Multivariate Galois Representations
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What do we study?

Let K be a p-adic field, i.e. a complete, discretely valued field of characteristic 0 whose residue field k is perfect of characteristic $p > 0$. Think about $\mathbb{Q}_p$. By $G_{\mathbb{Q}_p}$ we consider its absolute Galois group. Consider then its product

$$G_{\mathbb{Q}_{p,\Delta}} = G_{\mathbb{Q}_p} \times \cdots \times G_{\mathbb{Q}_p}$$

Δ a finite set. And then we indicate as a *multivariate Galois representation* as p-adic representation of $G_{\mathbb{Q}_{p,\Delta}} = G_\Delta$.

The case $\#\Delta = 1$ has been extensively studied by the work of Fontaine, Colmez, Berger...Kisin, Breuil in the framework of p-adic Hodge theory. And it had applications on some known cases of **Langlands correspondance** ($n = 2$) (via the theory of (φ, Γ)-modules (Montreal Functor...).

Moreover the study has opened the path to the use of the "tilting operation" i.e. the move from particular fields in $ch. = 0$ (deeply ramified) to objects in $ch = p$ which maintain all the relevant properties. Such a theory has been framed via Scholze's **perfectoid spaces**.

Why to study Multivariate Galois Representations?

The Perfectoid Setting. Diamonds

In the study of (classical) p-adic representation: one decomposes the Galois $G_{\mathbb{Q}_p} = G$ using a totally (deeply) ramified extension of $\mathbb{Q}_p$:

$$\mathbb{Q}_p(\mu_\infty) = \bigcup_{n \in \mathbb{N}} \mathbb{Q}_p(\mu^{(n)})$$

$\mu^{(n)}$ a p^n-root of unity.

This field admits an absolute Galois group H which corresponds to the Galois Group of a field in $ch = p$ ($\mathbb{F}_p((t))$, t is an "indeterminate"). And moreover $Gal(\mathbb{Q}_p(\mu_\infty)/\mathbb{Q}_p) = \mathbb{Z}_p^\times$.
t can be seen as a " geometric object" and then these arithmetic representations can be interpreted as coefficients (local systems) ...

Why to study Multivariate Galois Representations?

By Fontaine/Katz : if k is a field in $ch = p$: then the p-adic representations of the absolute Galois group of k are nothing but modules over the Cohen ring of k endowed with a Frobenius structure (φ). In our specific case $k = \mathbb{F}_p((t))$ we have as a Cohen ring the Amice's ring

$$\mathcal{E} = \{\sum_{i\in\mathbb{Z}} a_i t^i \ a_i \in \mathbb{Z}_p,\ a_i \to 0, i \to -\infty\}$$

We conclude that a p-adic Galois representation is equivalent to a φ module over the Amice-Fontaine ring $\mathcal{E}$ endowed with a *residual* $\Gamma = \mathbb{Z}_p^\times$-action a (φ, Γ)-module. ...The tangent action of Γ give a differential structure...so we may speak of coefficients hence of p-adic differential equations endowed with Frobenius structure.

Diamonds

But what should be a geometric space where to have a fundamental group as $G_{\mathbb{Q}_p}$? It should encompass both the *ch.p* and the residual action...This has been introduced by Scholze via the theory of Diamond. Let me give an example of what should be using the example of Weinstein (2016) on its interpretation as an object over a algebraically closed field. Consider $\mathbb{C}_p = C$: completion of the algebraic closure of $\mathbb{Q}_p$. Let D the open unit disk centered at 1 and $\mathbb{Z}_p$ structure $s \in \mathbb{Z}_p$ $a \mapsto a^s$. Consider then

$$\widetilde{D} = \varprojlim_{x \mapsto x^p} D$$

It is an Huber adic space, better: a perfectoid space. Consider then $\widetilde{D}^* = \tilde{D} \setminus \{1\}$, $\mathbb{Q}_p^*$ acts on it. We do have

Theorem (Weinstein, 2016)

The category of $\mathbb{Q}_p^*$ equivariant finite étale covering of $\widetilde{D}^*$ is equivalent to the category of finite étale $\mathbb{Q}_p$-algebras

..$\pi_1^{et}(\widetilde{D}^*/\mathbb{Q}_p^*) \simeq G = G_{\mathbb{Q}_p}$...if we could write it...

So: the geometric object we are talking about is $\widetilde{D}^*/\mathbb{Q}^*$....it is not a perfectoid neither -adic space.... But we do have a sheaf interpretation starting from the perfectoid space $\widetilde{D}^*$. It is define over a $ch = 0$ field let's go in its "tilting world". I.e. Consider the category *Perf* of perfectoid space in $ch.p$ endowed with a topology : the *pro-étale topology*.
By Yoneda embedding each object in *Perf* gives a sheaf. On $Sh(Perf)$ we do have the notion of étale finite morphism (if representable: the usual...).

On $Sh(Perf)$ we have quotient and product, consider $C^\flat$ as perfectoid field "tilt" of C. On it, take again the puntured open disk over $\mathbb{C}^\flat$ and doing the same as before getting the perfectoid space $\widetilde{D}^*_{C^\flat}$ (in the indeterminate t)...its untilted is $\widetilde{D}^*$. Now we can see it as an element of $Sh(Perf)$...we can take quotient ..then:

Definition

A Diamond is a sheaf on $Perf$ $\mathcal{F}$ which is isomorphic to the Coequalizer $Coeq(h_Z \rightrightarrows h_X)$ for some pro-étale equivalence relation $s, t : Z \rightrightarrows X$ for $X, Z \in Perf$.

.... we can have quotients, torsor, products..... The category of finite étale morphism over a connected diamond $\mathcal{F}$ on $Perf$ is a Galois category and we have a fundamental group $\pi_1^{et}(\mathcal{F})$. we can then define our aimed diamond $Spd\mathbb{Q}_p/F^{\mathbb{Z}}$...with its fundamental group ....and it has different presentations...
As a matter of fact $\widetilde{D}^*/\mathbb{Q}_p^*$ can be seen as linked to the extension to $SpdC$ of the **diamond** $Spd\mathbb{Q}_p/F^{\mathbb{Z}}$ and they share the same fundamental group (Scholze-Weinstein).

Construction of $Spd\mathbb{Q}_p/F^{\mathbb{Z}}$.
We may also consider the cyclotomic (perfectoid) field $\widehat{\mathbb{Q}_p(\mu_{p^\infty})})$ and its tilt

$$\mathbb{F}_p((t^{1/p^\infty}))$$

Because we have an action of $\mathbb{Z}_p^\times$ on $\widehat{\mathbb{Q}_p(\mu_{p^\infty})}$, we have also an interpretation in terms of sheaves of the quotient of the perfectoid space $Spa(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})^\flat$ via $\mathbb{Z}_p^\times$: this is exactly $Spd\mathbb{Q}_p$, viewed as a sheaf:

$$Spd\mathbb{Q}_p = (Spa(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})^\flat)^\diamondsuit/\mathbb{Z}_p^\times = (Spa(\mathbb{F}_p((t^{1/p^\infty}))))^\diamondsuit/\mathbb{Z}_p^\times$$

$(\alpha(t) = (1+t)^\alpha - 1)$ Because we are in $ch = p$ we have also a Frobenius hence the action of $F^{\mathbb{Z}}$.
The connection between the various definition has been done using the "courbe" of Fargues-Fontaine

In this framework even if $\mathbb{Q}_p$ has $ch = 0$ its $Spd\mathbb{Q}_p$ has $ch = p$, hence Frobenius. We consider the diamond

$$Spd\mathbb{Q}_p/F^{\mathbb{Z}} \times \cdots \times Spd\mathbb{Q}_p/F^{\mathbb{Z}} = (Spd\mathbb{Q}_{p,\Delta})/p.Fr$$

this should be the product of $Spd\mathbb{Q}_p$ but with independent Frobenius. Its Galois category of etale covering is given by finite étale (cover) $E \to (Spd\mathbb{Q}_p)^n$ equipped with commuting isomorphism $\gamma_i : E \to F_i^* E$ (F_i^* is the $i - th$ partial frobenius, and $\prod \gamma_i = F_E$ the absolute one) .Then in this framework (Scholze-Weinstein, Berkeley Lect)

$$\pi_1((Spd\mathbb{Q}_{p,\Delta})/p.Fr) \simeq G_{\mathbb{Q}_{p,\Delta}} = G_\Delta$$

its p-adic representations should be seen as the coefficients for this diamond...

p-adic Langlands
In the proof of p-adic classical correspondance by Colmez, Kisin, Emerton, Breuil, Paskunas......Colmez's functor uses Fontaine's theorem relating *mod*.p^h Galois representations $G_{\mathbb{Q}_p}$ to (φ, Γ)-modules over $\mathbb{Z}/p^h((t))$. Colmez managed to associate to any representations (by $h \to \infty$) of $\mathrm{GL}_2(\mathbb{Q}_p)$ a (φ, Γ) module over the p-adic completion which is the Amice Ring. How to have such a ring?
We consider the subgroup

$$N_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

and we can associate the Iwasawa algebra $\frac{\mathbb{Z}}{p^h\mathbb{Z}}[[N_0]] \simeq \frac{\mathbb{Z}}{p^h\mathbb{Z}}[[t]]$ seeing our $\mathrm{GL}_2(\mathbb{Q}_p)$ representation as a $\frac{\mathbb{Z}}{p^h\mathbb{Z}}[[t]]$-module....

Several attempts have been worked out by Breuil, Vigneras, Schneider to generalize such a tecnique to the case $\mathrm{GL}_n(\mathbb{Q}_p)$ ...In this case we should replace N_0 by

$$\begin{pmatrix} 1 & \mathbb{Z}_p & 0 & \dots & 0 & \\ 0 & 1 & \mathbb{Z}_p & 0 & \dots & 0 \\ 0 & 0 & 1 & & \dots & 0 \\ \dots & & & \dots & & \\ \dots & & & \dots & & \mathbb{Z}_p \\ 0 & & 0\dots & & & 1 \end{pmatrix}$$

Given a p-adic representation of $\mathrm{GL}_n(\mathbb{Q}_p)$ one may associate a (φ, Γ) -module over $\mathbb{Z}/p^h[[N_0]] \simeq \mathbb{Z}/p^h[[t_1, \dots, t_{n-1}]]$.... a multivariable setting .
..this has been the point of view of Zabradi. Of course the action is not anymore by $\Gamma \subset \mathbb{Z}_p^\times$ but by $\Gamma_\Delta \subset \mathbb{Z}_p^\times \times \dots \mathbb{Z}_p^\times$ $(\#\Delta = n-1)$...same for the frobenius action..each acts on one variable...
The initial approach (by Breuil and Paskunas) is a generic one. They associated to the above situation a one variable "case" over the Amice ring (i.e. t...which should be seen as $t_1 + \dots t_{n-1} = t$).

In Colmez's work once getting the framework of (φ, Γ)-modules he used all the power of Fontaine' theory on p-adic Galois representations and periods. This is what we are going to do today.

Some other work has been done by Colmez, Berger and Fourqueaux, but in the direction of using a different decomposition of the absolute Galois using different Lubin-Tate towers

Consider $\mathbb{Q}_p$ and $G_{\mathbb{Q}_p} = G$ its Galois group and Δ a finite set. G_Δ is the product. By C we can indicate a completion of an algebraic closure of $\mathbb{Q}_p$ (for our result we can have any complete extension of $\mathbb{Q}_p$).

We consider p-adic representations of this(fundamental) group. We have two lines to study such objects: via (φ, Γ)-modules or via periods rings.

The first: by Zabradi, Carter, Pal, Kedlaya. The second is the object of our work: we introduce $\mathrm{B}_{dR,\Delta}$ and $\mathrm{B}_{HT,\Delta}$ we then associated to a p-adic representation V a differential system $\mathrm{D}_{dif}(V)$ in $\#\Delta$-variable.... This system is trivial (has a full set of solutions) iff the G_Δ representation is de Rham. Moreover we will link such a differential module to the overconvergent $(\varphi_\Delta, \Gamma_\Delta)$-module built by Pal-Zabradi.

Classical Sen theory gives an equivalence

$$C\text{−representations of } G \simeq \mathbb{Q}_p(\mu_\infty)\text{−representation of } \Gamma$$

where $\Gamma = \mathrm{Gal}(\mathbb{Q}_p(\mu_\infty)/\mathbb{Q}_p) = \mathbb{Z}_p^\times$ and $\mathbb{Q}(\mu_\infty)$ is not complete...By $L = \widehat{\mathbb{Q}_p(\mu_\infty)}$ (the completion).

If we denote by $H = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\mu_\infty))$, we get $C^H = L$. We consider G_Δ and then $H_\Delta = \prod_{\alpha\in\Delta} H$. $G/H = \Gamma$, Γ_Δ. We have iso $\Gamma_\Delta \simeq \mathbb{Z}_\Delta^\times$.

If F is a closed in C : $\mathcal{O}_{F_\Delta}$ is p-adic completion of the tensor over $\mathbb{Z}_p$, $(\mathcal{O}_F)_\Delta^{\otimes}$.
$F_\Delta = \mathcal{O}_{F_\Delta}[1/p]$.
G_Δ (H_Δ, Γ_Δ) acts on C_Δ (L_Δ) components by components in commutative way. By applying Hochschild-Serre spectral "component after component"

Theorem

We have $\mathrm{H}^i(H_\Delta, C_\Delta) = \begin{cases} L_\Delta & \textit{if } i = 0 \\ 0 & \textit{if } i > 0 \end{cases}.$

Sen theory deals with C-representations of G: one first descends to L and then makes the "decompletion" to $\mathbb{Q}_p(\mu_\infty)$. We can define $\mathbb{Q}_p(\mu_\infty)_\Delta^{\otimes}$ (not complete!).

Our final result is the following (the full analogue of Sen theory)(all repr. are finite dim.)

Theorem

We have equivalences of categories

$$\mathrm{Rep}_{C_\Delta}(G_\Delta) \simeq \mathrm{Rep}_{L_\Delta}(\Gamma_\Delta), \quad W \mapsto W^{H_\Delta}$$

The inverse is just extension of scalar $X \mapsto C_\Delta \otimes_{L_\Delta} X$, *moreover*

$$\mathrm{Rep}_{L_\Delta}(\Gamma_\Delta) \simeq \mathrm{Rep}_{\mathbb{Q}_p(\mu_\infty)^{\otimes}_{\Delta}}(\Gamma_\Delta), \ X \mapsto X_{\mathrm{f}}$$

and the inverse is again extension of scalars.

The method uses decompletion and Tate generalizes formula.

The proof follows in some parts the analogue of the relative case i.e. Andreatta -Brinon. In particular one of the first results is the following

Theorem

The map

$$\varinjlim_{H} \mathrm{H}^1(G_\Delta/H, \mathrm{GL}_d(C_\Delta^H)) \to \mathrm{H}^1(G_\Delta, \mathrm{GL}_d(C_\Delta))$$

(the limit is taken in all open $H \trianglelefteq H_\Delta$) induced by inflation maps is bijective.

But this is not really the relative case: our case should be seen as a "finite discrete case".

In our theorem we also have X_{f}: if X is a L_Δ representation of Γ_Δ then X_{f} is the subset of all elements whose orbit under Γ_Δ generates a $\mathbb{Q}_{p\,\Delta}$ module of finite type. I.e. the union of all sub $\mathbb{Q}_{p\,\Delta}$-modules of X which are of finite type and stable by Γ_Δ. The proof uses the fact that $\mathbb{Q}_p(\mu_\infty)^{\otimes}_{\Delta}$-algebra L_Δ is faithfully flat. Moreover we used also the following fact if K is a finite and Galois extension of $\mathbb{Q}_p$, of Galois finite group G_K, then the algebra $K_\Delta = \otimes^{\Delta}_{\mathbb{Q}_p} K$ is isomorphic to the product K of $G_K^{\#\Delta-1}$ times (in particular: $K \otimes_{\mathbb{Q}_p} K$ is isomorphic to K^{G_K}). This allows us to reduce once we are at finite level to a discrete situation.

We are able to introduce **generalized Sen operator** φ_α, $\alpha \in \Delta$. We indicate by $\Gamma_\alpha \to \Gamma_\Delta$ the $\alpha - th$ inclusion (all the others are 1..). then if we have a $\mathbb{Q}_p(\mu_\infty)_\Delta$-representation of Γ_Δ : Y (say X_{f}) we have operators $(\varphi_\alpha)_{\alpha\in\Delta}$ such that for $y \in Y$, $(\varphi_\alpha)(y)$ is given by the limit for $\gamma \in \Gamma_\alpha$,

$$\varinjlim_{\gamma\to 1} \frac{\gamma(y) - y}{\log(\chi(\gamma))}$$

Multivariate period Rings

We can consider $\mathcal{O}_{C_\Delta}/(p) = (\mathcal{O}_C/(p))^{\otimes \Delta}$.The Frobenius map is surjective. We can take the tilt

$$\mathcal{O}^\flat_{C_\Delta} = \varprojlim_{x \to x^p} \mathcal{O}_{C_\Delta}$$

It is perfect and endowed with an action of G_Δ, We can take its Witt vectors and a map $\theta_\Delta : \mathrm{W}(\mathcal{O}^\flat_{C_\Delta}) \to \mathcal{O}_{C_\Delta}$,

$$(a_n) \mapsto \sum_0^\infty p^n a_n^{(n)}$$

We could have considered ${\mathcal{O}^\flat_C}^{\otimes \Delta}$ (tensor over $\mathbb{F}_p$) and the elements

$$\tilde{p}_\alpha = 1 \otimes \cdots \otimes \tilde{p} \otimes \dots 1$$

(iterated power of p roots of p)

$\alpha \in \Delta$: they generate the ideal $I_{\tilde{p}}$: ${\mathcal{O}^\flat_C}^{\otimes \Delta}$ is separated for the $I_{\tilde{p}}$ topology. We have that its completion is isomorphic to $\mathcal{O}^\flat_{C_\Delta}$. This allows us to conclude that

Lemma

For $\alpha \in \Delta$, let $\xi_\alpha = p - [\tilde{p}_\alpha] \in \mathrm{W}(\mathcal{O}^\flat_{C_\Delta})$, then $Ker(\theta_\Delta)$ is the ideal generated by $\{\xi_\alpha\}_{\alpha \in \Delta}$

$\mathrm{A}_{inf,\Delta}$ the completion of $\mathrm{W}(\mathcal{O}^\flat_{C_\Delta})$ with respect $\theta_\Delta^{-1}(p\mathcal{O}_{C_\Delta})$, the map θ_Δ extends and the G_Δ-action as well.
$\mathrm{B}^+_{dR,\Delta}$ is the completion of $\mathrm{A}_{inf,\Delta}$ by the $Ker(\theta_\Delta)$-topology. Again we have an action of G_Δ and an extension of θ_Δ.
In the classical case in B^+_{dR}, there is an element t which can be seen as the logarithm of a iterated $p-th$ roots of unity and generates $Ker(\theta)$, here we have elements t_α such that the $Ker(\theta_\Delta)$ is generated by the t_α's and it is a regular sequence. $Ker(\theta_\Delta)$ induces a filtration

Corollary

The morphism of C_Δ algebras $C_\Delta[X_\alpha]_{\alpha\in\Delta}$ to $\mathrm{grB}^+_{dR,\Delta}$ mapping X_α in t_α is an iso

If we denote by $t_\Delta = \prod t_\alpha$ we have

$$\mathrm{B}_{dR,\Delta} = \mathrm{B}^+_{dR,\Delta}[1/t_\Delta]$$

Again we have filtration on it (ind.by $\mathbb{Z}$). Its graded :$\mathrm{B}_{HT,\Delta} \simeq C_\Delta[t_\alpha, t_\alpha^{-1}]_{\alpha\in\Delta}$.

Definition

If $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\Delta)$, we put

$$\mathrm{D}_{dR}(V) = (\mathrm{B}_{dR,\Delta} \otimes_{\mathbb{Q}_p} V)^{G_\Delta},$$

It admits a G_Δ-equivariant map

$$\alpha_{dR}(V) : \mathrm{B}_{dR,\Delta} \otimes \mathrm{D}_{dR}(V) \to \mathrm{B}_{dR,\Delta} \otimes V$$

V is de Rham if this map is an isomorphism

result: If V is de Rham then $\mathrm{D}_{dR}(V)$ is free of rank $\dim_{\mathbb{Q}_p}(V)$ over $\mathbb{Q}_{p_\Delta}$. (same for HT)

Almost étale descent and decompletion

Put $\mathrm{L}^+_{dR,\Delta} = \mathrm{H}^0(H_\Delta, \mathrm{B}^+_{dR,\Delta})$. Then

$$\mathrm{l}^+_{dR,\Delta} = \mathbb{Q}_p(\mu_\infty)^{\otimes}_{\Delta}[[t_\alpha]]_{\alpha\in\Delta} \subset \mathrm{L}^+_{dR,\Delta}$$

They have an induced from $\mathrm{B}^+_{dR,\Delta}$ filtration: in original filtration the quotients of the graduation are free C_Δ-modules....hence to study $\mathrm{B}^+_{dR,\Delta}$ G_Δ representations one may reduce to the graded parts ...hence to C_Δ G_Δ -representation... by multi- Sen theory

Lemma

$$\mathrm{H}^1(\Gamma_\Delta, \mathrm{GL}_d(\mathrm{L}^+_{dR,\Delta})) \to \mathrm{H}^1(G_\Delta, \mathrm{GL}_d(\mathrm{B}^+_{dR,\Delta}))$$

Hence we deduce:

Theorem

We have equivalence of categories

$$\mathrm{Rep}_{\mathrm{B}^+_{dR,\Delta}}(G_\Delta) \simeq \mathrm{Rep}_{\mathrm{L}^+_{dR,\Delta}}(\Gamma_\Delta), \quad W \mapsto W^{H_\Delta}$$

Let X be a $\mathrm{L}^+_{dR,\Delta}$ -representation of Γ_Δ. If X is killed by $\mathrm{Fil}^{r+1}\mathrm{L}^+_{dR,\Delta}$ then X_{f} is the union of the $\mathbb{Q}_{p_\Delta}$-modules of finite type stable by Γ_Δ (for the case $r=0$...we are in the Sen theory...).
In general we put : $X_{\mathrm{f}} = \varprojlim_r (X/\mathrm{Fil}^{r+1}\mathrm{L}^+_{dR,\Delta}X)_{\mathrm{f}}$. The filtration is done via powers of t_α's...

Theorem

Let X be a free $\mathrm{L}^+_{dR,\Delta}$-module of rank d of Γ_Δ, then X_{f} is free of rank d over $\mathrm{l}^+_{dR,\Delta}$ and the map

$$\mathrm{L}^+_{dR,\Delta} \otimes_{\mathrm{l}^+_{dR,\Delta}} X_{\mathrm{f}} \to X$$

is an isomorphism of Γ_Δ . X_{f} is the union of all sub-$\mathrm{l}^+_{dR,\Delta}$-modules of X of finite type and stable by Γ_Δ.

Theorem

We have an equivalence of categories:

$$\mathrm{Rep}_{\mathrm{B}^+_{dR,\Delta}}(G_\Delta) \simeq \mathrm{Rep}_{\mathrm{l}^+_{dR,\Delta}}(\Gamma_\Delta)$$

$$W \mapsto (W^{H_\Delta})_{\mathrm{f}}$$

remember that $\mathrm{l}^+_{dR,\Delta} = \mathbb{Q}_p(\mu_\infty)^{\otimes}_{\Delta}[[t_\alpha]]_{\alpha\in\Delta} \subset \mathrm{L}^+_{dR,\Delta}$

So, if we start with a $p-adic$ Galois representation then after extending it to $\mathrm{B}^+_{dR,\Delta}$ we can reduce ourselves to a $\Gamma_\Delta \simeq (\mathbb{Z}_p^\times)^\Delta$-representation over a modules on the ring of power series in several variables... . $\mathrm{l}^+_{dR,\Delta}$ ....and then we can have a connection.

We denote Ω^+ the $\mathrm{l}^+_{dR,\Delta}$-modules of continuos differentials of $\mathrm{l}^+_{dR,\Delta}$ over $\mathbb{Q}_p(\mu_\infty)_\Delta$ having poles ≤ 1 at $t_\Delta = \prod t_\alpha$. It is free of basis dt_α/t_α.
If Y is a free module over $\mathrm{l}^+_{dR,\Delta}$. We say that Y admits a connection if there is a $\mathbb{Q}_p(\mu_\infty)^{\otimes}_\Delta$-linear map

$$\nabla_Y : Y \to Y \times_{\mathrm{l}^+_{dR,\Delta}} \Omega^+$$

with Leibniz rule $\nabla_Y(\lambda y) = y \otimes d\lambda + \lambda \nabla_Y(y)$.

If we take $Y \in \mathrm{Rep}_{\mathrm{l}^+_{dR,\Delta}}(\Gamma_\Delta)$ and $Y_r = \mathrm{l}^+_{dR,\Delta}/\mathrm{Fil}^{r+1}\mathrm{l}^+_{dR,\Delta} \otimes Y$ then it is an element of $\mathrm{Rep}_{\mathbb{Q}_p(\mu_\infty)_\Delta}(\Gamma_\Delta)$...hence it admits the Generalized Sen operators....and they are compatible as r grows... these infinitesimal actions commute together with the various φ_α at the different r of the filtration, we get $\nabla_{Y,\alpha}$ operator and together they make a connection ∇_Y which is integrable. If $\mathcal{R}_{\mathrm{l}^+_{dR,\Delta}}$ indicate the category of the modules with connection then a functor

$$\mathrm{Rep}_{\mathrm{l}^+_{dR,\Delta}}(\Gamma_\Delta) \to \mathcal{R}_{\mathrm{l}^+_{dR,\Delta}}$$

So: if $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\Delta)$. then $\mathrm{B}^+_{dR,\Delta} \otimes_{\mathbb{Q}_p} V \in \mathrm{Rep}\mathrm{B}^+_{dR,\Delta}(G_\Delta)$, then put

$$\mathrm{D}^+_{\mathrm{dif}}(V) = (\mathrm{B}^+_{dR,\Delta} \otimes_{\mathbb{Q}_p} V)^{H_\Delta}_{\mathrm{f}}$$

and $\mathrm{D}^+_{\mathrm{dif}}(V)^{\Gamma_\Delta} = (\mathrm{B}^+_{dR,\Delta} \otimes_{\mathbb{Q}_p} V)^{G_\Delta} = \mathrm{D}_{dR}(V)$. (no Frobenius)
As a dividend

Theorem

Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\Delta)$ then V is de Rham if and only if $\mathrm{D}^+_{\mathrm{dif}}(V)$ is trivial as a module with connection.

-we should have the same results starting from any extension of $\mathbb{Q}_p$ with perfect residue field....

$(\varphi_\Delta, \Gamma_\Delta)$-modules

From another point of view Pal , Zabradi, Kedlaya, Carter studied our representation along the lines of Cherbonnier-Colmez: by considering a decomposition of the absolute Galois group via the cyclotomic perfectoid (up to completion) field and then introducing a field in $ch = p$ hence a Frobenius...
The Galois of the cyclotomic tower is a Galois group of a field in $ch = p$ (again being over $\mathbb{Q}_p$ it is $\mathbb{F}_p((t))$ and then making an equivalence of categories between galois representations and (φ, Γ)-modules over a Cohen Ring of $\mathbb{F}_p((t))$: the Amice-Fontaine ring $\mathcal{E}$ the elements of $\mathcal{E}$ do not have radius of convergence. We can then introduce the bounded Robba Ring of overconvergents functions $(\mathcal{E}^\dagger)$. I.e.

$$\mathcal{R} = \bigcup_{r<1} \{ \sum_{i \in \mathbb{Z}} a_i x^i \mid a_i \in \mathbb{Z}_p \mid a_i r^i \to 0 \, as \, i \to -\infty \} = \bigcup_{r<1} \mathcal{R}_r$$

they are defined in an annulus.... Of course we can make also $\mathcal{R}_\Delta$ in several variable and we have

Theorem

(Cherbonnier-Colmez/Pal-Zabradi) We have an equivalence of categories given by the functor $D^\dagger$

$$\mathrm{Rep}_{\mathbb{Q}_p}(G_\Delta) \simeq Mod^{et}_{\mathcal{R}_\Delta}(\varphi_\Delta, \Gamma_\Delta)$$

of course we have the usual tangential action of Γ hence a differential module with Frobenius structure How to relate them to the differential module arising from periods? they don't have singularities while here they are defined in an annulus.....

This is the work done in one variable by Berger-Cherbonnier-Colmez and we are able to translate it in several variables. The idea is that one considers "the developments at non singular points in terms of $t_\alpha \in \mathrm{B}^+_{dR,\Delta}$-variables which is the log-of the iterated of the various $p-th$-roots of unity in each $\alpha \in \Delta$. In fact if $r \leq r_n = \frac{1}{(p-1)p^{n-1}}$ then (Cherbonnier-Colmez) we can make a map

$$\iota_n \, : \, \mathcal{R}_r \to \mathbb{Q}_p(\mu^{(n)})[[t]] \subset \mathrm{B}^+_{dR}$$

This is done by
$\iota_n(x) = \mu^{(n)}\exp(t/p^n)$. Putting together all these maps for $\alpha \in \Delta$ then we obtain for $r \leq \max(\frac{1}{(p-1)p^{n-1}}, \#\Delta/p^n)$

$$\iota_{n,\Delta} : \mathcal{R}_{r,\Delta} \to \mathbb{Q}_p(\mu_\infty)_\Delta[[t_\alpha]]_{\alpha\in\Delta} = \mathrm{l}^+_{dR,\Delta} \subset \mathrm{B}^+_{dR,\Delta}$$

As a matter of fact we end in $\mathbb{Q}_p(\mu^{(n)})_\Delta[[t_\alpha]]_{\alpha\in\Delta}\cdots$

The Pal-Zabradi-Cherbonnier-Colmez theorem says that starting with a $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_\Delta)$, then applying $\mathrm{D}^\dagger(V) \in Mod^{et}_{\mathcal{R}_\Delta}(\varphi_\Delta, \Gamma_\Delta)$, choosing an appropriate r we can find that $\mathrm{D}^\dagger(V)$ is defined in $\mathcal{R}_{r,\Delta}$, applying $\iota_{n,\Delta}$ we can compare it with the differential module we constructed with the "periods" method over $\mathrm{l}^+_{dR,\Delta}$: $\mathrm{D}_{\mathrm{dif}}(V)$. we then have

Theorem

We have a Γ_Δ equivariant isomorphism

$$\mathrm{l}^+_{dR,\Delta} \otimes_{\mathcal{R}_r} \mathrm{D}^\dagger(V) \simeq \mathrm{D}_{\mathrm{dif}}(V)$$

Fontaine's theory

Fontaine's theory with the introduction of rings such that $B^+_{crys,\Delta}$ or $B^+_{st,\Delta}$.

On other hand, along these lines, Berger was able to associated, in the classical case, to a de Rham representation a differential module endowed with Frobenius in one variable on an annulus (cutting all the singularities linked to the zeroes of the log).

Here we would like to extend such a result to the multivariable setting: note that we will deal with differential modules in several variables endowed with a Frobenius which is a product of Δ-independent Frobenius acting each one in one single variables: this should give a good amount of symmetries.

Berger-Colmez approach to locally analytic vectors vs Sen theory

In another direction: consider the approach introduced by Berger and Colmez of replacing classical Berger-Colmez-Sen decompletion procedure via the theory of **locally analytic vectors**

This has been used when dealing with the decomposition of the absolute Galois Group via different Lubin-Tate groups than the cyclotomic ones (the action is not given by a p-adic Lie group Γ, $\dim\Gamma = 1$ ...... > 1....): in this case the classical Sen theory could not help because of the lack of generalized trace maps... The vector space X_f we introduced before is not good, and it has to be replaced by X_{la} (if $\dim\Gamma = 1$ then $X_f = X_{la}$). using locally analytic vectors...

In our case the residual action is given by Γ_Δ: the dimension is bigger than one, but as we said the action should be thought as "discrete" not "relative". In fact we do have in our case too, the identification $X_f = X_{la}$ as $\mathrm{Rep}_{\mathbb{Q}_p(\mu_\infty)^{\otimes}_{\Delta}}(\Gamma_\Delta)$...we are trying to extend to $\dim\Gamma > 1$.