ARITHMETICAL FUNCTIONALS AND WEAK KÖNIG'S LEMMA

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ABSTRACT. By $\mathsf{RCA_0}$, we denote the system of recursive comprehension axioms with Σ^0_1 induction. $\mathsf{WKL_0}$ is $\mathsf{RCA_0}$ plus weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. In this paper, we first show that for any countable model (M, S) of $\mathsf{RCA_0}$, there exists a countable model (M, S') of $\mathsf{WKL_0}$ such that $S \cap S'$ consists of all Δ^0_1 subsets of M. By sophisticating this argument, we finally prove that if a functional is arithmetically definable in $\mathsf{WKL_0}$, it is already so in $\mathsf{RCA_0}$. More precisely, it is shown that if $\mathsf{WKL_0}$ proves $\forall x \forall X \exists ! Y \varphi(x, X, Y)$ with φ arithmetical, so does $\mathsf{RCA_0}$.

1. INTRODUCTION

A celebrating metamathematical theorem due to L. Harrington asserts that $\mathsf{WKL_0}$ is conservative over $\mathsf{RCA_0}$ for the arithmetical (in fact, Π^1_1) sentences. In other words, if an arithmetical theorem can be obtained by some analytical methods involving the compactness argument over computable mathematics, it is already provable without it. This result can be viewed as a computable analogue of the Gödel-Kreisel theorem on set theory, which asserts that if an arithmetical sentence can be proved in ZF with the axiom of choice, it is already provable without it.

Then it is natural to think of extending Harrington's conservation result to analytical sentences, since the Gödel-Kreisel theorem has been extended to the Σ^1_2 sentences by J. Shoenfield. However, it is obvious that $\mathsf{WKL_0}$ is not Σ^1_1 conservative over $\mathsf{RCA_0}$, since an instance of weak König's lemma is Σ^1_1.

In this context, we claim that if $\mathsf{WKL_0}$ proves $\forall x \forall X \exists ! Y \varphi(x, X, Y)$ with φ arithmetical, so does $\mathsf{RCA_0}$. Note that $\exists ! X \varphi(X)$ means that there exists a unique X satisfying $\varphi(X)$. This claim answers a problem posed by the first author [9] and attempted by some others [2], [10].

Let us see an application of our claim at first. The fundamental theorem of algebra, which asserts that any complex polynomial of any positive degree has a zero (or a unique non-void set of roots), can be stated in the form $\forall x \forall X \exists ! Y \varphi(x, X, Y)$ by using a canonical expression (i.e., the binary expansion) for the complex numbers. Most of popular proofs of the theorem use some analytical methods which can be easily formalized in $\mathsf{WKL_0}$ but not in $\mathsf{RCA_0}$. However, by our conservation result, it can be concluded without elaborating a computable solution that the fundamental theorem of algebra is already provable in $\mathsf{RCA_0}$.

By contrast, take a look at the statement that any continuous real function on the closed unit interval $[0, 1]$ has a maximum value. This sentence can not be expressed in the form $\forall x \forall X \exists ! Y \varphi(x, X, Y)$ with φ arithmetical. The point is that

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we can not determine arithmetically whether or not a set encodes a *total* continuous function in the terms of Simpson [8].

Now, we give rigorous definitions of the systems RCA_0 and WKL_0. The language $\mathcal{L}_2$ of second-order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and set variables $X, Y, Z, \ldots$ Numerical terms are built up from numerical variables and constant symbols $0, 1$ by means of binary operations $+$ and $\cdot$. Atomic formulas are $s = t$, $s < t$ and $s \in X$, where s and t are numerical terms. *Bounded* (Σ^0_0 or Π^0_0) formulas are constructed from atomic formulas by propositional connectives and bounded numerical quantifiers $(\forall x < t)$ and $(\exists x < t)$, where t does not contain x. A Σ^0_n formula is of the form $\exists x_1 \forall x_2 \ldots x_n \theta$ with θ bounded, and a Π^0_n formula is of the form $\forall x_1 \exists x_2 \ldots x_n \theta$ with θ bounded. All the Σ^0_n and Π^0_n formulas are the *arithmetical* (Σ^1_0 or Π^1_0) formulas. A Σ^1_n formula is of the form $\exists X_1 \forall X_2 \ldots X_n \varphi$ with φ arithmetical, and a Π^1_n formula is of the form $\forall X_1 \exists X_2 \ldots X_n \varphi$ with φ arithmetical.

The system of RCA_0 consists of

1. the ordered semiring axioms for $(\omega, +, \cdot, 0, 1, <)$,
2. Δ^0_1 comprehension scheme:

$$\forall x(\varphi(x) \leftrightarrow \psi(x)) \to \exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ is Σ^0_1, $\psi(x)$ is Π^0_1, and X does not occur freely in $\varphi(x)$,

3. Σ^0_1 induction scheme:

$$\varphi(0) \land \forall x(\varphi(x) \to \phi(x+1)) \to \forall x \varphi(x),$$

where $\varphi(x)$ is a Σ^0_1 formula.

Within RCA_0, we define $2^{<\mathbb{N}}$ to be the set of (codes for) finite sequences of 0's and 1's. A set $T \subseteq 2^{<\mathbb{N}}$ is said to be a *tree* (or precisely 0-1 *tree*) if any initial segment of a sequence in T is also in T. A *path* through T is a function $f : \mathbb{N} \to \{0, 1\}$ such that for each n, the sequence $f[n] = \langle f(0), f(1), \ldots, f(n-1)\rangle$ belongs to T. The axioms of WKL_0 consists of those of RCA_0 plus *weak König's lemma*: every infinite 0-1 tree T has a path.

The interest of WKL_0 has been well established through an ongoing program, called *Reverse Mathematics*. H. Friedman, S. G. Simpson and others have shown that numerous well-known theorems in different fields of mathematics are provably equivalent to WKL_0 over RCA_0 [8].

An $\mathcal{L}_2$-structure $(M, S, +_M, \cdot_M, 0_M, 1_M, <_M, \in_{M \times S})$ is simply denoted by (M, S) throughout this paper. We also write M for an $\mathcal{L}_1$ structure $(M, +_M, \cdot_M, 0_M, 1_M, <_M)$. If M is the set (or structure) of standard natural numbers, an $\mathcal{L}_2$-structure (M, S) is called an *ω-structure* or an *ω-model*. In particular, ω-models of WKL_0 are known as Scott systems and extensively studied by not a few people, e.g. Kaye [4].

In the next section, we prove that for any countable model (M, S) of RCA_0, there exists a countable model (M, S') of WKL_0 such that $S \cap S'$ is the set of Δ^0_1 subsets of M. This can be regarded as a non-ω extension of Kreisel's recursive hard core theorem, which asserts that the intersection of all ω-models of WKL_0 is the set of recursive sets.

In section 3, we refine the argument for the non-ω hard core theorem to show that for any countable model M of the axioms of ordered semirings and Σ^0_1 induction, there exist two countable models (M, S) and (M, S') of WKL_0 such that $S \cap S'$ is the set of Δ^0_1 subsets of M and in addition, $(M, S \cup S') \models \Sigma^0_1$ induction. The last condition implies that there exists a third model (M, S'') of WKL_0 such that

$S \cup S' \subseteq S''$. To see how to use this fact, let us suppose that WKL_0 proves a sentence $\exists! X \varphi(X)$ with φ arithmetical. Then both (M, S) and (M, S') have a unique set X satisfying $\varphi(X)$, and such unique sets must be identical (hence in Δ^0_1), for otherwise (M, S'') has more than one set satisfying φ. Therefore, RCA_0 also proves $\exists! X \varphi(X)$.

2. A non-ω hard core theorem

In this section, we first review the tree forcing argument originated by Jockusch-Soare [3] and used by L.Harrington for his conservation result on WKL_0. We then reinforce this argument with some other machinery to prove that for any countable model (M, S) of RCA_0, there exists a countable model (M, S') of WKL_0 such that $S \cap S'$ is the set of Δ^0_1 subsets of M.

Let (M, S) be an $\mathcal{L}_2$-structure. We say that $X \subseteq M$ is Δ^0_1 definable over (M, S), denoted $X \in \Delta^0_1(S)$, if there exist a Σ^0_1 formula φ and a Π^0_1 formula ψ both with parameters from $M \cup S$ such that

$$X = \{m \in M : (M, S) \models \varphi(m)\} = \{m \in M : (M, S) \models \psi(m)\}.$$

We write Δ^0_1 for $\Delta^0_1(\emptyset)$. It is easy to see that if (M, S) is a model of RCA_0, $\Delta^0_1(S) = S$.

Lemma 1. *Let (M, S) be an $\mathcal{L}_2$-structure which satisfies the axioms of ordered semirings and Σ^0_1 induction. Then $(M, \Delta^0_1(S))$ is a model of RCA_0.*

Proof. See the proof of Lemma IX.1.8 [8]. □

We now define basic notions of the tree forcing. Let (M, S) be a model of RCA_0. Let $\mathcal{T}_S$ be the set of all $T \in S$ such that

$$(M, S) \models T \text{ is an infinite 0-1 tree with no path.}$$

Note that $\mathcal{T}_S = \emptyset$ if and only if (M, S) is a model of WKL_0. We say that $D \subseteq \mathcal{T}_S$ is *dense* if for each $T \in \mathcal{T}_S$, there exists $T' \in D$ such that $T' \subseteq T$. For a set $\mathcal{D}$ of dense sets, P is said to be *generic* for $\mathcal{D}$ if for each set $D \in \mathcal{D}$, there exists $T \in D$ such that P is a path through T.

Lemma 2. *Let (M, S) be a countable model of RCA_0, and $\mathcal{D}$ a countable set of dense subsets of $\mathcal{T}_S$. Then each $T \in \mathcal{T}_S$ has a path generic for $\mathcal{D}$.*

Proof. If $\mathcal{D} = \{D_i : i \in \omega\}$ is a set of dense sets, we can easily construct a sequence of trees T_i $(i \in \omega)$ such that $T_0 = T, T_{i+1} \subseteq T_i$ and $T_{i+1} \in D_i$ for each $i \in \omega$. Then a path $P \subseteq \bigcap T_i$ is what we want. □

Lemma 3. *Let (M, S) be a model of RCA_0. For any tree $T \in \mathcal{T}_S$, there exists a path P through T such that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction.*

Proof. Let (M, S) be a model of RCA_0. Since Σ^0_1 induction is provably equivalent to bounded Σ^0_1 comprehension (cf. Remark II.3.11 [8]), it suffices to prove that any tree $T \in \mathcal{T}_S$ has a path P such that for each $m \in M$, $\{n \in M : n <_M m \wedge (M, S \cup \{P\}) \models \varphi(n, P)\} \in S$, where $\varphi(x, X)$ is a Σ^0_1 formula with parameters from $M \cup S$.

Let $\{\varphi_e(x, X) : e \in \omega\}$ enumerate all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X. Without loss of generality, we may assume that $\varphi_e(x, X)$ is of the form $\exists y \theta_e(x, X[y])$ with $\theta_e(x, s) \in \Sigma^0_0$, where $X[y]$ denotes the sequence $\langle f(0), f(1), \dots, f(y-1)\rangle$ with the characteristic function f of X.

For each $e \in \omega$ and $m \in M$, let $D^0_{e,m}$ be the set of all $T \in \mathcal{T}_S$ such that for any $n <_M m$, (M, S) satisfies either

1. $\forall s \in T \neg\theta_e(n, s)$, or
2. $\exists w \forall s \in T(\mathrm{lh}(s) = w \to \exists y \leq w \theta_e(n, s[y]))$,

where $\mathrm{lh}(s)$ denotes the length of sequence s, and $s[y]$ is the initial subsequence of s with the length y. Then it is not difficult to see that $D^0_{e,m}$'s are dense. (See Lemma IX.2.4 [8].)

Let $T \in \mathcal{T}_S$ be given. By Lemma 2, we can take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\}$. Fix any $e \in \omega$ and $m \in M$. Since P is generic, there is a tree $T' \in D^0_{e,m}$ which has a path P. Then, it is easy to see

$\{n \in M : n <_M m \wedge (M, S \cup \{P\}) \models \varphi_e(n, P)\}$

$= \{n \in M : n <_M m \wedge (M, S) \models \exists w \forall s \in T'(\mathrm{lh}(s) = w \to \exists y \leq w \theta_e(n, s[y])\}$.

The set on the right-hand side belongs to S by bounded Σ^0_1 comprehension for (M, S). Thus $(M, S \cup \{P\})$ also satisfies bounded Σ^0_1 comprehension. □

Lemma 4. *Let (M, S) be a countable model of* RCA_0*. For any infinite* 0-1 *tree $T \in S$, there exists a countable model (M, S') of* RCA_0 *such that $S \subseteq S'$ and T has a path in S'.*

Proof. It is obvious from Lemmas 1 and 3. □

Lemma 5. *Let (M, S) be a countable model of* RCA_0*. Then there exists a countable model (M, S') of* WKL_0 *such that $S \subseteq S'$.*

Proof. Use Lemma 4 repeatedly. □

Theorem 6. (*L. Harrignton*) *For any Π^1_1-sentence φ, if φ is a theorem of* WKL_0*, then φ is already a theorem of* RCA_0*. Especially, the first order part of* WKL_0 *is the same as that of* RCA_0*, i.e.,* $\mathbf{I\Sigma_1}$ (*Peano arithmetic with induction restricted to the Σ^0_1 $\mathcal{L}_1$-formulas*).

Proof. It easily follows Lemma 5 by the help of Gödel's completeness theorem. □

We now recall another important characterization of models of WKL_0.

Theorem 7. *There is a Π^0_1-formula $\psi(X, Y)$ with no free variables other than X and Y such that for any model (M, S) of* WKL_0 *and for any $A \in S$,*

(1) *there exists $W \in S$ such that $(M, S) \models \psi(A, W)$, and*

(2) *if $(M, S) \models \psi(A, W)$, then $(M, \{(W)_n : n \in M\}) \models$* WKL_0 *and $A \in \{(W)_n : n \in M\}$, where $(W)_n = \{k \in M : (k, n) \in W\}$ for each $n \in M$.*

Proof. See Lemma VII.2.9 [8]. □

The above theorem essentially says that WKL_0 proves the existence of a structure satisfying WKL_0. We notice that this assertion does not conflict with Gödel's second incompleteness theorem, since the structure need not be equipped with the satisfaction relation. Though the essence of the theorem is a kind of folklore, the above particular statement is due to S. Simpson [8]. See also [4], [5] for other accounts.

Since $\{(W)_n : n \in M\}$ is a proper subset of S in the theorem, it turns out that there is no minimal model of WKL_0 with a fixed first order part M. Later in this section, we will show that for a fixed M, the intersection of all such S's that $(M, S) \models \mathsf{WKL}_0$ is just $\mathbf{\Delta^0_1}$. If $M = \omega$, this fact has been known as Kreisel's hard core theorem.

Lemma 8. *Let (M, S) be a countable model of* RCA_0. *Let T be a tree in $\mathcal{T}_S$. Then, for each $A \subseteq M$ such that $A \notin S$, there exists a path P through T such that A is not in $\Delta^0_1(S \cup \{P\})$ and that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction.*

Proof. Let (M, S) be a countable model of RCA_0 and A a subset of M such that $A \notin S$. Let $\{\varphi_e(x, X) : e \in \omega\}$ enumerates all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X.

We first claim that for each $T \in \mathcal{T}_S$ and each pair $(e, d) \in \omega^2$, there exists a path Z through T such that $(M, \{Z\}) \models \Sigma^0_1$ induction and that (e, d) is not a $\Delta^0_1(S \cup \{Z\})$-index of A, i.e., $A \neq \{m : (M, S \cup \{Z\}) \models \varphi_e(m, Z)\}$ or $A \neq \{m : (M, S \cup \{Z\}) \models \neg\varphi_d(m, Z)\}$.

By way of contradiction, deny the claim. Then there exist a tree $T \in \mathcal{T}_S$ and a pair $(e, d) \in \omega^2$ such that if Z is a path through T and $(M, S \cup \{Z\}) \models \Sigma^0_1$ induction, then (e, d) is a $\Delta^0_1(S \cup \{Z\})$-index of A. By Lemma 5, we can construct a countable model (M, S') of WKL_0 such that $S \subseteq S'$. Then, for any path Z through T such that $Z \in S'$,

$$\begin{aligned} m \in A &\Leftrightarrow (M, S \cup \{Z\}) \models \varphi_e(m, Z) \\ &\Leftrightarrow (M, S') \models \varphi_e(m, Z). \end{aligned}$$

Hence, we have

$$m \in A \Leftrightarrow (M, S') \models \forall Z[Z \text{ is a path through } T \to \varphi_e(m, Z)].$$

Since "Z is a path through T" is expressed as a Π^0_1 formula, "Z is a path through $T \to \varphi_e(m, Z)$" is Σ^0_1, and so the whole formula $\forall Z[Z$ is a path through $T \to \varphi_e(m, Z)]$ is logically equivalent in (M, S') to a Σ^0_1 formula $\varphi'(m)$ (with parameters from $M \cup S$) by virtue of compactness of the Cantor space (cf. Lemma V.III.2.4 [8]). Since for any $m \in M$, $(M, S') \models \varphi'(m)$ if and only if $(M, S) \models \varphi'(m)$, we finally have

$$m \in A \Leftrightarrow (M, S) \models \varphi'(m).$$

Similarly, we have

$$m \in A \Leftrightarrow (M, S') \models \exists Z[Z \text{ is a path through } T \wedge \neg\varphi_d(m, Z)]\},$$

and so by compactness, there exists a Π^0_1-formula $\psi'(m)$ with parameters from $M \cup S$ such that

$$m \in A \Leftrightarrow (M, S) \models \psi'(m).$$

Therefore, A is in $\Delta^0_1(S)$, hence in S since (M, S) is a model of RCA_0. This contradicts with our assumption. Thus the claim is proved.

From now, we may assume that for each $e \in \omega$, Σ^0_1-formula $\varphi_e(x, X)$ is of the form $\exists y \theta_e(x, X[y])$ with $\theta_e(x, s) \in \Sigma^0_0$. For each $(e, d) \in \omega^2$, we define $D^A_{e,d}$ to be the set of all $T \in \mathcal{T}_S$ such that one of the followings holds for some $m \in M$:

A1. $m \in A \wedge (M, S) \models \forall s \in T \neg\theta_e(m, s)$,
A2. $m \notin A \wedge (M, S) \models \exists w \forall s \in T(\mathrm{lh}(s) = w \to \exists y \leq w \theta_e(m, s[y]))$,
A3. $m \in A \wedge (M, S) \models \exists w \forall s \in T(\mathrm{lh}(s) = w \to \exists y \leq w \theta_d(m, s[y]))$,
A4. $m \notin A \wedge (M, S) \models \forall s \in T \neg\theta_d(m, s)$.

To show that for each $(e, d) \in \omega^2$, $D^A_{e,d}$ is dense, we choose any $T \in \mathcal{T}_S$. By the above claim, there exists a path Z through T and $m \in M$ such that one of the following conditions holds:

B1. $m \in A \wedge (M, S \cup \{Z\}) \models \forall y \neg\theta_e(m, Z[y])$,
B2. $m \notin A \wedge (M, S \cup \{Z\}) \models \exists y \theta_e(m, Z[y])$,
B3. $m \in A \wedge (M, S \cup \{Z\}) \models \exists y \theta_d(m, Z[y])$,

B4. $m \notin A \wedge (M, S \cup \{Z\}) \models \forall y \neg \theta_d(m, Z[y])$.

First suppose that condition B1 holds. Let $T' = \{s \in T : \forall t \subseteq s \neg \theta_e(x,t)\}$. Then, $T' \in \mathcal{T}_S$, since $T' \in S$ and T' is an infinite subtree of T. It is also clear that A1 holds with T' (instead of T). Thus $T' \in D^A_{e,d}$. Next suppose that condition B2 holds. Take an initial segment t of Z such that $\theta_e(m,t)$. Then $T' = \{s \in T : s \subseteq t \text{ or } t \subseteq s\}$ clearly satisfies A2, hence $T' \in D^A_{e,d}$. The other two cases can be treated similarly. Hence, in any case, there exists a subtree T' of T such that $T' \in D^A_{e,d}$, which means that $D^A_{e,d}$ is dense.

Given a $T \in \mathcal{T}_S$, we take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\} \cup \{D^A_{e,d} : (e,d) \in \omega^2\}$, where $D^0_{e,m}$'s are the dense sets defined in the proof of Lemma 3. Then, $(M, S \cup \{P\})$ satisfies Σ^0_1 induction by the proof of Lemma 3. By way of contradiction, we assume that A is in $\mathbf{\Delta^0_1}(S \cup \{P\})$, that is, there exist e and d such that

$$A = \{m : (M, S \cup \{P\}) \models \exists y \theta_e(x, P[y])\} = \{m : (M, S \cup \{P\}) \models \forall y \neg \theta_d(x, P[y])\}.$$

Since P is generic, there exists $T' \in D^A_{e,d}$ with a path P. First suppose that condition A1 of the definition of $D^A_{e,d}$ holds for T'. Then there exists $m \in A$ such that $(M, S \cup \{P\}) \models \forall y \neg \theta_e(m, P[y])$, since $P \subset T'$. This contradicts with the above equation for A. Suppose that condition A2 holds for T'. Then there exists $m \notin A$ such that $(M, S \cup \{P\}) \models \exists y \theta_e(m, P[y])$, which is also absurd. Similarly, conditions A3 and A4 lead to a contradiction. Thus, we have shown that A is not in $\mathbf{\Delta^0_1}(S \cup \{P\})$. This completes the proof. □

Lemma 9. *Let (M,S) be a countable model of RCA_0, and C a countable set of subsets of M such that $C \cap S = \emptyset$. Then any tree $T \in \mathcal{T}_S$ has a path P such that $C \cap \mathbf{\Delta^0_1}(S \cup \{P\}) = \emptyset$ and that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction.*

Proof. Let (M,S) and C be as in the above statement. We define dense sets $D^0_{e,m}$ and $D^A_{e,d}$ as in the proofs of Lemmas 3 and 8. By Lemma 2, for each $T \in \mathcal{T}_S$, we can take a path P through T which is generic for $\{D^0_{e,m} : e \in \omega, m \in M\} \cup \{D^A_{e,d} : A \in C \wedge (e,d) \in \omega^2\}$. Then by the proofs of Lemmas 3 and 8, it is easy to see that $C \cap \mathbf{\Delta^0_1}(S \cup \{P\}) = \emptyset$ and that $(M, S \cup \{P\}) \models \Sigma^0_1$ induction. □

Lemma 10. *Let (M,S) be a countable model of RCA_0, and C a countable set of subsets of M such that $C \cap S = \emptyset$. Then there exists a countable model (M,S') of WKL_0 such that $S \subseteq S'$ and $S' \cap C = \emptyset$.*

Proof. Use the above lemma repeatedly. □

The next corollary is a generalized version of Kreisel's hard core theorem.

Theorem 11. *Let (M,S) be a countable model of RCA_0. Then there exists a countable model (M,S') of WKL_0 such that $S \cap S' = \mathbf{\Delta^0_1}$.*

Proof. By replacing S and C in Lemma 10 by $\mathbf{\Delta^0_1}$ and $S - \mathbf{\Delta^0_1}$, respectively, we obtain the theorem. □

Corollary 12. *Let M be a countable model of $\mathbf{I\Sigma_1}$. Then there exist uncountably many countable sets S of subsets of M such that (M,S) satisfies WKL_0.*

Proof. If there were only countably many of such S's, then, putting $C =$ (the union of all of them)-$\mathbf{\Delta^0_1}$, by Lemma 10 we could obtain another model (M,S') of WKL_0 such that $S' \cap C = \emptyset$, which is a contradiction. □

Finally, we remark that in our theorem, $(M, S \cup S')$ may not satisfy Σ^0_1 induction. In fact, there are two models (M, S) and (M, S') of Σ^0_1 induction such that $(M, S \cup S')$ does not satisfy Σ^0_1 induction. This fact is easily obtained from the following two theorems.

Theorem 13. (*Mytilinaios* [7]) *Let $M \models \mathbf{I\Sigma_1}$, and let W be a Σ_1 definable but not Δ_1 definable subset of M. Then there exist two Σ_1 definable subsets A, B of M such that W is in $\mathbf{\Delta^0_1(A, B)}$, but neither in $\mathbf{\Delta^0_1(A)}$ nor in $\mathbf{\Delta^0_1(B)}$.*

Theorem 14. (*Groszek et al.* [6]) *Let $M \models \mathbf{B\Sigma_2}$, and let A be a Σ_1 definable subset of M. Then $(M, \{A\}) \models \Sigma^0_1$ induction, or A is complete.*

Note that $\mathbf{B\Sigma_2}$ denotes the collection axioms for the Σ_2 formulas.

Theorem 15. *Let $M \models \mathbf{B\Sigma_2} \wedge \neg\mathbf{I\Sigma_2}$. Then there exist two Σ_1 definable subsets A, B of M such that both $(M, \{A\})$ and $(M, \{B\})$ satisfies Σ^0_1 induction, but $(M, \{A, B\})$ does not.*

Proof. Let $M \models \mathbf{B\Sigma_2} \wedge \neg\mathbf{I\Sigma_2}$, and let W be a complete Σ_1 subset of M. Then $(M, \{W\})$ does not satisfy Σ^0_1 induction, since $M \models \neg\mathbf{I\Sigma_2}$ and any Σ_2 set is $\Sigma^0_1(W)$. By Theorem 13, there exist two incomplete Σ_1 subsets A, B of M such that W is in $\mathbf{\Delta^0_1(A, B)}$. By Theorem 14, both $(M, \{A\})$ and $(M, \{B\})$ satisfies Σ^0_1 induction. But $(M, \{A, B\})$ does not satisfy Σ^0_1 induction, since $(M, \{W\})$ does not. □

3. A new conservation result

In this section, we prove our new conservation result by refining the preceding argument for the non-ω hard core theorem (Theorem 11). As a main lemma, we shall show that for any countable model M of the axioms of ordered semirings and Σ_1 induction, there exist two countable models (M, S) and (M, S') of $\mathsf{WKL_0}$ such that $S \cap S'$ is the set of Δ^0_1 subsets of M and in addition, $(M, S \cup S') \models \Sigma^0_1$ induction (Lemma 18).

Let (M, S) be a countable model of $\mathsf{RCA_0}$. Recall that $\mathcal{T}_S$ denotes the set of all $T \in S$ such that $(M, S) \models T$ is an infinite 0-1 tree with no path. Let $\mathcal{T}^2_S$ be the set of all pairs (T_0, T_1) of trees in $\mathcal{T}_S$. We define a partial order $\leq$ on $\mathcal{T}^2_S$ by $(T'_0, T'_1) \leq (T_0, T_1)$ if and only if $T'_0 \subseteq T_0$ and $T'_1 \subseteq T_1$. A pair (P_0, P_1) of subsets of M is a *pair of paths through* (T_0, T_1) if P_i is a path through T_i $(i = 0, 1)$. Then the notions of a *dense* set and a *generic* path can be defined for $(\mathcal{T}^2_S, \leq)$ in the same way as for $(\mathcal{T}_S, \subseteq)$.

Lemma 16. *Let (M, S) be a countable model of $\mathsf{RCA_0}$. For each tree $T \in \mathcal{T}_S$, there exist two paths P_0 and P_1 through T such that P_i is not in $\mathbf{\Delta^0_1}(S \cup \{P_{1-i}\})$ $(i = 0, 1)$, and $(M, S \cup \{P_0, P_1\}) \models \Sigma^0_1$ induction.*

Proof. Let (M, S) be a countable model of $\mathsf{RCA_0}$. Let $\{\exists y \theta_e(x, X[y]) : \theta_e(x, s) \in \Sigma^0_0, e \in \omega\}$ enumerate all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X. For each $(e, d) \in \omega^2$, we define $D^1_{e,d}$ to be the set of all $(T_0, T_1) \in \mathcal{T}^2_S$ such that (M, S) satisfies, for some $m \in M$, either

A1. $\forall s \in T_1[\mathrm{lh}(s) = m + 1 \to s(m) = 1] \wedge$

(i) $\forall t \in T_0 \neg\theta_e(m, t) \vee$

(ii) $\exists w \forall t \in T_0[\mathrm{lh}(t) = w \to \exists y < w \theta_d(m, t[y])]$ or

A2. $\forall s \in T_1[\mathrm{lh}(s) = m + 1 \to s(m) = 0] \wedge$

(i) $\forall t \in T_0 \neg\theta_d(m, t)) \vee$

(ii) $\exists w \forall t \in T_0[\mathrm{lh}(t) = w \to \exists y < w \theta_e(m, t[y])$.

To show that $D^1_{e,d}$'s are dense, fix any $(T_0, T_1) \in \mathcal{T}^2_S$. Let $P_1 \subseteq M$ be a path through T_1. Since $P_1 \notin S$, by Lemma 8 T_0 has a path P_0 such that P_1 is not in $\Delta^0_1(S \cup \{P_0\})$. Then there exists $m \in M$ such that one of the following conditions holds:

B1. $m \in P_1 \wedge (M, S \cup \{P_0\}) \models \forall y \neg\theta_e(m, P_0[y])$,
B2. $m \in P_1 \wedge (M, S \cup \{P_0\}) \models \exists y \theta_d(m, P_0[y])$,
B3. $m \notin P_1 \wedge (M, S \cup \{P_0\}) \models \exists y \theta_e(m, P_0[y])$,
B4. $m \notin P_1 \wedge (M, S \cup \{P_0\}) \models \forall y \neg\theta_d(m, P_0[y])$.

Suppose that condition B1 holds. Let $t_1 = P_1[m+1]$ and $T'_1 = \{s \in T_1 : s \subseteq t_1 \text{ or } t_1 \subseteq s\}$. Let $T'_0 = \{s \in T_0 : \forall t \subseteq s \neg\theta_e(m, t)\}$. Then it is clear that $(T'_0, T'_1) \in \mathcal{T}^2_S$ and A1.(i) holds with (T'_0, T'_1). Thus $(T'_0, T'_1) \in D^1_{e,d}$. Suppose that condition B2 holds. Take T'_1 as above. Let t_0 be an initial segment of P_0 with $\theta_d(m, t_0)$. Let $T'_0 = \{s \in T_0 : s \subseteq t_0 \text{ or } t_0 \subseteq s\}$. Then $(T'_0, T'_1) \in \mathcal{T}^2_S$ satisfies A1.(ii). Thus $(T'_0, T'_1) \in D^1_{e,d}$. The other two cases can be treated similarly. Hence $D^1_{e,d}$'s are dense.

We also define $D^2_{e,d}$ in the same way as $D^1_{e,d}$ but exchanging the roles of T_0 and T_1. Then $D^2_{e,d}$'s are also dense.

For $e \in \omega$, $m \in M$, we define $D^3_{m,e}$ to be the set of $(T_0, T_1) \in \mathcal{T}^2_S$ such that (M, S) satisfies $\forall n \leq m$ [

1. $\exists w \forall s_0 \in T_0 \forall s_1 \in T_1[\mathrm{lh}(s_0) = \mathrm{lh}(s_1) = w \to (\exists y < w \theta_e(n, s_0 \oplus s_1[y]))]$ or
2. $\forall s_0 \in T_0 \forall s_1 \in T_1 \neg\theta_e(n, s_0 \oplus s_1)$],

where $s_0 \oplus s_1$ is the 0-1 sequence t with length $\mathrm{lh}(s_0) + \mathrm{lh}(s_1)$ defined by $t(2n) = s_0(n)$ and $t(2n+1) = s_1(n)$. By the proof of Lemma 3, we can easily see that $D^3_{m,e}$'s are dense.

We now have a countable set $\mathcal{D} = \{D^1_{e,d},\ D^2_{e,d},\ D^3_{m,e} : m \in M,\ e, d \in \omega\}$ of dense sets. Let (P_0, P_1) be a pair of paths through (T, T) which is generic for $\mathcal{D}$. By the proof of Lemma 3, it is easy to see $(M, S \cup \{P_0, P_1\}) \models \Sigma^0_1$ induction since (P_0, P_1) is generic for $\{D^3_{m,e}\}$. Since (P_0, P_1) is generic for $\{D^1_{e,d}\}$, there exists $m \in P_1$ such that $\forall y \neg\theta_e(m, P_0[y]) \vee \exists y \theta_d(m, P_0[y])$, or $m \notin P_1$ such that $\exists y \theta_e(m, P_0[y]) \vee \forall y \neg\theta_d(m, P_0[y])$. Thus it is not the case that

$$P_1 = \{m : \exists y \theta_e(x, P_0[y])\} = \{m : \forall y \neg\theta_d(x, P_0[y])\}.$$

So P_1 is not in $\Delta^0_1(S \cup \{P_0\})$. In the similarly way, we can prove that $P_0 \notin \Delta^0_1(S \cup \{P_1\})$. This completes the proof. □

To prove our main theorem, we generalize Lemma 16 as follows.

Lemma 17. *Let (M, S) be a countable model of RCA_0. For any tree $T \in \mathcal{T}_S$, there exist two paths P_0 and P_1 through T such that $\{(P_0)_i : i \in M\} \cap \{(P_1)_j : j \in M\} \subseteq S$ and $(M, S \cup \{P_0, P_1\}) \models \Sigma^0_1$ induction.*

Proof. Let (M, S) be a countable model of RCA_0. Let $\{\exists y \theta_e(x, X[y]) : \theta_e(x, s) \in \Sigma^0_0, e \in \omega\}$ enumerate all the Σ^0_1-formulas with parameters from $M \cup S$ and no free variables other than x and X. Let $D^4_{j,e,d}$ be the set of $(T_0, T_1) \in \mathcal{T}^2_S$ such that $(Z)_j \in S$ for any path Z through T_1, or that there exists $m \in M$ such that (M, S) satisfies that

A1. $\forall s \in T_1[\mathrm{lh}(s) = \langle m, j \rangle + 1 \to s(\langle m, j \rangle) = 1] \wedge$
(i) $\forall t \in T_0 \neg\theta_e(m, t) \vee$
(ii) $\exists w \forall t \in T_0[\mathrm{lh}(t) = w \to \exists y < w \theta_d(m, t[y])$ or

A2. $\forall s \in T_1[\mathrm{lh}(s) = \langle m, j\rangle + 1 \to s(\langle m, j\rangle) = 0] \wedge$
(i) $\forall t \in T_0 \neg\theta_d(m, t)) \vee$
(ii) $\exists w \forall t \in T_0[\mathrm{lh}(t) = w \to \exists y < w \theta_e(m, t[y])$.

By straightforward adaptation of the proof of Lemma 16, $D^4_{j,e,d}$'s are dense.

Let $D^3_{m,e}$'s be the dense sets defined in the proof of Lemma 16 and let $\mathcal{D} = \{D^3_{m,e},\ D^4_{j,e,d} : e, d \in \omega,\ j, m \in M\}$. Then $\mathcal{D}$ is a countable set of dense sets. So we can take a pair (P_0, P_1) of paths through (T, T) which is a generic for $\mathcal{D}$. It is easy to see that $(M, S \cup \{P_0, P_1\}) \models \Sigma^0_1$ induction. Thus we only need to show that $\{(P_0)_i : i \in M\} \cap \{(P_1)_j : j \in M\} \subseteq S$.

To see this, we first claim that for each $j \in M$, either $(P_1)_j \in S$ or $(P_1)_j \notin \Delta^0_1(S \cup \{P_0\})$. Fix any $j \in M$ such that $(P_1)_j \notin S$. Since (P_0, P_1) is generic for $\{D^4_{j,e,d}\}$ and $(P_1)_j \notin S$, there exists $m \in M$ such that (M, S) either $(P_1)_j \neq \{m : (M, S \cup \{P_0\}) \models \exists y \theta_e(m, P_0[y])\}$ or $(P_1)_j \neq \{m : (M, S \cup \{P_0\}) \models \forall y \neg\theta_d(m, P_0[y])\}$. Since (e, d) is arbitrary, $(P_1)_j$ is not in $\Delta^0_1(S \cup \{P_0\})$.

Choose any $X \in \{(P_0)_i : i \in M\} \cap \{(P_1)_j : j \in M\}$. Then there exist i, j such that $X = (P_0)_i = (P_1)_j$. The above claim implies $X \in S$ since $(P_1)_j = (P_0)_i \in \Delta^0_1(S \cup \{P_0\})$. Hence $\{(P_0)_i : i \in M\} \cap \{(P_1)_j : j \in M\} \subseteq S$. This completes the proof. □

Lemma 18. *Let M be a countable model of $\mathbf{I\Sigma_1}$. Let A be a subset of M such that $(M, \{A\})$ satisfies Σ^0_1 induction. Then there exist three sets $S_1,\ S_2,\ S_3$ of subsets of M such that*
(1) *$(M, S_i) \models \mathsf{WKL_0}$ $(i = 1, 2, 3)$, and*
(2) *$S_1 \cap S_2 = \Delta^0_1(\{A\})$, and*
(3) *$S_1 \cup S_2 \subseteq S_3$.*

Proof. By Theorem 7, there is a Π^0_1 formula $\psi(X, Y)$ such that for any model (M, S) of $\mathsf{WKL_0}$ and for any $B \in S$, (i) there exists $W \in S$ such that $(M, S) \models \psi(B, W)$, and (ii) if $(M, S) \models \psi(B, W)$, then $(M, \{(W)_n : n \in M\}) \models \mathsf{WKL_0}$ and $B \in \{(W)_n : n \in M\}$. Let $(M, \{A\})$ be a countable model of the axioms of ordered semirings and Σ^0_1 induction, and let $S_0 = \Delta^0_1(\{A\})$. By the normal form theorem, write $\psi(X, Y)$ as $\forall n \theta(X[n], Y[n])$ where θ is Σ^0_0. Let T be the set of $\tau \in 2^{<M}$ such that $\forall \sigma \subseteq \tau \theta(A[\mathrm{lh}(\sigma)], \sigma)$. Then $T \in S_0$ and by (i), T is infinite. We claim that T has no path in S_0. To see this, suppose that T has a path P in S_0. By Lemma 5, let (M, S') be a model of $\mathsf{WKL_0}$ with $S_0 \subseteq S'$. By (ii), $(M, \{(P)_n : n \in M\}) \models \mathsf{WKL_0}$. But (M, S_0) is the least M-model of $\mathsf{RCA_0}$ containing A. This is a contradiction. Thus T has no path in S_0.

By Lemma 17, there exist two paths $P_1,\ P_2$ through T such that $\{(P_1)_i : i \in M\} \cap \{(P_2)_j : j \in M\} \subseteq S_0$ and $(M, \{A, P_1, P_2\}) \models \Sigma^0_1$ induction. By Lemma 5, we have a model (M, S_3) of $\mathsf{WKL_0}$ containing A, P_1 and P_2. Let $S_1 = \{(P_1)_i : i \in M\}$ and $S_2 = \{(P_2)_j : j \in M\}$. Then (M, S_1) and (M, S_2) are models of $\mathsf{WKL_0}$ such that $S_1 \cap S_2 = S_0$ and $S_1 \cup S_2 \subseteq S_3$. □

Theorem 19. *Let $\varphi(x, X, Y)$ be an arithmetical formula with exactly the free variables shown. If $\mathsf{WKL_0}$ proves $\forall x \forall X \exists! Y \varphi(x, X, Y)$, then so does $\mathsf{RCA_0}$.*

Proof. Let $\varphi(x, X, Y)$ be an arithmetical formula with exactly the free variables shown. Suppose that $\mathsf{WKL_0}$ proves $\forall x \forall X \exists! Y \varphi(x, X, Y)$ and $\mathsf{RCA_0}$ can not prove it. Then by Gödel's completeness theorem, there exists a countable model (M, S) of $\mathsf{RCA_0}$ such that $\neg \exists! Y \varphi(n, A, Y)$ holds in (M, S) for some $n \in M$ and $A \in S$.

First suppose that $\exists Y\varphi(n, A, Y)$ holds in (M, S). Then there exists more than one set in S which satisfies φ. By Lemma 5, there exists a model (M, S') of WKL_0 such that $S \subseteq S'$. Obviously, S' has at least two distinct sets which satisfy φ. Hence WKL_0 does not prove $\forall x\forall X\exists! Y\varphi(x, X, Y)$, which is a contradiction.

Next assume that $\forall Y\neg\varphi(n, A, Y)$ holds within (M, S). Let $S_0 = \Delta^0_1(\{A\})$. Then $\forall Y\neg\varphi(n, A, Y)$ holds within (M, S_0). By Lemma 18, there exist three sets S_1, S_2, S_3 of subsets of M such that (1) $(M, S_i) \models \mathsf{WKL}_0$ (i=1, 2, 3); (2) $S_1 \cap S_2 = S_0$; (3) $S_1 \cup S_2 \subseteq S_3$. By (1), there exists $B_i \in S_i$ such that $(M, S_i) \models \varphi(n, A, B_i)$ for $i = 1, 2$. Since there exists no set C in S_0 which satisfies $\varphi(n, A, C)$, we must have $B_1 \neq B_2$ by (2). So, $(M, S_3) \models \neg\exists! Y\varphi(n, A, Y)$ by (3). Since $(M, S_3) \models \mathsf{WKL}_0$, this contradicts with the assumption that WKL_0 proves $\forall x\forall X\exists! Y\varphi(x, X, Y)$. Then the proof is completed. □

Open problems. (1) If WKL_0 proves $\forall x\forall X\exists! Y\varphi(x, X, Y)$ and φ is Π^1_1, then does RCA_0 already prove it?

(2) If $\mathsf{WKL}_0+\Sigma^0_n$ induction proves $\forall x\forall X\exists! Y\varphi(x, X, Y)$ and φ is arithmetical, then does $\mathsf{RCA}_0+\Sigma^0_n$ induction already prove it ($n = 2, 3, \dots$) ? (cf. [1].)

We remark that (2)'s answer is known to be affirmative when $n = \infty$ (i.e., full arithmetical induction). To see this, let (M, S) be a model of ACA_0. By a formalized version of Kreisel's hard core theorem which is provable in ACA_0(Corollary VIII.2.26 [8]), for each $A \in S$, there exist W, $W' \in S$ such that $(M, \{(W)_n\})$ and $(M, \{(W')_n\})$ are models of WKL_0 and $\{(W)_n\} \cap \{(W')_n\} = \Delta^0_1(\{A\})$. The rest of the argument is the same as the proof of Theorem 19.

We also remark that Fernandes [2] solves the problem (2) with restriction to the case that $n = 2$ and φ is Σ^0_3. For more related problems, see [10].

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