$L^p$ WELLPOSEDNESS FOR THE DRIFT-DIFFUSION SYSTEM ARISING FROM THE SEMICONDUCTOR DEVICE SIMULATION

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ABSTRACT. We discuss a strong solution of the nonlinear parabolic system arose from the simulation for the semiconductor device design. This equation considering here is governing the electron and positive hole dynamics on the MOS FET for the Large Scaled Integral-Circuit (V-LSI). We show that the existence and uniqueness and stability of the strong solution in $L^p$ spaces and will discuss on the global existence.

1. INTRODUCTION

Huge amount of numerical and analytical effort was executed for designing the V-LSI. The mathematical theory for the simulation for the dynamics of the electric particles has been extensively studied (see for example and [12] and references there in). Due to the demand for more fine structure on the semiconductor devices, it is now required detailed microscopic mathematical analysis for the dynamics of the electron and holes. In the simplest model, dynamics of electron and hole density is governed by the drift-diffusion model which is described by the following coupled system of elliptic-parabolic equations.

\[
\begin{align*}
\partial_t n - \nabla \cdot (\nu_1 \nabla n - \mu_1 n \nabla \psi) &= f, & t > 0, x \in \mathbb{R}^n, \\
\partial_t p - \nabla \cdot (\nu_2 \nabla p + \mu_2 p \nabla \psi) &= f, & t > 0, x \in \mathbb{R}^n, \\
- \Delta \psi &= \frac{1}{\kappa} (p - n + g), & x \in \mathbb{R}^n, \\
n(0, x) &= n_0(x), & p(0, x) = p_0(x).
\end{align*}
\]

(1.1)

where $n(t, x)$ and $p(t, x)$ denote the electron and hole density in the semiconductor, $g(x)$ denotes difference of the donor and acceptor density which is given function and $c \nu_1$, $\nu_2$ are diffusion constants and $\mu_1$ and $\mu_2$ are coupled constants. $f = f(t, x)$ express the variation of the charge by the external current which is also a given function. The constant $\kappa > 0$ denotes so called the Debye length that stands for the shielding of the hole and electron particles. Mathematical study of this equation has been extensively developed,
for example, [20], [15], [9], [8], [3], [5], [6], [2] and reference therein. The main concern of these results is devoted to the initial boundary value problem for \((??)\). Since some similar equations also appear in the other context as Nernst-Plank equation in Astronomy, Keller-Siegel model in Chemotaxis, it is also interest to consider the Cauchy problem to (1.2). For simplicity, we introduce the following slightly simple system which is obtained from the above equation,

\[
\begin{align*}
\partial_t u - \nu \Delta u + \mu_1 \nabla (u \nabla \psi) &= f, \quad t > 0, x \in \mathbb{R}^n, \\
\partial_t v - \nu \Delta v - \mu_2 \nabla (v \nabla \psi) &= f, \quad t > 0, x \in \mathbb{R}^n, \\
- \Delta \psi &= v - u + g, \quad x \in \mathbb{R}^n, \\
\psi(0, x) &= \psi_0(x), \quad v(0, x) = v_0(x).
\end{align*}
\]

We first show that there exists a unique time local solution of (1.2) for any initial data in \(L^p\).

**Definition.** For \(1 \leq \theta \leq \infty\), we call a pair of exponent \((p, \theta)\) as the (Serrin) admissible pair of order \(s\) if

\[
\frac{2}{\theta} + \frac{n}{p} = 2 + s
\]

and the space \(L^\theta(I; L^p)\) as the Serrin class of the above scaling.

Note that if \(f = 0\) and \(g = 0\), the system of drift-diffusion holds its form under the scaling \(u(t, x) \rightarrow u_\lambda(t, x) = \lambda^2 u(\lambda t^2, \lambda x)\). Under this scaling, the space \(L^\theta(I; L^p)\) is invariant for the 0 order admissible pair \((p, \theta)\). Hence it is expectable that there is a local well-posedness result in the space of 0 order Serrin class.

**Theorem 1.1.** Suppose that

\[
1 < p < 2, \quad \text{for } n = 2
\]

\[
\frac{n}{2} \leq p < n, \quad \text{for } n \geq 3
\]

and \(1 \leq \theta < 2\), let \(f \in L^\theta(0, T; W^{1,p})\) and \(g \in W^{1,p}\). Then for \((u_0, v_0) \in L^p \times L^p\), there exists \(T > 0\) and a unique solution of (1.2) with the initial data \((u_0, v_0)\) such that \(u, v \in C([0, T]; L^p) \cap C((0, T); W^{2,p}) \cap C^1((0, T); L^p)\). When \(n = 2\) and \(p = \theta = 2\), the same result also holds.

For the case of lower exponent of \(p\), the regularity for the initial data is required for the well posedness.

**Theorem 1.2.** For \(0 < a \leq 1\), we let \(\frac{2n}{n+a+1} < p < \frac{n}{a+1}\), \(2a < \theta < \frac{2}{1-a}\). Suppose that \(f \in L^\theta(0, T; W^{1,p})\) and \(g \in W^{1,p}\). Then for \((u_0, v_0) \in W^{\sigma,p} \times W^{\sigma,p}\), there exists \(T > 0\) and unique solution of (1.2) with the initial data \((u_0, v_0)\) such that \(u, v \in C([0, T]; L^p) \cap C((0, T); W^{2,p}) \cap C^1((0, T); L^p)\).

**Remark.** There are many results for the existence and uniqueness problem for the initial boundary value problem corresponding to (1.2) in a bounded domain \(\Omega \subset \mathbb{R}^3\) with the
Neumann and other boundary conditions. See for example, [15], [3], [9], [8] and [12]. Also some abstract results can also cover some cases. See for this direction, [10], [11], [23]. Here the difference with our result than the previous one is that the domain is not bounded and the initial data is rather weaker. That is we assume the less regularity for the initial data. This is related to the initial behavior for the solution near t = 0.

Under more restricted situations, we can state the global existence for the system (1.2).

**Theorem 1.3.** Suppose that \( g \in L^\infty(\mathbb{R}^n) \), \( f \equiv 0 \) and we assume that \( 1 < p \leq 2 \) with \( \theta < 2 \). Suppose that the initial data satisfies the same assumption as in Theorem 1.1 and moreover they are positive definite, i.e., \( u_0(x), p_0(x) \geq 0 \). Then the solution obtained in Theorem 1.1 exists globally in time i.e., \((n, p) \in C([0, T); W^{2,p}) \cap C^1((0, \infty); L^2)\).

If \( p = 2 \), the equation is globally wellposed in \( C([0, \infty); W^{2,2}) \cap C^1((0, \infty); L^2)\).

We should also mention that there is a very much related research for a similar system. First, the drift-diffusion system equation has a strong analogy to the equation of the fluid mechanics. From the Navier-Stokes equation; the incompressible fluid dynamics is described by the well-known system:

\[
\begin{align*}
\frac{\partial}{\partial t} u - \nu \Delta u + u \cdot \nabla u + \nabla p &= f, & t > 0, x \in \mathbb{R}^n, \\
div u &= 0, & t > 0, x \in \mathbb{R}^n, \\
u_0(x) &= u_0(x), \end{align*}
\]

one can reduce the equation to the vorticity; By setting \( \omega = \text{rot} \ u \) we have

\[
\begin{align*}
\frac{\partial}{\partial t} \omega - \nu \Delta \omega + u \cdot \nabla \omega - \omega \cdot \nabla u &= \text{rot} f, & t > 0, x \in \mathbb{R}^n, \\
div \omega &= 0, & t > 0, x \in \mathbb{R}^n, \\
\omega(0, x) &= \text{rot} u_0(x),
\end{align*}
\]

Observing that the velocity \( u \) can be inverted by the vorticity by the Bio-Savourt law as

\[ u = \text{rot} \ (-\Delta)^{-1} \omega. \]

we see the drift-diffusion system (1.2) has an analogous structure of nonlinearity to the vorticity equation (1.5).

The other one is for the model of chemotaxis. The dynamics of the density of mucus is governed by the following system of parabolic equations called as the Keller-Segel system.

\[
\begin{align*}
\partial_t u - \nu \Delta u + \chi \nabla (u \nabla \psi) &= 0, & t > 0, x \in \Omega, \\
\mu \partial_t \psi - \nu \Delta \psi + \gamma \psi &= \alpha u, & t > 0, x \in \Omega, \\
u_0(x) &= u_0(x), & \psi_0(x) = \psi_0(x), \\
\frac{\partial u}{\partial n} &= \frac{\partial \psi}{\partial n} = 0 & x \in \partial \Omega,
\end{align*}
\]

where \( \mu, \nu, \chi, \gamma, \alpha \) are positive constants. When the parameter \( \mu = 0 \), the system is elliptic-parabolic and very close looking to the drift-diffusion model. However the appearance of positive constant \( \gamma > 0 \) makes the \( L^1 \) local wellposedness for (1.6) rather easier since it does not involve the singular integral operator. On the other hand for the global existence,
it is observed that the solution blows-up in a finite time in the case $\mu = 1$. Our system (1.2) is however more stable and there is no blow up solution like that in (1.6).

To solve those equations, we first invert them into the corresponding integral equation via the Duhamel principle. Namely we find a solution to

$$
\begin{aligned}
\begin{cases}
  u = e^{\nu t} u_0 - \int_0^t e^{\nu (t-t')} \Delta \left\{ \mu_1 \nabla (u(\tau) \nabla \psi(\tau)) - f(\tau) \right\} d\tau, \\
v = e^{\nu t} u_0 + \int_0^t e^{\nu (t-t')} \Delta \left\{ \mu_2 \nabla (u(\tau) \nabla \psi(\tau)) + f(\tau) \right\} d\tau,
\end{cases}
\end{aligned}
$$

(1.7)

$$
-\Delta \psi = v - u + g, \quad x \in \mathbb{R}^n.
$$

Since $-\Delta$ is invertible in $\mathbb{R}^n$ by the Newton potential

$$
(1.8) \quad \psi = (-\Delta)^{-1} (v - u + g) =
\begin{cases}
  \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{v(y) - u(y) + g(y)}{|x-y|^{n-2}} dy, & n \geq 3 \\
  \frac{1}{2\pi} \int_{\mathbb{R}^n} (v(y) - u(y) + g(y)) \log |x-y|^{-1} dy, & n = 2,
\end{cases}
$$

where $\omega_n$ denotes the surface area of the unit sphere, we can eliminate $\psi$ from the integral equation and obtain

$$
\begin{aligned}
\begin{cases}
  u = e^{\nu t} u_0 - \int_0^t e^{\nu (t-t')} \Delta \left\{ \mu_1 \nabla (u(\tau) \nabla (-\Delta)^{-1} (v - u + g)(\tau)) - f(\tau) \right\} d\tau, \\
v = e^{\nu t} u_0 + \int_0^t e^{\nu (t-t')} \Delta \left\{ \mu_2 \nabla (u(\tau) \nabla (-\Delta)^{-1} (v - u + g)(\tau)) + f(\tau) \right\} d\tau.
\end{cases}
\end{aligned}
$$

(1.9)

The solution can be constructed by the fixed point argument in the properly defined function space where the initial data only assumed in $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$.

2. Preliminary Lemma

We summarize some lemmas that will be used in the proof of the existence of solutions. The first one is the linear estimate for the heat kernel which is originally due to Serrin, Weissler and Giga.

**Proposition 2.1** (Serrin, Weissler, Giga). For $\nu > 0$, $1 \leq q \leq p \leq \infty \gamma = n/2(1/q - 1/p)$,

1. For $\phi \in L^q$ and $0 \leq \alpha < 1$,

$$
\|\nabla^\alpha e^{\nu t} \phi\|_p \leq C_0 |t|^{-\gamma-\alpha/2} \|\phi\|_q.
$$

Next we states some well-known general inequalities.

**Proposition 2.2** (Hardy-Littlewood-Sobolev). Let $f \in L^p$ and $I_\mu f = c_n |x|^{-(n-\mu)} * f$ where $0 < \mu < n$. Then we have

$$
\|I_\mu f\|_q \leq C_{n,\mu} \|f\|_p \quad \frac{1}{q} = \frac{1}{p} - \frac{\mu}{n}.
$$
For the proof, see Ziemer [24].
Proposition 2.2 immediately gives

Corollary 2.3. Let \( f \in L^p \) and \((-\Delta)^{-1}f = C_n I_2 f \) where \( n \geq 2 \). Then we have

\[
\|\nabla (-\Delta)^{-1}f\|_q \leq C_n \|f\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.
\]

3. PROOF OF EXISTENCE AND REGULARITY

We begin for the proof of Theorem 1.1. For the given initial data \((u_0, v_0) \in L^p \times L^p\), we choose \( M = C_1 \|u_0\|_p \vee C_1 \|f\|_{L^p(I;L^p)} \), \( N = C_1 \|v_0\|_p \vee C_1 \|f\|_{L^p(I;L^p)} \) and \( G = \|g\|_p \). The solution can be constructed in the complete metric space

\[ X_T = \left\{ (\phi, \psi) \in \left\{ C([0, T); L^p) \cap L^\theta(0, T; \dot{W}^{1,p}) \right\}^2 : \|\phi\|_X \leq 3M, \|\psi\|_X \leq 3N \right\}, \]

where

\[
\|\phi\|_X \equiv \sup_{t \in [0, T]} \|\phi\|_p + \left( \int_0^T \|\nabla\phi\|^\theta_p d\tau \right)^{1/\theta}.
\]

Let

(3.1)
\[
\begin{aligned}
\Phi(u, v)(t) &\equiv e^{\nu t}u_0 - \int_0^t e^{\nu(t-\tau)}\Delta \{ \mu_1 \nabla(u(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) - f(\tau) \} d\tau, \\
\Psi(u, v)(t) &\equiv e^{\nu t}v_0 + \int_0^t e^{\nu(t-\tau)}\Delta \{ \mu_2 \nabla(v(\tau)\nabla(-\Delta)^{-1}(v-u+g)(\tau)) + f(\tau) \} d\tau.
\end{aligned}
\]

The following estimate is essential for proving the existence.

Proposition 3.1. Let \((u_0, v_0) \in (L^p)^2\). Then for small \( T > 0 \) and \((u, v) \in X_T\), we have

(3.2) \[ \|\Phi(u, v)\|_X \leq 3M \]
(3.3) \[ \|\Psi(u, v)\|_X \leq 3N \]
(3.4) \[ \|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_X \leq \frac{1}{4} \{ \|u_1 - u_2\|_X + \|v_1 - v_2\|_X \} \]
(3.5) \[ \|\Psi(u_1, v_1) - \Psi(u_2, v_2)\|_X \leq \frac{1}{4} \{ \|u_1 - u_2\|_X + \|v_1 - v_2\|_X \} \]

Proof of Proposition 3.1. We first show (3.2) -(3.5) under the condition

\[
\frac{n}{2} < p < n, \quad \frac{2n}{n+1} \leq p < n.
\]
By Proposition 2.1 and Corollary 2.3

(3.6)
\[
\|\Phi(u, v)\|_{p} \leq \|u_{0}\|_{p} + C\mu_{1} \int_{0}^{t} (t - \tau)^{-\gamma - 1/2} \|u(t)\|_{p} \|\nabla(-\Delta)^{-1}(v - u + g)(\tau)\|_{p} \, d\tau
\]
\[
+ C \int_{0}^{t} (t - \tau)^{-\gamma} \|f(\tau)\|_{p} \, d\tau
\]
\[
\leq M + C\mu_{1} \int_{0}^{t} (t - \tau)^{-\gamma - 1/2} \|u(t)\|_{p} \|\nabla(-\Delta)^{-1}(v - u + g)(\tau)\|_{q} \, d\tau
\]
\[
+ C \int_{0}^{t} (t - \tau)^{-\gamma} \|f(\tau)\|_{p} \, d\tau
\]
\[
\leq M + C\mu_{1} \int_{0}^{t} (t - \tau)^{-\gamma} \|u(t)\|_{p} \|(v - u + g)(\tau)\|_{p} \, d\tau
\]
\[
+ C \int_{0}^{t} (t - \tau)^{-\gamma} \|f(\tau)\|_{p} \, d\tau,
\]
where
\[
\gamma = n \left(\frac{1}{2} - \frac{1}{p} \right) \in [0, 1/2), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n} \in (0, 1],
\]
and it is required that $1/r \in (0, 1]$ and $\gamma \in [0, 1), 1/q \in (0, 1)$. This can be satisfied under the condition

(3.7)
\[
\frac{n}{2} < p < n
\]
\[
\frac{2n}{n + 2} \leq p < n
\]

Thus using Proposition 2.2 for $I = [0, T)$,

(3.8)
\[
\|\Phi(u, v)\|_{L^{\infty}(I; L^{p})} \leq M + C\mu_{1} \sup_{\tau \in I} \|u(\tau)\|_{p} \|t\|^{-\gamma - 1/2} \|(v - u + g)(\tau)\|_{p} \|L^{\infty}(I)\]
\[
+ \|\nabla(-\Delta)^{-1}\|_{L^{\infty}(I)}
\]
\[
\leq M + C\mu_{1} \sup_{\tau \in I} \|u(\tau)\|_{p} \|v(t)\|_{L^{p}(I; L^{p})} + \|u(\tau)\|_{L^{p}(I; L^{p})} + \|g\|_{L^{p}(I; L^{p})}
\]
\[
+ C\|f(\tau)\|_{L^{1/\gamma}(I; L^{r})}
\]
\[
\leq M + C\mu_{1} T^{1/\rho} \sup_{\tau \in I} \|u(\tau)\|_{p} \|v(t)\|_{L^{p}(I; L^{p})} + \sup_{\tau \in I} \|u(\tau)\|_{p} + \|g\|_{p}
\]
\[
+ CT^{\beta} \|f(\tau)\|_{L^{\rho}(I; L^{r})}
\]
\[
\leq M + C(\mu_{1} T^{1/\rho} + T^{1/\theta})(3M + 3N + G),
\]
where
\[
\frac{1}{\rho} = \frac{1}{2} - \gamma \equiv \delta > 0 \quad \beta = \frac{1}{\theta} - 1 + \gamma.
\]
The last condition requires $\frac{n}{2} < p$ and $\theta < 2$. Hence by choosing $T$ properly small, we have

\[(3.9) \quad \|\Phi(u, v)\|_{L^\infty(I; L^p)} \leq \frac{3}{2} M.\]

Similarly for $\gamma = \frac{n}{2}(1/r - 1/p)$,

\[
\|\nabla \Phi(u, v)\|_p \leq C_0 t^{-1/2} \|u_0\|_p \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \|\nabla u(\tau)\|_p \|\nabla (-\Delta)^{-1}(v-u+g)(\tau)\|_q d\tau \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \|f(\tau)\|_r d\tau \\
\leq C_0 t^{-1/2} \|u_0\|_p + C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \{ \|\nabla u(\tau)\|_p \|\nabla (-\Delta)^{-1}(v-u+g)(\tau)\|_q \\
+ \|u(\tau)\|_p \|\nabla^2 (-\Delta)^{-1}(v-u+g)(\tau)\|_p \} d\tau \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-\alpha/2} \|f(\tau)\|_r d\tau,
\]

(by Corollary 2.3 and Sobolev’s inequality)

\[
\leq C_0 t^{-1/2} \|u_0\|_p + C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \|\nabla u(\tau)\|_p \|(v-u+g)(\tau)\|_p d\tau \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \|u(\tau)\|_q \|(v-u+g)(\tau)\|_p d\tau \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-\alpha/2} \|f(\tau)\|_r d\tau,
\]

(by the Sobolev inequality)

\[
\leq C_0 t^{-1/2} \|u_0\|_p + C_\mu_1 \int_0^t (t-\tau)^{-\gamma-1/2} \|\nabla u(\tau)\|_p \|(v-u+g)(\tau)\|_p d\tau \\
+ C_\mu_1 \int_0^t (t-\tau)^{-\gamma-\alpha/2} \|f(\tau)\|_r d\tau,
\]

where

\[
\gamma = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{p} \right), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.
\]
Thus we have for $I = [0, T)$ and $\theta \in [1, 2)$,

\begin{equation}
(3.10) \quad \|\nabla \Phi(u, v)\|_{L^p(I; L^p)} \leq M + C\mu_{1} \sup_{\tau \in I} \{ \|u(\tau)\|_{p} + \|v(\tau)\|_{p} + \|g\|_{p}\} \|u\|_{L^p(I)}^{-\gamma - 1/2} \|\nabla u(\tau)\|_{p} \|\nabla v(\tau)\|_{p} \|\nabla u\|_{L^p(I)} + C\|f\|_{L^p(I)} \leq M + C\mu_{1} \|u\|_{L^p(I)} + \|v\|_{L^p(I)} + \|g\|_{p} \|\nabla u\|_{L^p(I)} \leq M + C\mu_{1} T^{1/p - 1/\theta} \|u(\tau)\|_{L^p(I; L^p)} + \|v(\tau)\|_{L^p(I; L^p)} + \|g\|_{p} \|\nabla u\|_{L^p(I; L^p)} + C T^{1/p - 1/\theta} \|f\|_{L^p(I; L^p)} \leq M + 9C(\mu_{1} + 1)T^{\delta} M(M + N + G) + C\mu_{1} (T^{1/p} + T^{\delta})M,
\end{equation}

where

\[
\frac{1}{r} = 2 - \frac{1}{n}, \quad \frac{1}{\rho} - \frac{1}{\theta} = \frac{1}{2} - \gamma = \frac{1}{2} + \frac{1}{2p} \equiv \delta > 0.
\]

Hence again by choosing $T$ sufficiently small, we conclude

\begin{equation}
(3.11) \quad \|\nabla^\alpha \Phi(u, v)\|_{L^p(I; L^p)} \leq \frac{3}{2} M.
\end{equation}

Inequalities (4.4) and (4.6) imply (3.2). The inequality (3.3) follows in a similar way.

Next we consider the third inequality (3.4). Let $(u_1, v_1)$ and $(u_2, v_2) \in X_T$.

\begin{equation}
(3.12) \quad \|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_p \leq C\mu_{1} \int_{0}^{T} \|\nabla e^{\alpha \Delta} (u_1(\tau) - u_2(\tau))\|_{p} d\tau + C\mu_{1} \int_{0}^{T} \|\nabla e^{\alpha \Delta} u_2(\tau)\|_{p} \|\nabla (-\Delta)^{-1}(v_1 - u_1 + g)(\tau)\|_{q} d\tau + C\mu_{1} T^{1/p} \sup_{\tau \in I} \|u_1(\tau) - u_2(\tau)\|_{p} \sup_{\tau \in I} \|v_1(\tau)\| + \|u_1(\tau)\|_{p} + \|g\|_{p} + C\mu_{1} T^{1/p} \sup_{\tau \in I} \|u_2(\tau)\| \sup_{\tau \in I} \|v_1 - v_2\|_{p} + \|u_1 - u_2\|_{p} \leq C\mu_{1} T^{1/p} \{ (M + N + G)\|u_1 - u_2\|_{X} + M\|v_1 - v_2\|_{X}\}
\end{equation}

where

\[
\gamma = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{p} \right), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.
\]
\begin{equation}
\|\nabla \Phi(u_1, v_1) - \nabla \Phi(u_2, v_2)\|_p \\
\leq C \mu_1 \int_0^t (t - \tau)^{-\gamma - \frac{1}{2}} \left\{ \|\nabla (u_1(\tau) - u_2(\tau))\|_p \|\nabla (-\Delta)^{-1}(v_1 - u_1 + g)(\tau)\|_q \\
+ \|u_2(\tau)\|_q \|\nabla^2 (-\Delta)^{-1}(v_1 - v_2 - u_1 + u_2)(\tau)\|_p \right\} d\tau.
\end{equation}

Similarly to (4.5)

\begin{equation}
\|\nabla (\Phi(u_1, v_1) - \Phi(u_2, v_2))\|_{L^p(I;L^p)} \\
\leq C \mu_1 \sup_{\tau \in I} \{\|u_1(\tau)\|_p + \|v_1(\tau)\|_p + \|g\|_p\} \|\nabla (u_1 - u_2)\|_{L^p(I;L^p)} \\
+ C \mu_1 \{\|\nabla (u_1 - u_2)\|_{L^p(I;L^p)} + \|\nabla (v_1 - v_2)\|_{L^p(I;L^p)}\} \|u_2\|_{L^\infty(I;L^p)} \\
\leq C \mu_1 T^d (3M + 3N + G) \|\nabla (u_1 - u_2)\|_{L^p(I;L^p)} \\
+ 3C \mu_1 T^d M \{\|\nabla (u_1 - u_2)\|_{L^p(I;L^p)} + \|\nabla (v_1 - v_2)\|_{L^p(I;L^p)}\} \\
\leq 3C(\mu_1 + 1) T^d (M + N + G)(\|u_1 - u_2\|_{X} + \|v_1 - v_2\|_{X}).
\end{equation}

Hence (3.12) and (3.14) imply (3.4). (3.5) is similarly obtained.

**Proof of Theorem 1.1, continued.** Since Proposition 3.1 shows that for properly small $T > 0$,

$$(\Phi(u, v), \Psi(u, v)) : X_T \rightarrow X_T$$

is a contraction mapping. By the Banach fixed point theorem, there exists a unique fixed point $(u, v) \in X_T$ such that

\begin{equation}
u(t) = \Phi(u, v)(t) = e^{\nu t \Delta} u_0 - \int_0^t e^{\nu(t-\tau)\Delta} \{\mu_1 \nabla (u(\tau)) \nabla (-\Delta)^{-1}(v - u + g)(\tau) - f(\tau)\} d\tau,
\end{equation}

\begin{equation}
v(t) = \Psi(u, v)(t) = e^{\mu t \Delta} v_0 + \int_0^t e^{\mu(t-\tau)\Delta} \{\mu_2 \nabla (v(\tau)) \nabla (-\Delta)^{-1}(v - u + g)(\tau) + f(\tau)\} d\tau
\end{equation}

hold.

Now for any small $0 < \eta < T$, $(u(\eta), v(\eta)) \in W^{1,p}$ and we exchange the initial data such as $(u_1, v_1) = (u(\eta), v(\eta))$. Then the regularity of solution follows in the similar argument of the above.

For the given initial data $(u_0, v_0) \in W^{1,p} \times W^{1,p}$, we choose $M = C_1 \|u_1\|_p \vee C_1 \|f\|_{L^p(I;W^{1,p})}$, $N = C_1 \|v_1\|_p \vee C_1 \|f\|_{L^p(I;W^{1,p})}$ and $G = \|g\|_{W^{1,p}}$, where $I = [0, T']$. The solution can be constructed in the complete metric space

$$Y_T = \left\{ (\phi, \psi) \in \{C([0, T']; W^{1,p}) \cap L^p(0, T'; \dot{W}^{2,p}) \}^2 : \|\phi\|_Y \leq 3M, \|\psi\|_Y \leq 3N \right\}.$$
where
\[ \| \phi \|_Y \equiv \sup_{t \in [0, T]} \| \phi \|_{W^{1,p}} + \left( \int_0^T \| \Delta \phi \|_p^\theta d\tau \right)^{1/\theta}. \]

Let
\begin{align*}
\Phi(u, v)(t) &\equiv e^{\nu t} \Delta u_1 - \int_0^t e^{\nu(t-\tau)} \Delta \left\{ \mu_1 \nabla (u(\tau) \nabla (-\Delta)^{-1}(v-u+g)(\tau)) - f(\tau) \right\} d\tau, \\
\Psi(u, v)(t) &\equiv e^{\nu t} \Delta v_1 + \int_0^t e^{\nu(t-\tau)} \Delta \left\{ \mu_2 \nabla (v(\tau) \nabla (-\Delta)^{-1}(v-u+g)(\tau)) + f(\tau) \right\} d\tau.
\end{align*}

The following estimate is essential for proving the existence.

**Proposition 3.2.** Let \((u_1, v_1) \in (W^{1,p})^2\). Then for small \(T' > 0\) with \(T' < T\) and \((u, v) \in Y_{T'}\), we have
\begin{align*}
\| \Phi(u, v) \|_Y &\leq 3M, \quad (3.17) \\
\| \Psi(u, v) \|_Y &\leq 3N, \quad (3.18) \\
\| \Phi(u_1, v_1) - \Phi(u_2, v_2) \|_Y &\leq \frac{1}{4} \{ \| u_1 - u_2 \|_Y + \| v_1 - v_2 \|_Y \}, \quad (3.19) \\
\| \Psi(u_1, v_1) - \Psi(u_2, v_2) \|_Y &\leq \frac{1}{4} \{ \| u_1 - u_2 \|_Y + \| v_1 - v_2 \|_Y \}. \quad (3.20)
\end{align*}

**Proof of Proposition 3.2.** The proof is very much similar to the one of Proposition 3.1. For example,
\begin{align*}
\| \Phi(u, v) \|_{W^{1,p}} &\leq \| u_0 \|_{W^{1,p}} + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \| u(\tau) \nabla (-\Delta)^{-1}(v-u+g)(\tau) \|_{W^{1,\gamma}} d\tau \\
&\quad + C \int_0^t (t-\tau)^{-\gamma} \| f(\tau) \|_{W^{1,\gamma}} d\tau \\
&\leq M + C\mu_1 \int_0^t (t-\tau)^{-\gamma-\frac{1}{2}} \| u(\tau) \|_{W^{1,p}} \| (v-u+g)(\tau) \|_{W^{1,p}} d\tau \\
&\quad + C \int_0^t (t-\tau)^{-\gamma} \| f(\tau) \|_{W^{1,\gamma}} d\tau,
\end{align*}
where
\[ \gamma = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{p} \right), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \]
Thus for $I = [0, T')$,

\[(3.22)\]
\[
\|\Phi(u, v)\|_{L^\infty(I; W^{1,p})} \leq M + C\mu_1 T^{1/p} \sup_{\tau \in I} \|u(\tau)\|_{W^{1,p}}
\]
\[
\times (\sup_{\tau \in I} \|v(\tau)\|_{W^{1,p}} + \sup_{\tau \in I} \|u(\tau)\|_{W^{1,p}} + \|g\|_{W^{1,p}})
\]
\[
+ CT^\beta \|f(\tau)\|_{L^p(I; W^{1,p})}
\]
\[
\leq M + C(\mu_1 T^{1/p} + T^{1/\theta})3M(3M + 3N + G),
\]

where

\[
\rho = \frac{1}{2} - \gamma, \quad \beta = \frac{1}{\theta} - 1 + \gamma.
\]

The last condition requires $\theta < 2$.

Similarly for $\gamma = \frac{n}{2}(1/r - 1/p)$,

\[(3.23)\]
\[
\|\Delta\Phi(u, v)\|_p \leq C_0 t^{-1/2}\|u_0\|_{W^{1,p}}
\]
\[
+ C\mu_1 \int_0^t (t - \tau)^{-1/2} \|
\nabla \{u(\tau)\nabla (-\Delta)^{-1}(v - u + g)(\tau)\}\|_{W^{1,r}} d\tau
\]
\[
+ C \int_0^t (t - \tau)^{-1/2} \|f(\tau)\|_{W^{1,r}} d\tau
\]
\[
\leq C_0 t^{-1/2}\|u_0\|_{W^{1,p}} + C\mu_1 \int_0^t (t - \tau)^{-1/2} \|
\nabla u(\tau)\|_p \|(v - u + g)(\tau)\|_{W^{1,p}} d\tau
\]
\[
+ C \int_0^t (t - \tau)^{-\alpha/2} \|f(\tau)\|_{W^{1,r}} d\tau,
\]

where

\[
\gamma = \frac{n}{2}(\frac{1}{r} - \frac{1}{p}), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}.
\]

Hence we obtain for small $T'$ that

\[
\|\Phi(u, v)\|_Y \leq 3M.
\]

The estimate (3.18) follows similarly. For the difference estimates (3.19) and (3.20), the proof is also valid for the estimates (3.12) to (3.14) changing $\|\cdot\|_p$ into $\|\cdot\|_{W^{1,p}}$.

\[\square\]

**Proof of Theorem 1.1, concluded.** By Proposition 3.2, one can show the existence of the solution of (3.16) with the initial data $(u_1, v_1)$ which has a better regularity over $(\eta, T)$ than the one obtained in $X_T$. Namely it is constructed a smooth pair of solutions $(u, v) \in C([\eta, T']; W^{1,p}) \cap L^\theta([\eta, T'); W^{2,p})$. This solution coincides of the original solution by the uniqueness of the solution of (3.16) in $X_T$. We then exchange the initial data and obtain the regularity up to the time interval $(0, T)$.

After reiterating these procedure, we obtain a regular solution $(u, v)$. It is then standard to show that the solution of (3.16) satisfies the equation (1.2).
4. Proof of Existence and Regularity

Proof of Theorem 1.2. Let $0 < \sigma < 1$. For the given initial data $(u_0, v_0) \in W^{\sigma,p} \times W^{\sigma,p}$, we change $M = C_1\|u_0\|_{W^{\sigma,p}} + C_1\|f\|_{L^p(I;L^p)}$, $N = C_1\|v_0\|_{W^{\sigma,p}} + C_1\|f\|_{L^p(I;L^p)}$ and $G = \|g\|_p$ and we also modify the complete metric space as

$$X^\sigma_T = \left\{ (\phi, \psi) \in \left\{ C([0,T); W^{\sigma,p}) \cap L^p(0,T; W^{1,p}) \right\}^2; \|\phi\|_X \leq 3M, \|\psi\|_X \leq 3N \right\},$$

where

$$\|\phi\|_X \equiv \sup_{t \in [0,T]} \|\phi\|_{W^{\sigma,p}} + \left( \int_0^T \|\nabla \phi\|_{L^p}^2 \right)^{1/2}.$$

By changing this norm, the similar estimate in Proposition 3.1 can be obtained. In fact, By Proposition 2.1 and Corollary 2.3 and Gagliard-Nirenberg inequality.

\begin{equation}
(4.1)
\|\Phi(u,v)\|_p \leq \|u_0\|_p + C \mu_1 \int_0^t (t - \tau)^{-\gamma-1/2} \|u(\tau)\nabla(-\Delta)^{-1}(v - u + g(\tau))\|_p d\tau
\end{equation}

\begin{equation}
+ C \int_0^t (t - \tau)^{-\gamma} \|f(\tau)\|_r d\tau
\end{equation}

\begin{equation}
\leq M + C \mu_1 \int_0^t (t - \tau)^{-\gamma-1/2} \|u(\tau)\|_p \|\nabla(-\Delta)^{-1}(v - u + g(\tau))\|_q d\tau
\end{equation}

\begin{equation}
+ C \int_0^t (t - \tau)^{-\gamma} \|f(\tau)\|_r d\tau,
\end{equation}

\begin{equation}
\leq M + C \mu_1 \int_0^t (t - \tau)^{-\gamma} \|u(\tau)\|_p^{2-a} \|(v - u + g(\tau))\|_p^{2-a} d\tau
\end{equation}

\begin{equation}
+ C \int_0^t (t - \tau)^{-\gamma} \|f(\tau)\|_r d\tau,
\end{equation}

where

$$\gamma = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{p} \right) \in [0, 1/2), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{q'} - \frac{1}{n} \in (0, 1],$$

$$0 < a < 1, \frac{1}{q'} = \frac{1}{p} + \frac{a}{2} - \frac{a}{2} \in (0, 1], \quad \alpha \leq a \leq 1.$$

and it is required that $1/r \in (0, 1]$ and $\gamma \in [0, 1), 1/q \in (0, 1)$. This can be satisfied under the condition

\begin{equation}
(4.2)
\frac{2n}{n + a + 1} \leq p < \frac{n}{a + 1}
\end{equation}
Thus using Proposition 2.2 for $I = [0, T)$,

(4.3)

\[ \| \Phi(u, v) \|_{L^{\infty}(I; L^p)} \leq M + C \mu_1 \sup_{\tau \in I} \| u(\tau) \|_p \| |t|^{-\gamma} \ast |(v - u + g)(\tau)|_p \|_{L^{\infty}(I)} \]
\[ + \| |t|^{-\gamma} \ast f(\tau)\|_{L^{\infty}(I)} \]
\[ \leq M + C \mu_1 \sup_{\tau \in I} \| u(\tau) \|_p \| v(\tau) \|_{L^{p'}(I; L^p)} + \| u(\tau) \|_{L^{p}(I; L^p)} + \| g \|_{L^{p}(I; L^p)} \]
\[ + C \| f(\tau) \|_{L^{1/p}(I; L^r)} \]
\[ \leq M + C \mu_1 T^{1/p} \sup_{\tau \in I} \| u(\tau) \|_p (\sup_{\tau \in I} \| v(\tau) \|_p + \sup_{\tau \in I} \| u(\tau) \|_p + \| g \|_p) \]
\[ + CT^{3} \| f(\tau) \|_{L^{p}(I; L^r)} \]
\[ \leq M + C(\mu_1 T^{1/p} + T^{1/\theta})3M(3M + 3N + G), \]

where

\[ \frac{1}{p} = \frac{1}{2} - \gamma \equiv \delta > 0 \quad \beta = \frac{1}{\theta} - 1 + \gamma. \]

The last condition requires \( \frac{n}{2} < p \) and \( \theta < 2 \). Hence by choosing \( T \) properly small, we have

(4.4)

\[ \| \Phi(u, v) \|_{L^{\infty}(I; L^p)} \leq \frac{3}{2} M. \]

Similarly for \( \gamma = \frac{n}{2}(1/r - 1/2) \),

\[ \| \nabla \Phi(u, v) \|_p \leq C_0 t^{-1/2} \| \nabla u_0 \|_p \]
\[ + C \mu_1 \int_0^t (t - \tau)^{-\gamma - \frac{1}{2}} \| \nabla \{ u(\tau) \nabla (-\Delta)^{-1}(v - u + g)(\tau) \} \|_r d\tau \]
\[ + C \int_0^t (t - \tau)^{-1/2} \| f(\tau) \|_r d\tau \]
\[ \leq C_0 t^{-1/2} \| u_0 \|_p + C \mu_1 \int_0^t (t - \tau)^{-\gamma - \frac{1}{2}} \{ \| \nabla u(\tau) \|_p \| \nabla (-\Delta)^{-1}(v - u + g)(\tau) \|_q \}
\[ + \| u(\tau) \|_q \| \nabla^2 (-\Delta)^{-1}(v - u + g)(\tau) \|_p \} d\tau \]
\[ + C \int_0^t (t - \tau)^{-\gamma - \alpha/2} \| f(\tau) \|_r d\tau \]

(by Corollary 2.3 and Sobolev’s inequality)
\[ \leq C_0 t^{-1/2} \| u_0 \|_p + C_{\mu_1} \int_0^t (t - \tau)^{-\gamma/2} \| \nabla u(\tau) \|_p \| (v - u + g)(\tau) \|_p d\tau \]
\[ + C_{\mu_1} \int_0^t (t - \tau)^{-\gamma/2} \| u(\tau) \|_q \| (v - u + g)(\tau) \|_p d\tau \]
\[ + C \int_0^t (t - \tau)^{-\gamma/2} \| f(\tau) \|_r d\tau, \]

(by the Sobolev inequality)

\[ \leq C_0 t^{-1/2} \| u_0 \|_p + C_{\mu_1} \int_0^t (t - \tau)^{-\gamma/2} \| \nabla u(\tau) \|_p \| (v - u + g)(\tau) \|_p d\tau \]
\[ + C \int_0^t (t - \tau)^{-\gamma/2} \| f(\tau) \|_r d\tau, \]

where

\[ \gamma = \frac{n}{2} \left( \frac{1}{r} - \frac{1}{p} \right), \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{n}. \]

Thus we have for \( I = [0, T) \) and \( \theta \in [1, 2) \),

\[ (4.5) \]
\[ \| \nabla \Phi(u, v) \|_{L^p(I; L^p)} \leq M + C_{\mu_1} \sup_{\tau \in I} \left\{ \| u(\tau) \|_p + \| v(\tau) \|_p + \| g \|_p \right\} \| t \|^{-\gamma/2} \| \nabla u(\tau) \|_{L^p(I)} \]
\[ + C \| t \|^{-\gamma/2} \| f(\tau) \|_{L^r(I)} \]
\[ \leq M + C_{\mu_1} \left\{ \| u \|_{L^\infty(I; L^p)} + \| v \|_{L^\infty(I; L^p)} + \| g \|_p \right\} \| \nabla u \|_{L^p(I; L^p)} \]
\[ + C \| f \|_{L^p(I; L^r)} \]
\[ \leq M + C_{\mu_1} T^{1/p - 1/\theta} \left\{ \| u(\tau) \|_{L^\infty(I; L^p)} + \| v(\tau) \|_{L^\infty(I; L^p)} + \| g \|_p \right\} \| \nabla u \|_{L^p(I; L^p)} \]
\[ + C T^{1/p - 1/\theta} \| f(\tau) \|_{L^p(I; L^r)} \]
\[ \leq M + 9C(\mu_1 + 1)T^\theta M(M + N + G) + C_{\mu_1}(T^{1/p} + T^\delta)M, \]

where

\[ \frac{1}{r} = \frac{2}{p} - \frac{1}{n}, \quad \frac{1}{\rho} - \frac{1}{\theta} = \frac{1}{2} - \gamma = 1 - \frac{n}{2p} \equiv \delta > 0. \]

Hence again by choosing \( T \) sufficiently small, we conclude

\[ (4.6) \]
\[ \| \nabla^\alpha \Phi(u, v) \|_{L^p(I; L^p)} \leq \frac{3}{2} M. \]

Inequalities (4.4) and (4.6) imply (3.2). The inequality (3.3) follows in a similar way.

5. A PRIORI ESTIMATES FOR THE GLOBAL EXISTENCE

In this section, we give an apriori estimate for the solution. The space where we have the an apriori estimate is in \( L^2(\mathbb{R}^n) \). For simplicity, we let \( \mu_1 = \mu_2 = 1 \ (n, p) \).
Multiply the first equation of (5.1) by $v$ and integral by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \int_{\mathbb{R}^n} v \nabla w \cdot \nabla \psi dx + \int_{\mathbb{R}^n} vw \Delta w dx = 0.
\]

By the third equation of (5.1),

\[
\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} w \nabla v \cdot \nabla \psi dx + \int_{\mathbb{R}^n} vw(w + g) dx = 0.
\]

Similarly

\[
\frac{1}{2} \frac{d}{dt} \{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v |w|^2 dx + 2 \int_{\mathbb{R}^n} gw dx = 0.
\]

Adding (5.3) and (5.4)

\[
\frac{1}{2} \frac{d}{dt} \{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v |w|^2 dx + \int_{\mathbb{R}^n} gw dx = 0
\]

\[
\frac{1}{2} \frac{d}{dt} \{\|v(t)\|_2^2 + \|w(t)\|_2^2\} + \|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v |w|^2 dx
\]

\[
= - \int_{\mathbb{R}^n} gw dx \leq \|g\|_\infty \|v\| \leq C(\|v\|_2^2 + \|w\|_2^2)
\]

\[
\frac{d}{dt} \{\|v(t)\|_2^2 + \|w(t)\|_2^2\} \leq C(\|v(t)\|_2^2 + \|w(t)\|_2^2)
\]

\[
\|v(t)\|_2^2 + \|w(t)\|_2^2 \leq (\|v_0\|_2^2 + \|w_0\|_2^2)e^{Ct}
\]

\[
\frac{d}{dt} \{\|v(t)\|_2^2 + \|w(t)\|_2^2\} \leq 2\|\nabla v(t)\|_2^2 + \|\nabla w(t)\|_2^2 + \int_{\mathbb{R}^n} v |w|^2 dx
\]

\[
\leq C(\|v_0\|_2^2 + \|w_0\|_2^2)e^{Ct}
\]
Integrate over \([0,t]\) in \(t\) variable,
\[
||v(t)||_2^2 + ||w(t)||_2^2 + 2 \int_{0}^{t} \{||\nabla v(t)||_2^2 + ||\nabla w(t)||_2^2\}d\tau + 2 \int_{0}^{t} d\tau \int_{\mathbb{R}^n} v|w|^2dx 
\leq (||v_0||_2^2 + ||w_0||_2^2)e^{Ct}
\]

(5.11)

It now follows the estimate for \(n(t)\) and \(p(t)\) that
\[
||n(t) + p(t)||_2^2 + ||n(t) - p(t)||_2^2 + 2 \int_{0}^{t} \{||\nabla (n(t) + p(t))||_2^2 + ||\nabla (n(t) - p(t))||_2^2\}d\tau 
+ 2 \int_{0}^{t} d\tau \int_{\mathbb{R}^n} (n(t) + p(t))|n(t) - p(t)|^2dx 
\leq (||n_0 + p_0||_2^2 + ||n_0 - p_0||_2^2)e^{Ct}.
\]

Namely
\[
||n(t)||_2^2 + ||p(t)||_2^2 + 2 \int_{0}^{t} \{||\nabla n(t)||_2^2 + ||\nabla p(t)||_2^2\}d\tau 
+ \int_{0}^{t} d\tau \int_{\mathbb{R}^n} (n(t) + p(t))|n(t) - p(t)|^2dx 
\leq (||n_0||_2^2 + ||p_0||_2^2)e^{Ct}.
\]

(5.13)

All the above procedure is valid for the sufficiently smooth solution. Now by wellposed-ness theorem, we have for the a priori estimate in \(L^2\).

**Proposition 5.1.** Suppose that \(g \in L^\infty(\mathbb{R}^n)\) and let \((n_0, p_0) \in L^2 \times L^2\). Then the solution obtained in Theorem 1.1 \((n,p)\) satisifies the following estimate.
\[
||n(t)||_2^2 + ||p(t)||_2^2 + 2 \int_{0}^{t} \{||\nabla n(t)||_2^2 + ||\nabla p(t)||_2^2\}d\tau 
+ \int_{0}^{t} d\tau \int_{\mathbb{R}^n} (n(t) + p(t))|n(t) - p(t)|^2dx 
\leq (||n_0||_2^2 + ||p_0||_2^2)e^{Ct}.
\]

(5.14)

According to Proposition 5.1, we now obtain the global existence for the positive solution in 3 dimensional case as follows.

**Theorem 5.2.** Suppose that \(g \in L^\infty(\mathbb{R}^3)\), \(f \equiv 0\) and let \((n_0, p_0) \in L^2 \times L^2\). Moreover we assume that the initial data are positive definite, i.e, \(n_0(x), p_0(x) \geq 0\). Then the solution obtained in Theorem 1.1 \((n,p)\) in \(C([0,T]; W^{2,2}) \cap C^1((0,T); L^2)\) is globally exists.

**Proof of 5.2.** By the maximum principle, the local solution obtained in Theorem 1.1 is positive for all time. i.e., \(n(t,x) \geq 0\) and \(p(t,x) \geq 0\). Then by the a priori estimate Proposition 5.1, we have
\[
||n(t)||_2^2 + ||p(t)||_2^2 + 2 \int_{0}^{t} \{||\nabla n(t)||_2^2 + ||\nabla p(t)||_2^2\}d\tau 
\leq (||n_0||_2^2 + ||p_0||_2^2)e^{Ct}.
\]

(5.15)

and this implies the solution can not blow up in a finite time. This concludes the theorem.
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