak{gl}_3(\mathbb{R}), K_3; H_{\pi^{(3)},K_3} \otimes V_\xi^\vee) \cong (H_{\pi^{(3)},K_3} \otimes L^{(3)}(\boldsymbol{w_\lambda}; \mathbb{C}) \otimes \bigwedge^i \mathcal{P}_{3,\mathbb{C}}^*)^{\mathrm{SO}_3(\mathbb{R})}$

$\mathcal{A}$: integral domain of characteristic 0.
$\mathcal{A}[X, Y, Z; A, B, C]_w$ of homogenous polynomials of degree w in each variables X, Y, Z and A, B, C.
Define an action $\varrho_{\boldsymbol{w}}^{(3)}$ of $\mathrm{GL}_3(\mathcal{A})$ on $\mathcal{A}[X, Y, Z; A, B, C]_w$ as follows:

$$\varrho_{\boldsymbol{w}}^{(3)}(g)P(X, Y, Z; A, B, C) = (\det g)^w P((X, Y, Z)g; (A, B, C)^{\mathrm{t}}g^{-1}).$$

$$\iota_w := \frac{\partial^2}{\partial X \partial A} + \frac{\partial^2}{\partial Y \partial B} + \frac{\partial^2}{\partial Z \partial C} : \mathcal{A}[X, Y, Z; A, B, C]_w \to \mathcal{A}[X, Y, Z; A, B, C]_{w-1}.$$

$L^{(3)}(\boldsymbol{w}; \mathcal{A}) := \mathrm{Ker}\iota_w$.
$L^{(3)}(\boldsymbol{w}; \mathbb{C})$ has the highest weight $(2w, w, 0)$.
$V_\xi^\vee = L^{(3)}(\boldsymbol{w_\lambda}; \mathbb{C})$ for $w_{\boldsymbol{\lambda}} = \frac{l_3}{2} - 1$.

Eichler-Shimura map for GL_3

Construct elements in $(H_{\pi^{(3)},K_3} \otimes L^{(3)}(\boldsymbol{w_\lambda};\mathbb{C}) \otimes \bigwedge^i \mathcal{P}^*_{3,\mathbb{C}})^{\mathrm{SO}_3(\mathbb{R})}, \quad (i=2,3)$

(element in $L^{(3)}(\boldsymbol{w_\lambda};\mathbb{C})$)

$$(Xz_1+Yz_2+Zz_3)^{w_{\boldsymbol{\lambda}}} \otimes (Az_1+Bz_2+Cz_3)^{w_{\boldsymbol{\lambda}}} \begin{pmatrix} v_3^{(3,\delta)} & \dots & v_{-3}^{(3,\delta)} \end{pmatrix}$$
$$= \begin{pmatrix} v^{\boldsymbol{\lambda}}_{\lambda_3} & v^{\boldsymbol{\lambda}}_{\lambda_3-1} & \dots & v^{\boldsymbol{\lambda}}_{-\lambda_3} \end{pmatrix} \mathcal{P}(X,Y,Z,A,B,C)$$

Then $\mathcal{P}(X,Y,Z,A,B,C) \in \mathrm{M}_{2\lambda_3+1,7}(\mathbb{C}[X,Y,Z;A,B,C]_{w_{\boldsymbol{\lambda}},w_{\boldsymbol{\lambda}}})$.

(element in $\bigwedge^i \mathcal{P}^*_{3,\mathbb{C}}$)

$\mathcal{P}_{3,\mathbb{C}}$: the cpxif. of the complement of the fixed part of the Cartan involution of $\mathfrak{gl}_3(\mathbb{R})$.

$V^{(3)}_{(3,0)} \longrightarrow \bigwedge^i \mathcal{P}^*_{3,\mathbb{C}};\, v^{(3),((3,0))}_{\pm\mu} \longmapsto \boldsymbol{\omega}^i_{\pm\mu}$ $(0 \le \mu \le 3)$.

Fix a coordinate $g = \begin{pmatrix} y_1(g)y_2(g) & y_1(g)x_2(g) & x_3(g) \\ 0 & y_1(g) & x_1(g) \\ 0 & 0 & 1 \end{pmatrix} \in Y^{(3)}_{\mathcal{K}_3}$.

Eichler-Shimura map for GL_3

The section $\boldsymbol{\omega}^i_{\pm\mu}(g)$ at g is given by the following formula:

$\varsigma_1 := \mathrm{d}y_1$, $\varsigma_2 := \mathrm{d}y_2$, $\varsigma_3 := \mathrm{d}x_1, \varsigma_4 := \mathrm{d}x_2, \varsigma_5 := \mathrm{d}x_3, \varsigma_{j,j'} := \varsigma_j \wedge \varsigma_{j'}$.

$$\boldsymbol{\varsigma}^2 = \begin{pmatrix} \varsigma_{2,1} & \varsigma_{2,0} & \varsigma_{2,-1} & \varsigma_{2,-2} & \varsigma_{1,0} & \varsigma_{1,-1} & \varsigma_{1,-2} & \varsigma_{0,-1} & \varsigma_{0,-2} & \varsigma_{-1,-2} \end{pmatrix}$$

$\begin{pmatrix} \boldsymbol{\omega}^2_3(g) & \dots & \boldsymbol{\omega}^2_{-3}(g) \end{pmatrix} = \boldsymbol{\varsigma}^2 Q^2$, where

$$Q^2 = \begin{pmatrix}
0 & \frac{1}{2y_1y_2} & 0 & 0 & 0 & -\frac{1}{2y_1y_2} & 0 \\
0 & 0 & \frac{x_2+\sqrt{-1}y_2}{2y_1^2y_2} & 0 & \frac{x_2-\sqrt{-1}y_2}{2y_1^2y_2} & 0 & 0 \\
0 & -\frac{\sqrt{-1}}{2y_1y_2} & 0 & 0 & 0 & -\frac{\sqrt{-1}}{2y_1y_2} & 0 \\
0 & 0 & -\frac{1}{2y_1^2y_2} & 0 & -\frac{1}{2y_1^2y_2} & 0 & 0 \\
\frac{x_2+\sqrt{-1}y_2}{8y_1y_2^2} & 0 & -\frac{x_2-5\sqrt{-1}y_2}{8y_1y_2^2} & 0 & -\frac{x_2+5\sqrt{-1}y_2}{8y_1y_2^2} & 0 & \frac{x_2-\sqrt{-1}y_2}{8y_1y_2^2} \\
0 & -\frac{\sqrt{-1}}{4y_2^2} & 0 & \frac{\sqrt{-1}}{2y_2^2} & 0 & -\frac{\sqrt{-1}}{4y_2^2} & 0 \\
-\frac{1}{8y_1y_2^2} & 0 & \frac{1}{8y_1y_2^2} & 0 & \frac{1}{8y_1y_2^2} & 0 & -\frac{1}{8y_1y_2^2} \\
\frac{\sqrt{-1}x_2-y_2}{8y_1y_2^2} & 0 & -\frac{3(\sqrt{-1}x_2+y_2)}{8y_1y_2^2} & 0 & \frac{3(\sqrt{-1}x_2-y_2)}{8y_1y_2^2} & 0 & -\frac{\sqrt{-1}x_2+y_2}{8y_1y_2^2} \\
0 & 0 & 0 & \frac{\sqrt{-1}}{y_1^2y_2} & 0 & 0 & 0 \\
\frac{\sqrt{-1}}{8y_1y_2^2} & 0 & -\frac{3\sqrt{-1}}{8y_1y_2^2} & 0 & \frac{3\sqrt{-1}}{8y_1y_2^2} & 0 & -\frac{\sqrt{-1}}{8y_1y_2^2}
\end{pmatrix}.$$

We have a similar formula for 3-forms $\boldsymbol{\omega}^3_{\pm\mu}$.

Eichler-Shimura map for GL_3

Then the image of $\boldsymbol{f}$ via the Eichler-Shimura map for GL_3 is described as

$$\delta^{(3),i}(\boldsymbol{f}) := \begin{pmatrix} f^{\boldsymbol{\lambda}}_{\lambda_3} & f^{\boldsymbol{\lambda}}_{\lambda_3-1} & \dots & f^{\boldsymbol{\lambda}}_{-\lambda_3} \end{pmatrix} \mathcal{P}(X,Y,Z,A,B,C) \begin{pmatrix} \boldsymbol{\omega}^i_3 \\ \vdots \\ \boldsymbol{\omega}^i_{-3} \end{pmatrix},$$

which gives a class of $H^i_{\mathrm{cusp}}(Y^{(3)}_{\mathcal{K}_3}, \mathcal{L}^{(3)}(\boldsymbol{w}_{\boldsymbol{\lambda}};\mathbb{C}))$.
This is an analogue of Eichler-Shimura map for GL_2 :

$$\delta(f) = f(z)(X - zY)^{k-2}\mathrm{d}z.$$

Branching rule

$\mathcal{A}$: integral domain of characteristic 0 s.t. $w \in \mathcal{A}^{\times}$.
For each $0 \leq k, l \leq w = \frac{l_3}{2} - 1$, $0 \leq l \leq w_1^-$ and $P \in L^{(3)}(\boldsymbol{w}; \mathcal{A})$, set

$$(\nabla_{k,l}P)(X,Y) = \frac{1}{k!\,l!}\frac{\partial^{k+l}P}{\partial Z^k \partial C^l}(X,Y,0\,;-Y,X,0).$$

Then $\nabla_{k,l}$ defines the following $\mathrm{GL}_2(\mathcal{A})$-equivariant map:

$$\boldsymbol{\nabla} = (\nabla_{k,l})_{0\leq k,l\leq w} : L^{(3)}(\boldsymbol{w};\mathcal{A})|_{\mathrm{GL}_2(\mathcal{A})} \longrightarrow \bigoplus_{k,l=0}^{w} L^{(2)}(2w-k-l,l;\mathcal{A}).$$

Let $(k,l) = (l_3 - m, m - l_2 - 1), \boldsymbol{n}_m = (l_2 - 1, m - l_2 - 1)$.

$$\nabla^{\boldsymbol{n}_m} := \nabla_{k,l} : L^{(3)}(\boldsymbol{w}\,;\mathcal{A})|_{\mathrm{GL}_2(\mathcal{A})} \longrightarrow L^{(2)}(\boldsymbol{n}_m;\mathcal{A}).$$

Combining these descriptions with the explicit formula of the archimedean zeta integral for $\mathrm{GL}_3 \times \mathrm{GL}_2$ due to Hirano-Ishii-Miyazaki (to appear in Mem. of AMS), we obtain the formula for $I(m, \pi^{(3)} \times \pi^{(2)})$.

Yoshida's periods invariants for GL_3

(The refinement of Januszewski's interpolation formula)
Immediate from the main theorem.

(Period relation)
Main theorem and Deligne's conjecture imply $\dfrac{\Gamma(m,\mathcal{M})}{\Omega_{\pi^{(3)}}\Omega^{\pm}_{\pi^{(2)}}} \sim_{\overline{\mathbb{Q}}^\times} \dfrac{1}{c^+(\mathcal{M}(m))}$.
By the interlace condition $(0 < l_2 < l_3)$ and Yoshida's description of periods of tensor product of motives, we observe

$$c^{\pm}(\mathcal{M}) \sim_{\overline{\mathbb{Q}}^\times} (2\pi\sqrt{-1})^{\frac{l_3}{2}} c^+(\mathcal{M}[\pi^{(3)}])c^-(\mathcal{M}[\pi^{(3)}])c^{\pm}(\mathcal{M}[\pi^{(2)}])$$

The modular symbol method for GL_2 and Deligne's conjecture imply

$$\Omega^{\pm}_{\pi^{(2)}} \sim_{\overline{\mathbb{Q}}^\times} c^{\pm}(\mathcal{M}[\pi^{(2)}]).$$

Combining these, we obtain $\Omega_{\pi^{(3)}} \sim_{\overline{\mathbb{Q}}^\times} (2\pi\sqrt{-1})^{\frac{l_3}{2}} c^+(\mathcal{M}[\pi^{(3)}])c^-(\mathcal{M}[\pi^{(3)}])$.

Yoshida's periods invariants for GL_n

Expectation (Ishii-Miyazaki (arXiv:2006.04095, base field is tot. imag.))

$$I_\infty(m, \pi^{(n+1)}, \pi^{(n)}) \sim_{\overline{\mathbb{Q}}^\times} L_\infty(m, \mathcal{M}).$$

Yoshida (AJM, 2002) introduces certain period invariants $c_p(\mathcal{M})$ $(p = 0, \ldots, t)$ of $\mathcal{M}$. $(t := \sharp$ of jumps to of the Hodge filt. $c_0(\mathcal{M}) = \delta(\mathcal{M})$.) The interlace condition and Deligne's conjecture yields that

Expectation

$$\Omega^{\pm}_{\pi^{(n)}} \sim_{\mathbb{Q}^\times} (2\pi\sqrt{-1})^{\sum_{i=1}^{n} \frac{l_1^{(n)} - l_i^{(n)}}{2}(n-i)} \prod_{p=1}^{\lfloor \frac{n}{2} \rfloor} c_p(\mathcal{M}[\pi^{(n)}]) \times \begin{cases} 1, & (n : \text{odd}), \\ c^{\pm}(\mathcal{M}[\pi^{(n)}]), & (n : \text{even}). \end{cases}$$

This explains the motivic back ground of Raghuram-Shahidi's Whittaker periods for GL_n.

Expectations in progress

We expect that the main theorem will be applied to the following problems.

Remark

1. The Kummer congruences for $\mathrm{GL}_3 \times \mathrm{GL}_2$ should be given.
2. The further study of $\Omega_{\pi^{(3)}}$ might be interesting, since it is a product of Deligne's periods. For instances, refine the following works:
   - **(alg. of critical values for GL_3)** Raghuram-Sachdeva (Contrib. Math. Comp. Sci., 2017).
   - **(p-adic L-functions for GL_3)** Mahnkopf (Compos. 2005).
3. The motivic interpretation of the periods of $\pi^{(n)}$ for the top degree cohomology should be given. (Prasanna-Venkatesh (arXiv:1609.06370), Balasubramanyam-Raghuram (AJM, 2017), S. Y. Chen(arXiv:2012.00625)).

Thank you for your attention.