Rigidity theorems for universal and symplectic universal lattices

thesis by

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Abstract

Cohomological rigidity theorems (with Banach coefficients) for some matrix groups G over general rings are obtained. Main examples of these groups are (finite index subgroups of) *universal lattices* $\mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ for m at least 3 and *symplectic universal lattices* $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])$ for m at least 2 (where k is finite). The results includes the following for certain large m:

(1) The first group cohomology vanishing with any isometric L^p or p-Schatten coefficients, where p is any real on $(1, \infty)$. This is strictly stronger than having Kazhdan's property (T).

(2) The injectivity of the comparison map in degree 2 from bounded to ordinary cohomology, with coefficients as in item (1) *not* containing trivial one.

As a corollary, homomorphim rigidity (, namely, the statement that every homomorphism from G has finite image) is established with the following targets: circle diffeomorhisms with low regularity; mapping class groups of surfaces; and outer automorhisms of free groups. These results can be regarded as a generalization of some previously known rigidity theorems for higher rank lattices (Bader–Furman–Gelander–Monod; Burger–Monod; Farb–Kaimanovich–Masur; Bridson–Wade) to the case of certain general matrix group cases, which are *not* realizable as lattices in algebraic groups. Note that G above does *not* usually satisfy the Margulis finiteness property.

Finally, quasi-homomorphims are studied on special linear groups over euclidean domains. This concept has relation to item (2) above for trivial coefficient case, and to the conception of the stable commutator length. In particular, a question of M. Abért and N. Monod, which was for instance stated at ICM 2006, is answered for large degree case, and a new example of groups with the following intriguing features is provided: having infinite commutator width; but the stable commutator length vanishing.

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Chapter 0

Notation and convention

Unless otherwise stating, we always assume that *all topological groups in this paper are locally compact and σ-compact* (our definition of the compactness contains being Hausdorff). Throughout this thesis, we also assume all rings are associative and unital, all representations and actions of a topological group on Banach spaces are strongly continuous, and all subsets and subgroups of a topological group are closed. We use the terminology representations for linear representations. We basically use the symbol Γ, G, H, N, and Λ for topological groups; S for a subset of a topological group; ρ for a representation on a Banach space; π for a unitary representation; B, E and for Banach spaces; $\mathfrak{H}$ for a Hilbert space; $\mathcal{C}$ for a class of Banach spaces; A and R for rings.

We use the following symbols, which are standard in mathematical literatures:

- $\mathbb{Z}[x_1, \ldots, x_k]$: the (commutative) polynomial ring of k independent generators over $\mathbb{Z}$ (, as a discrete ring);
  $\mathbb{Z}\langle x_1, \ldots, x_k\rangle$: the noncommutative polynomial ring of k independent generators over $\mathbb{Z}$ (, as a discrete ring)
  (for k a natural number)

- $\mathbb{F}_q$: the finite field of order q (for q a positive power of a prime)

- $S(E)$: the unit sphere;
  $B(E)$: the unit ball;
  $\mathbb{B}(E)$: the Banach algebra of all bounded linear operators;
  $O(E)$: the group of linear isometries on E;
  E^*: the dual Banach space
  (for a Banach space E)

- $U(\mathfrak{H})$: the group of unitaries (for a Hilbert space $\mathfrak{H}$)

- $T \mapsto T^*$: the adjoint operation $\mathbb{B}(\mathfrak{H}) \to \mathbb{B}(\mathfrak{H})$ (for a Hilbert space $\mathfrak{H}$)

- $B^{\rho(G)}$: the subspace in B of $\rho(G)$-invariant vectors (for a Banach representation (ρ, B) of a group G)

- $|S|$: a number (for a finite set S)

- l_S: a word length $G \to \mathbb{Z}_{\geq 0}$ (for a group and a symmetric generating subset S)

- $\mathrm{diam}(X)$: the diameter of X, namely, the maximum of distances between two vertices in X

  (for a finite connected graph X)

- μ: a (nonzero) left Haar measure (for a group G)

- 1_G: trivial representation (for a group G) (therefore for a G-representation (B, ρ), the condition of $\rho \supseteq 1_G$ means $B^{\rho(G)} \neq 0$).

- λ_G: left regular representation on $L^p(G) = L^p(G, \mu)$ (for a group G and p in $(1, \infty)$)
  Unless otherwise stating, we consider the case of $p = 2$.

- $A \triangle B$: symmetric deference, $:= (A \setminus B) \cup (B \setminus A)$ (for a set A and B)

- $\widehat{G}$: the unitary dual of G (for a group G)

- $\pi \succeq 1_G$: the weak containment of 1_G (in the Fell topology on $\widehat{G}$), equivalently, π having almost invariant vectors (for a unitary representation $(\pi, \mathfrak{H})$ of a group G)

- $\mathcal{K}(G; S)$: the Kazhdan constant (for a group G and a compact subset S)

- $\mathcal{K}(G, N; S)$ the relative Kazhdan constant (for a group G, $N \leqslant G$ and a compact subset S of G)

- $\xi \mapsto \xi^*$: the duality mapping $S(B) \to S(B^*)$ (for a *uniformly smooth* Banach space B)

- $\rho^\dagger$: the contragredient representation $G \to \mathbb{B}(B^*)$ (for a Banach G-representation (ρ, B))

- e_G (or simply, e): the group unit of G;
  1_R (or simply, 1): the ring unit of R

- $\lim_\omega$: ultralimit (of bounded sequences), or ultraproduct (of sequences of metric spaces with base points) (for a fixed non-principal ultrafilter ω)

- $\mathrm{M}_m(R)$: the ring of $m \times m$ matrices;
  I_m: the ring unit of $\mathrm{M}_m(R)$;
  $\mathrm{GL}_m(R)$: the multiplicative group of invertible matrices in $\mathrm{M}_m(R)$
  (for $m \geq 2$ and a ring R)

- $W \to {}^tW$: the transpose map on $\mathrm{M}_m(A)$ (for $m \geq 2$ and a commutative ring A)

- $\mathrm{SL}_m(A)$: the multiplicative group of matrices in $\mathrm{M}_m(A)$ of determinant 1 (for $m \geq 2$ and A a commutative ring)

- $\mathrm{E}_m(R)$: the elementary group inside $\mathrm{GL}_m(R)$;
  $E_{i,j}(r)$: the elementary matrix in $\mathrm{M}_m(R)$ with the (i,j)-th entry r
  (for for $m \geq 2$ and a ring R; and $1 \leq i \leq m$, $1 \leq j \leq m$, $i \neq j$, and $r \in R$)

- $\mathrm{U}_m(R)$: the normal subrgoup of $\mathrm{GL}_m(R)$ generated by all unipotent matrices in $\mathrm{GL}_m(R)$

- $\mathrm{E}_m(R) \ltimes R^m \trianglerighteq R^m$: these groups are respectively identified with

$$\left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{E}_m(R),\ v \in R^m \right\} \quad \trianglerighteq \quad \left\{ \left(\begin{array}{c|c} I_m & v \\ \hline 0 & 1 \end{array} \right) : v \in R^m \right\}$$

  (for $m \geq 2$ and a ring R)

- $\mathrm{E}_{m_0}(R) \leqslant \mathrm{E}_m(R)$, $\mathrm{SL}_{m'}(A) \leqslant \mathrm{SL}_m(A)$: by these we mean the inclusions are respectively realized as the subgroups sit *in the left upper corner*. Namely, for instance we realize $\mathrm{E}_{m_0}(R) \leqslant \mathrm{E}_m(R)$ as

$$\left\{ \begin{pmatrix} W & 0 \\ 0 & I_{m-m_0} \end{pmatrix} : W \in \mathrm{E}_{m_0}(R) \right\} \leqslant \mathrm{E}_m(R).$$

  (For $m \geq m_0 \geq 2$, a ring R, and a commutative ring A)

- $\mathrm{St}_m(R)$: the Steinberg group (for a ring R and $m \geq 3$)

- C_p: the space of p-Schatten class operators (for p) (although the space itself is defined also for $p = 1, \infty$, we always assume $p \in (1, \infty)$, as mentioned in below).

- $H^\bullet(G; \rho, B)$, $H_\mathrm{b}^\bullet(G; \rho, B)$: respectively, group cohomology and group bounded cohomology with Banach coefficient (for a *discrete* group G and a Banach G-representation (ρ, B))

- $H^\bullet(G)$, $H_\mathrm{b}^\bullet(G)$: respectively, group cohomology, and group bounded cohomology with the trivial real coefficient $(\mathbb{R}, 1_G)$ (for a *discrete* group G)

- $H_{\mathrm{c}}^{\bullet}(G;\rho,B)$, $H_{\mathrm{cb}}^{\bullet}(G;\rho,B)$: respectively, *continuous* group cohomology, and *continuous* group bounded cohomology with Banach coefficient (for a *topological* group G and a Banach G-representation (ρ,B))

- $H_{\mathrm{b}}^{\bullet}(G;\rho,B) \to H^{\bullet}(G;\rho,B)$, $H_{\mathrm{cb}}^{\bullet}(G;\rho,B)$, $H_{\mathrm{c}}^{\bullet}(G;\rho,B)$: the comparison maps in degree $\bullet$ (for a dicrete/topological group G and a Banach G-representation (ρ,B));
  $H_{\mathrm{b}}^{\bullet}(G) \to H^{\bullet}(G)$: the comparison map in degree $\bullet$, with the trivial real coefficient (for a discrete group G)

- $QH(G)$: the ($\mathbb{R}$-vector) space of quasi-homomorphisms;
  $\widetilde{QH}(G)$: the *actual* ($\mathbb{R}$-vector) space of quasi-homomorhisms
  (for a discrete group G)

- $HQH(G)$: the ($\mathbb{R}$-vector) space of homogeneous quasi-homomorphisms;
  $\widetilde{HQH}(G)$: the *actual* ($\mathbb{R}$-vector) space of homogeneous quasi-homomorhisms
  (for a discrete group G)

- $[g,h]$: a single commutator, $:= ghg^{-1}h^{-1}$ (for $g.h \in G$ and a group G)

- $[G,G]$: the commutator subgroup;
  cl ($:[G,G] \to \mathbb{Z}_{\geq 0}$), scl ($:[G,G] \to \mathbb{R}_{\geq 0}$): respectively the commutator length and the stable commutator length on $[G,G]$
  (for a discrete group G)

- S^1: the unit circle on $\mathbb{R}^2$, identified with $[-\pi,\pi)$

- $\mathrm{Diff}_{+}^{1+\alpha}(\mathrm{S}^1)$: the group of orientation preserving circle homeomorphisms which are $(1+\alpha)$-Hölder diffrentiable

- Σ_g, $\Sigma_{g,l}$: a compact oriented connected surface respectively of genus g (closed), and of genus g and punctures l (for $g,l \geq 0$) (therefore $\Sigma_g = \Sigma_{g,0}$);
  $\mathrm{MCG}(\Sigma)$: the mapping class group, as a discrete group (for a surface Σ)

- F_n: the free group of rank n;
  $\mathrm{Aut}(F_n)$, $\mathrm{Out}(F_n)$: respectively the automorphism group and the outer automorphism group of F_n, as discrete groups
  (for $n \geq 2$ finite)

- $L \star L'$: the free product (for groups L and L')

- $\overline{\mathrm{IA}}_n$: the kernel of $\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})$ (for $n \geq 2$)

The following notation, convention and uses of symbols in this thesis may not be standard:

- For a simple algebraic group G over a local field, by *rank* we mean the local rank of G, namely, the dimension of maximal split torus in G (for example, for any $m \geq 1$ the group $\mathrm{Sp}_{m,1}$ is of rank 1 in our definition).

- By a *totally higher rank* (*algebraic*) *group*, we mean a group of the following form: $G = \Pi_{i=1}^{m} \mathbf{G}_i(k_i)$, where k_i are local fields, $\mathbf{G}_i(k_i)$ are k_i-points of Zariski connected simple k_i-algebraic groups (with finite center), and each simple factor $\mathbf{G}_i(k_i)$ has rank ≥ 2.

- By a *totally higher rank lattice*, we mean a lattice in a totally higher rank algebraic group.

- By a *higher rank lattice*, we mean a lattice Γ in a group the form $G = \Pi_{i=1}^{m} \mathbf{G}_i(k_i)$ where k_i are local fields and $\mathbf{G}_i(k_i)$ are k_i-points of Zariski connected simple k_i-algebraic groups with finite center and with $\sum_i \mathrm{rank} \mathbf{G}_i(k_i) \geq 2$, which satisfies the following condition: for every i such that $\mathrm{rank} \mathbf{G}_i(k_i) = 1$, the image of Γ by the projection $G \twoheadrightarrow \mathbf{G}_i(k_i)$ is dense in $\mathbf{G}_i(k_i)$.
  Note that essential examples of higher rank lattices which may not be of totally higher rank are *irreducible* lattices in a higher rank algebraic group. Here a higher rank algebraic group is a group of the form $G = \Pi_{i=1}^{m} \mathbf{G}_i(k_i)$ above with $\sum_i \mathrm{rank} \mathbf{G}_i(k_i) \geq 2$, and a lattice Γ in G is irreducible if each image of Γ by the projection into $\mathbf{G}_i(k_i)$ is dense.
  For instance, $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ is a higher rank lattice (can be realized as an irreducible lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$), but *not* a totally higher rank lattice. And even though $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is a lattice in the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ of rank 2, this is *not* a higher rank lattice in our definition.

- Let p represent any real in $(1, \infty)$ (it is important in this thesis that we *exclude the case of* $p = 1$ *and* $p = \infty$).

- Let k represent any natural number (for $k = 0$, we mean by $\mathbb{Z}[x_1, \ldots, x_k]$ the ring of integers $\mathbb{Z}$).

- Let $\mathcal{H}$ denote the class of all Hilbert spaces.

- Let $[\mathcal{H}]$ denote the class of all Banach spaces which admit compatible norms to those of Hilbert spaces.

- The symbol $\mathbb{K}$ is used for a local field (we allow archimedean local fields as well).

- For a real $M \geq 1$, let $\mathcal{H}_M$ denote the class of all Banach spaces which admit compatible norms to those of Hilbert spaces with the norm ratio $\leq M$.

Here for two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a Banach space, the norm ratio between $\|\cdot\|_1$ and $\|\cdot\|_2$ is defined by the following formula:

$$\max\left\{\sup_{x\neq 0}\frac{\|x\|_1}{\|x\|_2}, \sup_{x\neq 0}\frac{\|x\|_2}{\|x\|_1}\right\}.$$

- $\langle\cdot,\cdot\rangle$: the *duality* $B\times B^*\to\mathbb{C}$ (for a Banach space B)
- $\langle\cdot|\cdot\rangle$: the inner product $\mathfrak{H}\times\mathfrak{H}\to\mathbb{C}$ (for a Hilbert space $\mathfrak{H}$)
- S^{-1}: the set of all elements of the form s^{-1} with $s\in S$ (for a subset S of a group G)
- S^n: the set of all elements of the form $g=s_1\cdots s_n$ with $s_1,\ldots,s_n\in S$ (for a subset S of a group G, and $n\geq 1$)
  also we set $S^0:=\{e_G\}$.
- For p, let $\mathcal{L}_p$ denote the class of all L^p spaces on any (σ-additive) measure.
- $\rho\succeq 1_G$: ρ having almost invariant vectors (for a Banach G-representation (ρ,B))
- $|\rho|$: $=\sup_{g\in G}\|\rho(g)\|$ (for uniformly bounded representation of a group G)
- $d_{\|\cdot\|}$, $r_{\|\cdot\|}$: respectively, the modulus of convexity $(0,2)\to\mathbb{R}_{\geq 0}$, and the modulus of smoothness $\mathbb{R}_{>0}\to\mathbb{R}_{\geq 0}$ (for a Banach space $(B,\|\cdot\|)$)
- $B'_{\rho(N)}$: the natural complement of $B^{\rho(N)}$, as defined in (for a uniformly smooth Banach space B, an isometric G-representation (B,ρ), and $N\trianglelefteq G$)
- $\mathcal{K}(G,N;S,\rho)$: the relative Kazhdan constant for property (T_B) (for $N\trianglelefteq G$, a compact subset S of G, and a isometric Banach G-representation (ρ,B))
- $\overline{\mathcal{K}}(G,N;S;M)$: the generalized relative Kazhdan constant for uniformly bounded representation (for $N\trianglelefteq G$, a compact subset S of G, and M a real ≥ 1)
- $\mathrm{Cay}(G;S)$: a Cayley graph (for a finitely generated group G and a finite generating set S). In this thesis, we connect edges on $\mathrm{Cay}(G;S)$ by *right* multiplication of $s\in S$, and consider the isometric *left* G-action on $\mathrm{Cay}(G;S)$.
- δ_S: displacement function (for a finite subset S of a group G, and an isometric action of G on a metric space X)
- $QH_{\mathrm{c}}(G;B,\rho)$: the (vector) space of continuous quasi-cocycles (with Banach coefficient);
  $\widetilde{QH}_{\mathrm{c}}(G;B,\rho)$: the *actual* (vector) space of continuous quasi-cocycles (with Banach coefficient)
  (for a topological group G and a Banach G-representation (B,ρ))

- $QH(G; B, \rho)$: the (vector) space of quasi-cocycles (with Banach coefficient);
  $\widetilde{QH}(G; B, \rho)$: the *actual* (vector) space of quasi-cocycles (with Banach coefficient)
  (for a discrete group G and a Banach G-representation (ρ, B))

- $\mathrm{sr}(R)$: the stable range (for a ring R)

  NOTE: there is an inconsistency of ± 1 in the definition of stable range in literatures.

- J_m: the alternating matrix

$$J_m := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

  in M_{2m} (for $m \geq 1$)

- $\mathrm{Sp}_{2m}(A)$: the multiplicative group of symplectic matrices in $\mathrm{M}_{2m}(A)$ *associated with the alternating matrix* J_m, $:=\{g \in \mathrm{M}_{2m}(A) : {}^t g J_m g = J_m\}$;
  $\mathrm{Ep}_{2m}(A)$: the elementary symplectic group inside $\mathrm{Sp}_{2m}(A)$ *associated with the alternating matrix* J_m
  (for $m \geq 1$ and A a commutative ring)
  Note that the choice of the alternating matrix (, namely, J_m) is *not* a standard one in studies of symplectic groups.

- $\mathrm{Sp}_{2m_0}(A) \leqslant \mathrm{Sp}_{2m}(A)$ (or, $\mathrm{Sp}_{2m_0}(A) \hookrightarrow \mathrm{Sp}_{2m}(A)$): by this we mean the inclusion is realized as

$$\left\{ \begin{pmatrix} P & 0 & Q & 0 \\ 0 & I_{m-m_0} & 0 & 0 \\ R & 0 & S & 0 \\ 0 & 0 & 0 & I_{m-m_0} \end{pmatrix} : \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}_{2m_0}(A) \right\} \leqslant \mathrm{Sp}_{2m}(A);$$

  $\mathrm{SL}_{m_0}(A) \leqslant \mathrm{Sp}_{2m}(A)$ (or, $\mathrm{SL}_{m_0}(A) \hookrightarrow \mathrm{Sp}_{2m}(A)$): by this we mean the inclusion is realized as

$$\left\{ \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & I_{m-m_0} & 0 & 0 \\ 0 & 0 & {}^tW^{-1} & 0 \\ 0 & 0 & 0 & I_{m-m_0} \end{pmatrix} : W \in \mathrm{SL}_{m_0}(A) \right\} \leqslant \mathrm{Sp}_{2m}(A).$$

  (for $m \geq m_0 \geq 1$ and A a commutative ring)

- $S^{m*}(A^m)$: the additive group of all symmetric matrices in $\mathrm{M}_m(A)$ (for $m \geq 2$ and A a commutative ring)

- $\mathrm{E}_m(A) \ltimes S^{m*}(A^m) \trianglerighteq S^{m*}(A^m)$: these groups are respectively identified with

$$\left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & {}^tW^{-1} \end{array} \right) : W \in \mathrm{E}_m(A),\ v \in S^{m*}(A^m) \right\} \trianglerighteq \left\{ \left(\begin{array}{c|c} I_m & v \\ \hline 0 & I_m \end{array} \right) : v \in S^{m*}(A^m) \right\}$$

  (for $m \geq 2$ and a commutative ring A)

- U_mR; L_mR: the subgroups of $\mathrm{E}_m(R)$ respectively consisting of all unit upper triangle matrices; all unit lower triangle matrices (for a ring R and $m \geq 2$)

Finally, we shall define the following properties in terms of B:

- *property* (T_B) (for a group G)
  [Definition 3.2.2],
  relative property (T_B) (for a group pair $N \trianglelefteq G$)
  [Definition 3.2.2];

- *property* (F_B) (for a group G)
  [Definition 3.2.4],
  relative property (F_B) (for a group pair $N \leqslant G$)
  [Definition 3.2.4];

- the *Shalom property for* (F_B) (for a discrete group G)
  [Definition 5.3.1];

- *property* (FF_B) (for a group G)
  [Definition 7.1.3],
  relative property (FF_B) (for a group pair $N \leqslant G$, or a pair $Q \subseteq G$)
  [Definition 7.1.3];

- *property* $(\mathrm{FF}_B)/\mathrm{T}$ (for a group G)
  [Definition 8.1.1],
  relative property $(\mathrm{FF}_B)/\mathrm{T}$ (for a group pair $N \leqslant G$, or a pair $Q \subseteq G$)
  [Definition 8.1.1, Definition 8.1.6],
  property (TT)/T (for a group G)
  [Definition 8.1.1, Definition 8.3.4],
  relative property (TT)/T (for a group pair $N \leqslant G$, or a pair $Q \subseteq G$)
  [Definition 8.1.1, Definition 8.1.6, Definition 8.3.4].

When letting (P_B) represent any of these properties, we define the property $(P_{\mathcal{C}})$ in terms of a class $\mathcal{C}$ of Banach spaces as follows: having $(P_{\mathcal{C}})$ denotes having (P_B) for all $B \in \mathcal{C}$.

Chapter 1

Introduction and main results

The special linear group $G = \mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ over the commutative polynomial ring with k variables over $\mathbb{Z}$ (where $m \geq 3$) is called the *universal lattice* by Y. Shalom in [Sha1]. Here in this thesis the symbol k is used for representing any finite natural number (usually $k \geq 1$. We state if we allow $k = 0$). It was a long standing problem to determine whether this group satisfies a property so-called *Kazhdan's property* (T).

Kazhdan's property (T), which was first introduced in a paper [Kaz] of D. Kazhdan in 1967, represents certain forms of rigidity of a group, and now plays an important role in wide range of mathematical fields (we will see in Chapter 2 the definition, basic properties, and some examples of applications). The original definition of property (T) is stated in terms of weak containment of the trivial representation. The celebrated Delorme–Guichardet theorem [Del], [Gui] states that for locally compact and σ-compact groups, property (T) is equivalent to a property so-called *property* (FH), which is defined as follows: a group G is said to have *property* (FH) if every (continuous) affine isometric action of G on a Hilbert space has a global fixed point. This definition of property (FH) is identical to the condition of *first* (*continuous*) *group cohomology vanishing with any unitary coefficient*. Therefore as we mentioned, property (T) represents *extreme rigidity* of groups.

We go back to the question we raised in above. In Kazhdan's original paper [Kaz] of property (T), he shown that any "*totally higher rank algebraic group*" (we refer to Chapter 0 for the definition: roughly speaking, *rank* means local field rank, and *totally* means each simple factor has rank at least 2) and "*totally higher rank lattices*" enjoy this property. For instance, the special linear group $\mathrm{SL}_m(\mathbb{Z})$ has property (T) for $m \geq 3$ because it is a lattice in a simple algebraic group $\mathrm{SL}_m(\mathbb{R})$ of real rank $m-1(\geq 2)$. The special linear group $\mathrm{SL}_m(\mathbb{Z}[\sqrt{2}])$ also enjoys property (T) for $m \geq 3$ because this group can be realized as a lattice in $\mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_m(\mathbb{R})$. However, Kazhdan's proof is deeply based on representation theory of semisimple algebraic groups, and it gives no information whether the special group $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[x])$ enjoys property (T). This is one background of the question to determine whether universal lattices have property (T). Around 2006 and 2007, finally Shalom and L. Vaserstein answered this question affirmatively:

Theorem 1.0.1. (*Shalom* [Sha5]*, Vaserstein* [Vas2]; *Theorem* 5.4.1 *in this thesis*) *The universal lattice, namely, the group* $G = \mathrm{SL}_{m\geq 3}(\mathbb{Z}[x_1,\ldots,x_k])$ *has property* (T). *This means: for any unitary representation* π *of* G*, one has*

$$H^1(G;\pi) = 0.$$

We have more words on the motivation to focus on this problem. It is easy to see by definition that property (T) passes to group quotients. Therefore once Theorem 1.0.1 has been proved, this then immediately implies that groups such as $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[1/p])$ (here p is a prime number); $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[\sqrt{2},\sqrt{3}])$; $\mathrm{SL}_{m\geq 3}(\mathbb{F}_q[x])$ ($\mathbb{F}_q$ is the field of order q and q is a positive power of a prime); and $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[t,t^{-1}])$ have property (T). Note that in four examples above, all but last one are totally higher rank (hence arithmetic (or S-arithmetic)) lattices, and Kazhdan's theorem applies. However, the last one in the examples above cannot be realized as an arithmetic lattice, and property (T) for this group had not been obtained before the Shalom–Vaserstein theorem. In general, Theorem 1.0.1 implies property (T) for *elementary groups* $\mathrm{E}_{m\geq 3}(A)$ over *any* commutative and finitely generated ring (we always assume that rings are associative and unital). Here the elementary group $\mathrm{E}_m(A)$ is defined as the multiplicative group of $m\times m$ matrices generated by elementary matrices (for details, we will see in Chapter 4. In the examples above, the groups should be elementary groups, but they coincide in these case). Therefore, property (T) for universal lattices can be regarded as the *universal* result for elementary groups over a commutative finitely generated rings, which includes special linear groups over a ring of integers. This is the reason why groups $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[x_1,\ldots,x_k])$ are called *universal lattices*: they are *universal* for *lattices* of the form $\mathrm{SL}_m(\mathcal{O})$.

As we mentioned in above, the group $\mathrm{SL}_{m\geq 3}(\mathbb{Z}[t,t^{-1}])$ (and a universal lattice itself) can*not* be realized as a lattice in semi-simple algebraic group. This follows from the following argument: on an (irreducible) higher rank lattice, there is an extremely strong constraint, which is called the *Margulis finiteness property*:

every normal subgroup is either finite or of finite index.

This contradicts the fact that the group above contains an infinite group with infinite index (we will see in Lemma 4.1.12). Note that by the Margulis arithmeticity theorem, any (irreducible) higher rank lattice is arithmetic. In these views, Theorem 1.0.1 can be regarded as a *non-arithmetization* of extreme rigidity of totally higher rank lattices (of certain form).

We proceed to the next (but closely related) topic *property* ($\mathrm{T_B}$) and *property* (F_B), where B is a (given) Banach space or a class of Banach spaces. In 2007, Bader–Furman–Gelander–Monod [BFGM] investigated similar properties to property (T) (and property (FH))in the broader framework of general Banach spaces B. They named the Kazhdan type property and the fixed point property respectively property (T_B) and property (F_B) (we will see the precise definitions in Section 3). Property (F_B) is a straight generalization of property (FH), and defined as *first* (*continuous*) *group cohomology vanishing with any isometric coefficient on* B. The classes of Banach spaces of our main interest are the class $\mathcal{L}_p$ ($p \in (1,\infty)$ is given) and $[\mathcal{H}]$. Here the former denotes the class of all L^p spaces on any measure spaces; and the latter denotes the class of all Banach spaces which have compatible norms to ones of Hilbert spaces. The reason why we are interested in these cases is, then property (F_B) is strictly stronger than Kazhdan's property (T), and hence it represents *even much more extreme rigidity* of groups. Indeed, there are plenty of groups with property (T) which are known to *fail* to have property (F_B) for above class. For instance, P. Pansu [Pan] shown the group $\mathrm{Sp}_{m,1}$ of real rank 1, which is known to have property (T) if $m \geq 2$, *fails* to have property $(\mathrm{F}_{\mathcal{L}_p})$ as soon as $p > 4m+2$ (therefore fore instance, property $(\mathrm{F}_{\mathcal{L}_p})$ with $(\infty >) p > 10$ is *no longer* equivalent to property (FH)=property $(\mathrm{F}_{\mathcal{L}_2})$). Moreover G. Yu [Yu2] shown that for every (Gromov-)hyperbolic group H (we will give the definition in Subsection 2.6.4), including of wide range of groups with property (T), it has corresponding $p \gg 2$ such that H admits a (metrically) proper cocycle on an ℓ^p-space. If a cocycle is a coboundary, then it is bounded (we are considering *isometric* coefficients). Therefore, existence of proper (which means, "diverging at infinity") cocycle represents *strong negation* of property $(\mathrm{F}_{\mathcal{L}_p})$, hence, *soft* (or, *well-deformed*) feature of groups. On property $(\mathrm{F}_{[\mathcal{H}]})$, Shalom has shown in his unpublished work, that every rank 1 groups, including $\mathrm{Sp}_{m,1}$, *fails* to have this property.

On the other hand, in [BFGM], Bader–Furman–Gelander–Monod proved the following theorem and revealed that (F_B) is stronger than (T_B) in general, but that totally higher rank groups and lattices *remain* to have property $(\mathrm{F}_{\mathcal{L}_p})$ and $(\mathrm{F}_{[\mathcal{H}]})$ (the assertion in item (*iii*) for property $(\mathrm{F}_{[\mathcal{H}]})$ is due to Shalom):

Theorem 1.0.2. ([BFGM]) *Let G be a locally compact and σ-compact group.*

(*i*) *For any Banach space B, property* (F_B) *implies property* (T_B).

(*ii*) *Property* (T) *is equivalent to property* $(\mathrm{T}_{\mathcal{L}_\mathrm{p}})$, *where $p \in (1,\infty)$. It is also equivalent to property* $(\mathrm{F}_{\mathcal{L}_\mathrm{p}})$, *where $p \in (1,2]$.*

(*iii*) *Any totally higher rank groups G and any totally higher rank lattices Γ have property, in the sense in Chapter* 0 *possess property* $(\mathrm{F}_{\mathcal{L}_\mathrm{p}})$ *for $1 < p < \infty$ and property* $(\mathrm{F}_{[\mathcal{H}]})$.

Therefore, it is now natural to ask the following questions:

Do universal lattices have property (F_{L^p}) $(1 < p < \infty)$ *and property* $(\mathrm{F}_{[\mathcal{H}]})$*?*

Also, specially property ($\mathrm{F}_{\mathcal{L}_{\mathrm{p}}}$) has application to group actions on the circle. We will come back to this point later in this introduction.

The first part of this thesis gives the answer to this question, with some (slight) degree condition. The answer is *affirmative*, and the following is the precise statement:

Theorem A. ([Mim1]; *Theorem* 6.3.1 *in this thesis*) *Let* $p \in (1, \infty)$. *If* $m \geq 4$, *then for any* p (*and any* k) *the universal lattice* $G = \mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ *possesses property* ($\mathrm{F}_{\mathcal{L}_p}$) *and property* ($\mathrm{F}_{[\mathcal{H}]}$) *in the sense in Definition* 3.2.4. *This means: for any* ρ, *an isometric* G*-representation on an* L^p *space or a uniformly bounded* G*-representation on a Hilbert space, one has*

$$H^1(G; \rho) = 0.$$

It may be reasonable to expect that Theorem A remains true in the case of $m = 3$. However at the present the author has no idea how to settle this problem. As we will see in Chapter 4 and Chapter 5, in the proof of Theorem 1.0.1 Shalom employs the equivalence of property (T) and property (FH), and deduces property (FH) from a certain "relative version" of property (T). However in our setting ($B = \mathcal{L}_p$, $[\mathcal{H}]$), as we have seen in above, property (T_B) does *not* imply property (F_B). This means, an easy imitation of the proof of Theorem 1.0.1 does *not* provide with Theorem A. We need some idea which overcomes the gap from property (T_B) to property (F_B). We will examine this in Section 6.3.

Secondly, we consider a *quasification* of property (F_B): it means we consider maps which are cocycle *up to bounded error*. This conception has natural connection to the concept of *bounded cohomology* [Gro1], [Mon1]. Here bounded cohomology is defined by restricting each cochains to be bounded. We will see in Chapter 7. By naming after so-called *property* (TT) of N. Monod (which states any *quasi*-cocycle into unitary representation is bounded), we define a notion of *property* (FF_B) by strengthening property (F_B). We note the following: since there is a natural injection from bounded cochains to ordinary cochians, this map induces a natural map from (continuous) bounded to (continuous) ordinary group cohomology (with isometric Banach coefficient (ρ, B) of a group G),

$$\Psi^{\bullet}_{\mathrm{cb}} \colon H^{\bullet}_{\mathrm{cb}}(G; \rho, B) \to H^{\bullet}_{\mathrm{c}}(G; \rho, B).$$

This map is called the *comparison map*, and in general it is *neither* injective *nor* surjective. Property (FF_B) for a group G specially implies that for any isometric G-representation ρ, the comparison map degree 2

$$\Psi^{2}_{\mathrm{cb}} \colon H^{2}_{\mathrm{cb}}(G; \rho, B) \to H^{2}_{\mathrm{c}}(G; \rho, B)$$

is *injective*.

We examine whether universal lattice has some *confined* property of property (FF_B), which we name "*property* (FF_B)/T."

Theorem B. ([Mim1]; *Theorem* 8.1.10 *and Theorem* 8.3.5 *in this thesis*) *Let* $G = \mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ *be universal lattice. Let* $p \in (1, \infty)$.

(*i*) *If* $m \geq 4$, *then for any* p, G *possesses property* $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ *and property* $(\mathrm{FF}_{[\mathcal{H}]})/\mathrm{T}$ *in the sense in Definition* 8.1.1. *In particular the following hold: for any* ρ, *an isometric* G*-representation on an* L^p *space or a uniformly bounded* G*-representation on a Hilbert space, if moreover* $\rho \not\supseteq 1_G$, *then one has*

- (*a*) *both* $H^1(G; \rho) = 0$,
- (*b*) *and the comparison map in degree* $2 : H_b^2(G; \rho) \to H^2(G; \rho)$ *is injective.*

(*ii*) *For* $m \geq 3$, G *possesses property* $(\mathrm{TT})/\mathrm{T}$ (*, that is, property* $(\mathrm{FF}_{\mathcal{H}})/\mathrm{T}$).

As the statement of above theorem suggests, "$(\mathrm{FF}_B)/\mathrm{T}$" means "property (FF_B) *modulo the trivial linear part.*"

In the third part of this thesis, we consider the case of $B = C_p$, which denotes the space of *p-Schatten class operators* acting on a separable Hilbert space. This can be seen an analogy to property $(\mathrm{F}_{\mathcal{L}_p})$ (or $(\mathrm{FF}_{\mathcal{L}_p})$) in noncommutative L^p-setting. Some properties on commutative L^p space are *no longer* true in noncommutative setting, and one of them is crucial to establish property $(\mathrm{F}_{\mathcal{L}_p})$. Explicitly, the conditional negative definiteness of the kernel on L^p space with $p \in [1, 2]$ is not valid for C_p. However, by extending a previous work of M. Pushnigg [Pus] for totally higher rank groups and lattices, we show the following:

Theorem C. ([Mim3]; *Theorem* 8.2.6 *in this thesis*) *If* $m \geq 4$, *then for any* $p \in (1, \infty)$, *any finite index subgroup* Γ *of* $\mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ *has property* $(\mathrm{FF}_{C_p})/\mathrm{T}$. *In particular,* Γ *above has property* (F_{C_p}). *Here* C_p *denotes the space of* p*-Schatten class operators on a separable Hilbert space.*

For any p, *any totally higher rank algebraic group and any lattice therein has property* (F_{C_p}).

In the proof of this theorem, we state criteria on a class of Banach spaces for which universal lattice (with degree≥ 4) has property $(\mathrm{FF}_B)/\mathrm{T}$ and property (F_B). The class (or a single Banach space) fulfilling these criterion contains the cases of $\mathcal{L}_p$, $[\mathcal{H}]$, and C_p. We hope this will provide with further study on rigidity theory for universal lattices.

In the fourth part of this thesis, we consider the symplectic version of universal lattice, namely, the group $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])$ $(m \geq 2)$, and call it *symplectic universal lattice*. As is often in various fields of mathematics, the bahavior of symplectic group is more complicated than that of special linear group. However, by employing a result of a work [EJK] in progress of Ershov, Jaikin-Zapirain, and Kassabov (of property (T) for symplectic universal lattices), we obtain the following theorem:

Theorem D. ([Mim4]; *Theorem* 9.4.1 *and Theorem* 9.3.1 *in this thesis*) *Let* $G = \mathrm{Sp}_{2m}(\mathbb{Z}[x_1,\ldots,x_k])$ *be a symplectic universal lattice.*

(*i*) *If* $m \geq 3$, *then for any* $p \in (1,\infty)$, *G possesses property* $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ *and property* $(\mathrm{FF}_{C_p})/\mathrm{T}$. *In particular, G has property* $(\mathrm{F}_{\mathcal{L}_p})$ *and property* (F_{C_p}).

(*ii*) *For* $m \geq 2$, *G possesses property* (TT)/T.

Specially, property (TT)/T (this in particular implies that the comparison map

$$\Phi^2_{\mathrm{b}}\colon H^2_{\mathrm{b}}(G;\pi) \to H^2(G;\pi)$$

is injective for any unitary representation *which satisfies* $\pi \not\supseteq 1_G$) plays a significant role in application, as we shall see in below. We also note that property (TT)/T (as well as property (T)) passes to group quotients and finite index subgroups. Thus we obtain property (TT)/T for any finite index subgroups in the *elementary symplectic group* $\mathrm{Ep}_{2m}(A)$ over a finitely generated commutative ring (we will see for details in Chapter 9).

In the fifth part of this thesis, we consider quasi-cocycles with trivial (real) coefficient, which are called *quasi-homomorphisms.* Recall that property $(\mathrm{FF}_B)/\mathrm{T}$ means "modulo the trivial lienar part." Hence, in studies above, we escape from dealing with quasi-homomorphisms, which are turned out to be in fact the most tough types of quasi-cocyles for universal lattices. At the moment, we have not succeeded in establishing vanishing results of qusai-homomorphisms on universal lattices. Nevertheless, we have obtained the following result (Theorem E) on elementary groups over *euclidean domains* (then in fact elementary groups coincide with special linear groups). This result covers, for instance, $\mathrm{SL}_{m\geq 6}(K[x])$ for K being an *arbitrary* (commutative) field, and has interesting application. On quasi-homomorphisms, or equivalently in some sense, on the kernel of comparison map

$$\Psi^2_{\mathrm{b}}\colon H^2_{\mathrm{b}}(G) \to H^2(G),$$

(here coefficients are the trivial real coefficients) the following result is known as *Bavard's duality theorem*:

Theorem 1.0.3. (*Bavard* [Bav]) *Let G be a discrete group. Then the following are equivalent:*

(*i*) *The comparison map* $H^2_{\mathrm{b}}(G) \to H^2(G)$ *is injective.*

(*ii*) *The* stable commutator length $\mathrm{scl}\colon [G,G] \to \mathbb{R}_{\geq 0}$, *which is defined as*

$$[G,G] \ni g \mapsto \mathrm{scl}(g) := \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n},$$

vanishes identically.

If the commutator length on $[G,G]$ is bounded, then by the definition above, the stable commutator length vanishes identically (on $[G,G]$). M. Abért has asked whether there exists a counterexample of the converse, and Monod stated this question in his ICM invited lecture [Mon2]. This question is now known as a question of Abért and Monod, and our theorem provides with a natural class of counterexamples of the converse:

Theorem E. ([Mim2]; *Theorem* 10.5.1 *in this thesis*) *Let A be a euclidean domain. Then for $m \geq 6$, $G = \mathrm{SL}_m(A)$, as a discrete group (possibly uncountable), fulfills the following: the comparison map*

$$H_b^2(G; 1_G, \mathbb{R}) \to H^2(G; 1_G, \mathbb{R})$$

is injective. Equivalently, the stable commutator length vanishes identically on $[G,G](= G)$.

In particular, if K is a (commutative) field of infinite transcendence degree over its subfield (for instance, $K = \mathbb{C}$), then for $m \geq 6$ the group $G = \mathrm{SL}_m(K[x])$ enjoys the following two properties:

(*i*) *The commutator width of $G(= [G,G])$ is infinite; namely,*

$$\sup_{g\in G} \mathrm{cl}(g) = \infty.$$

(*ii*) *The stable commutator length vanishes identically on $[G,G]$ $(= G)$; namely, for any $g \in G$,*

$$\lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n} = 0.$$

We note that A. Muranov [Mur] has shown there exists a 2-generated simple group with infinite commutator width and with the stable commutator length vanishing. He employs small cancellation theory, and it has completely different background to our one.

We also note that there exists a *countable* field K satisfying the assumption of the latter part of Theorem E. We will examine these topics in Chapter 10.

In the final part of this thesis, we apply our theorems to group actions. There are two applications. One is to *group actions on the circles.* By combining a theorem of A. Navas [Nav1], [Nav2] with our property $(\mathrm{F}_{\mathcal{L}_p})$ result, we obtain the following theorem:

Theorem F. ([Mim1], [Mim4]; *Theorem* 11.1.2 *in this thesis*) *Let Γ be a finite index subgroup either of $\mathrm{SL}_m(\mathbb{Z}[x_1,\ldots,x_k])$ $(m \geq 4)$ or of $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1,\ldots,x_k])$ $(m \geq 3)$. Then for any $\alpha > 0$, every homomorphism*

$$\Gamma \to \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$$

has finite image. Here the symbol $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ *means the group of orientation preserving* $(1+\alpha)$*-differentiable* (*in the sense of Hölder continuity*) *diffeomorphisms on the unit circle.*

Note that by a lemma of A. Selberg, in both cases there exists such Γ which is torsion free.

The next and final application is to *homomorphism rigidity into mapping class groups of surfaces and into* (*outer*) *automorphism groups of free groups.* Those objects have strong connection to group actions on low dimensional manifolds (here by *homomorphism rigidity* we mean the property that every homomorphism from the group into a target group has finite image). We note that for higher rank lattices, homomorphism rigidity into those groups are respectively obtained by Farb and Masur [FaMas] (into mapping class groups); and Bridson–Wade [BrWa] (into automorphsim groups of free groups). We note that in their proof, the Margulis finiteness property for higher rank lattices plays a key role. However, as we have mentioned in above, this property is *not* valid for universal lattices or symplectic universal lattices. Also, we mention the following: Bridson–Wade [BrWa] have shown that if a group is $\mathbb{Z}$*-averse*, then homomrphism rigidity into the groups above holds. Here a group is said to be *not* $\mathbb{Z}$*-averse* if there exists finite index subgroup which has a normal subgroup mapping onto $\mathbb{Z}$. We have, however, universal lattices and symplectic universal lattices are *not* $\mathbb{Z}$-averse (see Lemma 11.2.26). Therefore, the following result is a new result:

Theorem G. ([Mim4]; *Theorem* 11.5.1 *in this thesis*) *Let* Γ *be a finite index subgroup either of* $\mathrm{SL}_m(\mathbb{Z}[x_1,\ldots,x_k])$ $(m\geq 3)$ *or of* $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1,\ldots,x_k])$ $(m\geq 2)$. *Then for any* $g\geq 0$ *and* $n\geq 2$*, every homomorphism*

$$\Phi\colon \Gamma\to \mathrm{MCG}(\Sigma_g)$$

and every homomorphism

$$\Psi\colon \Gamma\to \mathrm{Out}(F_n)$$

have finite image. Here Σ_g *denotes a compact closed connected oriented surface of genus* g *and* $\mathrm{MCG}(\Sigma_g)$ *denotes the mapping class group. The symbol* F_n *denotes the free group of rank* n (*here* n *is finite*) *and* $\mathrm{Out}(F_n)$ *denotes the outer automorphism group of* F_n.

For the proof of Theorem G, property (TT)/T for universal and symplectic universal lattices is one key. Counterpart are the study of quasi-cocycles on $\mathrm{MCG}(\Sigma)$ and $\mathrm{Out}(F_n)$; and subgroup classification for these groups. By combining our results with deep results of U. Hamenstädt [Ham], Bestvina–Bromberg–Fujiwara [BBF]; and McCarthy–Papadopoulos [McPa], and Handel–Mosher [HaMo], we establish the theorem.

In Theorem G, the restriction on m is optimal. Also, the case of $\mathrm{Out}(F_n)$ targets implies homomorphism rigidity with $\mathrm{Aut}(F_n)$ targets; and with $\mathrm{MCG}(\Sigma_{g,l})$ targets. Here $\Sigma_{g,l}$ denotes a compact oriented surface with g genus and l punctures, and here we assume $l \geq 1$. This is because these groups inject into $\mathrm{Out}(F_{n'})$ for sufficiently large n'. See Subsection 11.2.1.

It is worth making a remark that Theorem G for universal lattice cases can be deduced from much easier argument. In fact, we have the following theorem along that shortcut argument:

Theorem H. ([Mim4]; *Theorem* 11.6.4 *in this thesis*) *Let* Γ *be a finite index subgroup of noncommutaive universal lattice* $\mathrm{E}_m(\mathbb{Z}\langle x_1, \ldots, x_k\rangle)$ $(m \geq 3)$. *Then for any* $g \geq 0$ *and* $n \geq 2$, *every homomorphism* $\Phi \colon \Gamma \to \mathrm{MCG}(\Sigma_g)$ *and every homomorphism* $\Psi \colon \Gamma \to \mathrm{Out}(F_n)$ *have finite image.*

However at the moment, there seems to be a gap to extend the proof of Theorem H to symplectic universal lattice cases.

Therefore, Theorem G for symplectic universal lattices can be regarded as the high-end of this thesis. Theorem G together with Theorem H is *non-arithmetization* of Farb–Masur and Bridson–Wade theorems. Specially, in [FaSh], Farb and Shalen appealed to homomorphism rigidity for higher rank lattices with $\mathrm{Out}(F_n)$ target in order to obtain rigidity results on group actions on a 3-dimensional manifold. It may be possible our theorems give some extension of their results *beyond arithmetic lattice groups.*

Finally, in Appendix, we make an estimation of a generalization of relative Kazhdan constant to uniformly bounded representation cases (on Hilbert spaces) as follows:

Proposition I. ([Mim4]; *Proposition* I.0.5 *in this thesis*) *Let* $A_k = \mathbb{Z}[x_1, \ldots, x_k]$ *and set* $G = \mathrm{E}_2(A_k) \ltimes A_k^2$ *and* $N = A_k^2 \trianglelefteq G$. *Set* S *be the set of all unit elementary matrices in* G $(\subset \mathrm{SL}_3(A_k))$ *in the sense in Definition* 4.1.13. *Let* $M \geq 1$ *be a positive real. Then there is an inequality*

$$\overline{\mathcal{K}}(G, N; S; M) > (15k + 100)^{-1} M^{-6}.$$

In the case of $k = 0$, *one has* $\overline{\mathcal{K}}(SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; F; M) > (21M^6)^{-1}$. *Here the symbol* $\overline{\mathcal{K}}(G, N; S; M)$ *denotes the* generalized relative Kazhdan constant for uniformly bounded representations*, which is defined in Definition* 3.5.2.

Organization of this paper: Chapter 2 is for basic and fundamental facts on property (T) of Kazhdan. If the reader is familiar with this topic, this chapter can be omitted. Chapter 3 is on property (T_B) and property (F_B) of Bader–Furman–Gelander–Monod, and the reader also consult the original paper [BFGM]. In Chapter 4, we define elementary groups over rings, and universal lattices. Also we see a celebrated argument by Shalom to prove property (T) from bounded generation [Sha1]. Vaserstein's bounded generation [Vas2], stated there, is a powerful tool throughout this thesis. In Chapter 5, we introduce another effective tool, called *Shalom's machinery* [Sha5], [Mim1]. To obtain this machinery, examine ultraproducts of metric spaces and that of isometric actions. This study has important application to reduced group cohomology. Chapter 6 is devoted to study of property (F_B) for universal lattices, in which Theorem A is proven. In Chapter 7, we introduce property (FF_B), which is a generalization of property (TT), and see connection to bounded cohomology. In chapter 8, we introduce a notion of property (FF_B)/T, and prove Theorem B. Moreover we prove Theorem 8.1.7, which is a quite strong tool in deducing Property (FF_B)/T (, although the proof is very elementary). We show Theorem C with the aid of this. Chapter 9 is utilized for definitions of elementary symplectic groups, and symlectic universal lattices. There we prove Theorem D, with use of Theorem 8.1.7. In Chapter 10, we consider quasi-homomorphisms and stable commutator lengths. We prove Theorem E. Chapter 11 is devoted for the proofs of Theorem F and Theorem G. The proof of Theorem G is involved and requires a number of deep facts on mapping class groups and automorphism groups of free groups. We briefly see them and complete the proof of Theorem G. We also verify Theorem H. In Appendix, we give a proof of Proposition I.

Hereafter, unless otherwise stating, we use and keep the notation and convention as in Chapter 0.

Chapter 2

Kazhdan's property (T)

In this chapter, we collect basic facts on property (T) of D. A. Kazhdan [Kaz]. One main goal is to show the Delorme–Guichardet theorem, which states Kazhdan's property (T) is equivalent to a certain cohomological property called property (FH) for locally compact and σ-compact groups. First, we give definitions of those two properties and see some permanence properties, including heredity to lattices. The other goal is to show totally higher rank groups and lattices, such as $\mathrm{SL}_3(\mathbb{R})$ and $\mathrm{SL}_3(\mathbb{Z})$, enjoy property (T). Main references of this chapter are Chapter 1, Chapter2, Appendix C, and Appendix E in a book of Bekka–de la Harpe–Valette [BHV].

2.1 Original definition

Kazhdan's property (T) has two famous formulations, namely, the original definition of (T), and the definition of property (FH). In this section, we state the original definition of Kazhdan, and see some properties of groups with (T).

2.1.1 Definition

We say a bounded operator U on a Hilbert space $\mathfrak{H}$ is a *unitary* if U has a bounded inverse and if U preserves the inner product (, namely, for any $\xi, \eta \in \mathfrak{H}$ $\langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle$). Equivalently, if $U^*U = UU^* = I$, where I is the identity operator on $\mathfrak{H}$. The set $U(\mathfrak{H})$ of all unitary operators on $\mathfrak{H}$ becomes a group with composition. A unitary representation of a group G is a group homomorphism $G \to U(\mathfrak{H})$ on some Hilbert space $\mathfrak{H}$. Recall from Chapter 0 that we always assume a group G are locally compact and σ-compact, and a representation π is strongly continuous, namely, for every $\xi \in \mathfrak{H}$, the map $G \to \mathfrak{H}; g \mapsto \pi(g)\xi$ is continuous.

Definition 2.1.1. Let $(\pi, \mathfrak{H})$ be a unitary representation of a group G.

(*i*) For a subset $S \subseteq G$ and $\kappa > 0$, a vector $\xi \in \mathfrak{H}$ is said to be (S, κ)-*invariant* if

$$\sup_{s \in S} \|\xi - \pi(s)\xi\| < \kappa \|\xi\|.$$

(*ii*) We say π has *almost invariant vectors* if for any *compact* subset $S \subseteq G$ and $\kappa > 0$, there exists an (S, κ)-invariant vector.

(*iii*) We write $\pi \succeq 1_G$ and say π *weakly contains trivial representation* if π has almost invariant vectors.

Remark 2.1.2. (*i*) The zero vetor 0 is *not* (S, κ)-invariant for any $S \subseteq G$ and $\kappa > 0$ by definition.

(*ii*) Let $S \subseteq G$ and $\kappa > 0$ and $n \geq 2$. Then if ξ is (S, κ)-invarinant for a unitary representation $(\pi, \mathfrak{H})$, then ξ is $(\overline{S \cup S^{-1}}, \kappa)$-invariant and $(S^n, \kappa/n)$-invariant for π. (Here we refer to Chapter 0 for the definition of S^n.) Indeed, the first assertion is trivial because for any $g \in G$ and $\eta \in \mathfrak{H}$ $\|\eta - \pi(g)\eta\| = \|\pi(g^{-1})\eta - \eta\|$ holds. The second assertion follows from the following argument, for instance for $n = 2$: for any $g_1, g_2 \in G$ and any $\eta \in \mathfrak{H}$,

$$\begin{aligned}&\|\eta - \pi(g_1g_2)\eta\| = \|(\eta - \pi(g_1)\eta) + (\pi(g_1)\eta - \pi(g_1g_2)\eta)\| \\ \leq &\|\eta - \pi(g_1)\eta\| + \|\pi(g_1)\eta - \pi(g_1)\pi(g_2)\eta\| = \|\eta - \pi(g_1)\eta\| + \|\eta - \pi(g_2)\eta\|.\end{aligned}$$

Here recall $\pi(g_1)$ is a unitary.

From this point, we use this argument without mentioning. Note that this is available in a more general setting, in the case of (ρ, B) being an isometric representation on a Banach space.

We state the following basic observation, which is trivial in the case of discrete groups. We say a subset S of a group G is *generating* (or, S *generates* G) if $\bigcup_{n \in \mathbb{N}}(S \cup S^{-1})^n = G$. We say S is *symmetric* if $S = S^{-1}$. We say a group G is *compactly generated* (respectively, *finitely generated*) if there exists a compact (respectively finite) generating set.

Lemma 2.1.3. *Let G be a compactly generated group. Then there exists a symmetric compact generating set $\tilde{S}$ containing the group unit e. Moreover, for any compact subset $K \subseteq G$ there exists $m \in \mathbb{N}$ such that $K \subseteq \tilde{S}^m$.*

Proof. By assumption, there exists a compact generating set S. Hence one can set $\tilde{S} = \overline{S \cup S^{-1} \cup e}$. For the second assertion, observe that $G = \bigcup_{n \in \mathbb{N}} \tilde{S}^n$. With recalling that we always assume G is locally compact (with the Hausdorff property), we apply the Baire category theorem. Therefore there exists $n \in \mathbb{N}$ such that $\tilde{S}^n$ contains an open neighborhood U of e. By considering the open covering $K \subseteq \bigcup_{g \in K} gU$, we obtain a finite subcovering $K \subseteq \bigcup_{1 \leq i \leq l} g_iU$ (for some $l \in \mathbb{N}$). For each $1 \leq i \leq l$ there exists $n_i \in \mathbb{N}$ such that $g_i \in \tilde{S}^{n_i}$. Therefore, if one sets m as the maximum of n_i $(1 \leq i \leq l)$ and n, then this m works. ☐

By combining Lemma 2.1.3 and item (*ii*) of Remark 2.1.2, we obtain the following corollary.

Corollary 2.1.4. *Suppose a group G is compactly generated and let S be a compact generating set. Then for any unitary G-representation $(\pi, \mathfrak{H})$, π has almost invariant vectors if and only if for every $\kappa > 0$ there exists (S, κ)-invariant vectors.*

We shall see in Theorem 2.2.1 that every group with property (T) (see in Definition 2.1.5) is compactly generated.

Recall that we write $\pi \supseteq 1_G$ (and say π contains trivial representation) if 1_G is a subrepresentation of π, equivalently, if $\mathfrak{H}^{\pi(G)} \neq 0$ (namely, there exists a non-zero $\pi(G)$-invariant vector). For any unitary representation π, there is an implication "$\pi \supseteq 1_G \Rightarrow \pi \succeq 1_G$." We make a remark the converse "$\pi \succeq 1_G \Rightarrow \pi \supseteq 1_G$" is usually *false* for an infinite dimensional unitary representation π, as we will see in Example 2.1.6.

Definition 2.1.5. A group G is said to have *Kazhdan's property* (T) if for any unitary representation π of G, $\pi \succeq 1_G$ implies $\pi \supseteq 1_G$. Equivalently, if for any unitary representation π of G, whenever π has almost invariant vectors, π has a non-zero G-invariant vector.

Groups with (T) are also called *Kazhdan groups.*

Example 2.1.6. The group $\mathbb{Z}$ is *not* Kazhdan. More precisely, we claim $\lambda_{\mathbb{Z}} \succeq 1_{\mathbb{Z}}$, here $\lambda_{\mathbb{Z}}$ denotes the left regular representation $\mathbb{Z} \to U(\ell^2(\mathbb{Z}))$ (, namely, for $m \in \mathbb{Z}$ and $f \in \ell^2(\mathbb{Z})$, $\lambda(m)(f(n)) := f(-m+n)$.) Indeed, take a sequence of subsets $S_n := [-n, n] \cap \mathbb{Z}$ indexed by $n \in \mathbb{N}$. Then it is easy to see that for any $m \in \mathbb{Z}$,

$$\lim_{n \to \infty} \frac{|(m \cdot S_n) \triangle S_n|}{|S_n|} = 0.$$

Here $m\cdot$ means the left action of m as an element of the additive group $\mathbb{Z}$ (, namely, "$m+$"), $\triangle$ means the symmetric difference, and $|\cdot|$ means the number of sets. This means, for any finite set S and $\kappa > 0$, there exists $n \in \mathbb{N}$ such that for a unit vector $\xi := (|S_n|)^{-1/2} \cdot \chi_{S_n}$ in $\ell^2(\mathbb{Z})$ satisfies

$$\sup_{m \in S} \|\lambda_{\mathbb{Z}}(m)\xi - \xi\| < \kappa,$$

where χ is the characteristic function. This shows the claim. On the other hand, it is immediate that $\lambda_{\mathbb{Z}} \not\supseteq 1_{\mathbb{Z}}$.

This argument can be extended to the case of (locally compact) *amenable* groups, including all abelian groups, nilpotent groups, and solvable groups. For details of amenable groups, see Subsection 2.5.2.

2.1.2 Kazhdan constant

In the definition of Kazhdan's property (T), the choice of pairs (S, κ) may a priori depend on the choices of unitary representations π. The following lemma states one can take (S, κ) universal for π's: all unitary representations.

Lemma 2.1.7. *For a group G, the following are equivalent:*

(*i*) *The group G has* (T).

(*ii*) *There exist a* compact *subset $S \subseteq G$ and $\kappa > 0$ such that the following holds: for any unitary G-representation $(\pi, \mathfrak{H})$, whenever π has an (S, κ)-invariant vector, $\pi \supseteq 1_G$ holds.*

Proof. Condition (*ii*) implies condition (*i*) is trivial by definition. For the converse, we use a direct-sum argument. Suppose G does not satisfy condition (*ii*). Let I be the set of all pairs $\mu = (S, \kappa)$, where $S \subseteq G$ is compact and $\kappa > 0$, such that there exists a unitary G-representation $(\pi_\mu, \mathfrak{H}_\mu)$ with $\pi_\mu \not\supseteq 1_G$ which has a unit (S, κ)-invariant vector ξ_μ. Set $(\pi, \mathfrak{H}) := (\bigoplus_{\mu \in I} \pi_\mu, \bigoplus_{\mu \in I} \mathfrak{H}_\mu)$ be the ℓ^2-sum of the unitary representations. Then we claim that $\pi \succeq 1_G$ but that $\pi \not\supseteq 1_G$. Indeed, the first assertion follows from $\xi_\mu \in \mathfrak{H}_\mu$ being (S, κ)-invariant. For the second assertion, suppose $\eta = \bigoplus_{\mu \in I} \eta_\mu$ is $\pi(G)$-invariant. Then each $\eta_\mu \in \mathfrak{H}_\mu$ is $\pi_\mu(G)$-invariant. Hence all η_μ's are zero and $\eta = 0$. Thus G cannot have (T). $\square$

We mention that in some other literature (such as [BHV]), Kazhdan's property (T) is defined as condition (*ii*) in Lemma 2.1.7.

Remark 2.1.8. The symbol $\pi \succeq 1_G$ (weak containment) comes from the concept of the *Fell topology* on sets of unitary representations. This concept is in general defined in terms of *positive definite functions* (in other words, of diagonal matrix coefficients), see Definition 2.2.11. We will shortly treat positive definite functions in Subsection 2.4.1, and here we only mention the following fact (for the proof, see [BHV, Theorem 1.2.5]). For a group G, $\hat{G}$ denotes the *unitary dual* of G, namely, the set of all equivalence class of irreducible G-unitary representations.

Theorem 2.1.9. (Kazhdan [Kaz], Wang [Wan]) *For a group G, the following are all equivalent:*

(*i*) *The group G has property* (T).

(*ii*) *The trivial representation 1_G is isolated in $\hat{G}$ with respect to the Fell topology.*

(*iii*) *Every finite dimensional irreducible unitary representation is isolated in $\hat{G}$ with respect to the Fell topology.*

The very original definition of (T) in [Kaz] is condition (*ii*) in Theorem 2.1.9. The terminology "(T)" was named after this condition that the trivial representation , "T", is isolated , "( )."

Definition 2.1.10. Let G be a group.

(i) For $S \subseteq G$ and $\kappa > 0$, we say the pair (S, κ) is a *Kazhdan pair* for G if the following holds true: for any unitary G-representation π, whenever π has an (S, κ)-invariant vector, $\pi \supseteq 1_G$ holds.

(ii) For a compact subset $S \subseteq G$, the *Kazhdan constant* $\mathcal{K}(G; S)$ is defined as the supremum of $\kappa > 0$ such that (S, κ) is a Kazhdan pair for G. If there is no such $\kappa > 0$, we set $\mathcal{K}(G; S) = 0$.

In other words, we define

$$\mathcal{K}(G; S) := \inf_{(\pi, \mathfrak{H})} \inf_{\xi \in S(\mathfrak{H})} \sup_{s \in S} \|\xi - \pi(s)\xi\|,$$

where $(\pi, \mathfrak{H})$ moves among all unitary G-representations *with* $\pi \not\supseteq 1_G$, and $S(\mathfrak{H})$ denotes the unit sphere of $\mathfrak{H}$ (recall Chapter 0).

As a corollary of Lemma 2.1.7, we have the following:

Corollary 2.1.11. *A group G has* (T) *if there exists a compact subset S such that $\mathcal{K}(G; S) > 0$.*

One importance of Kazhdan constants is due to the following lemma. This states that if G has (T) and (S, κ) is a Kazhdan pair, then for any unitary G-representation and for any vector, how close to being G-invariant is controlled uniformly by a behavior on S.

Lemma 2.1.12. *Let G be a Kazhdan group and $S \subseteq G$ be a compact subset such that $\mathcal{K} = \mathcal{K}(G; S) > 0$. Let $\epsilon > 0$. Then for any unitary G-representation $(\pi, \mathfrak{H})$, if $\xi \in \mathfrak{H}$ is (S, ϵ)-invariant, then there exists a $\pi(G)$-inavarinat vector ξ_0 such that*

$$\|\xi - \xi_0\| < \mathcal{K}^{-1}\epsilon\|\xi\|.$$

In particular, there is an inequality:

$$\textit{for any } g \in G,\ \|\xi - \pi(g)\xi\| < 2\mathcal{K}^{-1}\epsilon\|\xi\|.$$

Proof. The point here is there is the following canonical decomposition of $\mathfrak{H}$, *as G-representation spaces*: $\mathfrak{H} = \mathfrak{H}^{\pi(G)} \oplus (\mathfrak{H}_{\pi(G)})^{\perp}$. Here $\mathfrak{H}^{\pi(G)}$ is the space of $\pi(G)$-invariant vectors (see Chapter 0), and $(\mathfrak{H}_{\pi(G)})^{\perp}$ denotes the orthogonal complement of $\mathfrak{H}^{\pi(G)}$. The G-invariance of $(\mathfrak{H}_{\pi(G)})^{\perp}$ follows from the unitary of representation π. Note that the following restriction of π

$$\pi' \colon G \to U((\mathfrak{H}_{\pi(G)})^{\perp})$$

satisfies $\pi' \not\supseteq 1_G$.

Decompose $\xi = \xi_0 + \xi_1$ such that $\xi_0 = (I - P)\xi \in \mathfrak{H}^{\pi(G)}$ and $\xi_1 = P\xi \in (\mathfrak{H}_{\pi(G)})^{\perp}$. Here P is the (orthogonal) projection $P \in \mathbb{B}(\mathfrak{H})$ onto $(\mathfrak{H}_{\pi(G)})^{\perp}$. By construction, for any $g \in G$, $\xi - \pi(g)\xi = \xi_1 - \pi'(g)\xi_1$ holds, and we have

$$\sup_{s \in S} \|\xi_1 - \pi'(s)\xi_1\| < \epsilon \|\xi\|$$

by assumption. On the other hand, by the definition of the Kazhdan constant, we have

$$\sup_{s \in S} \|\xi_1 - \pi'(s)\xi_1\| \geq \mathcal{K}\|\xi_1\|.$$

Thus we conclude $\|\xi_1\| < \mathcal{K}^{-1}\epsilon\|\xi\|$, as desired. For the second part, we have for any $g \in G$,

$$\|\xi - \pi(g)\xi\| = \|\xi_1 - \pi'(g)\xi_1\| \leq 2\|\xi_1\| < 2\mathcal{K}^{-1}\epsilon\|\xi\|.$$

□

In last part of this subsection, we see trivial examples of Kazhdan groups, more precisely, compact groups. We need the following lemma, but we shall prove this in more general setting in Chapter 3 (see Lemma 3.1.8).

Lemma 2.1.13. *Let X be a* bounded *subset in a Hilbert space $\mathfrak{H}$. Then there exists a* unique *closed ball with the minimum radius which includes X. We define the* Chebyshev center *of X as the center of this ball.*

Corollary 2.1.14. *For any group G, $(G, 1)$ is a Kazhdan pair.*
In particular, every compact group has property (T)*.*

There is a remark: it is known that in fact $(G, \sqrt{2})$ is a Kazhdan pair for G. For the proof, see [BHV, Proposition 1.1.5].

Proof. Suppose a unitary G-representation $(\pi, \mathfrak{H})$ has $(G, 1)$-invariant vector ξ. Consider the orbit $X = \pi(G)\xi$, which is a bounded subset in $\mathfrak{H}$. Lemma 2.1.13 applies and there exists the Chebyshev center η of X. By uniqueness of η and unitary of π, $\eta \in \mathfrak{H}^{\pi(G)}$. Finally, $\eta \neq 0$ holds because $\sup_{\zeta \in X} \|\zeta - \xi\| < \sup_{\zeta \in X} \|\zeta - 0\| = \|\xi\|$ by assumption. □

2.1.3 Relative (T)

The following relative version of property (T) was explicitly introduced by G. A. Margulis [Mar1] for an application to expander graphs (see Section 2.7). Now this concept is known to play an important role, as well as the original property (T).

Definition 2.1.15. Let G be a group and $N \leqslant G$ be a subgroup.

(i) The pair $G \geqslant N$ is said to have *relative property* (T) if the following condition is satisfied: for any unitary G-representation $(\pi, \mathfrak{H})$, whenever $\pi \succeq 1_G$, $\mathfrak{H}^{\pi(N)} \neq 0$ holds.

(ii) A pair (S, κ) ($S \subseteq G$ and $\kappa > 0$) is called a *relative Kazhdan pair* for $G \geqslant N$ if the following holds: for any unitary G-representation $(\pi, \mathfrak{H})$, whenever π has an (S, κ)-invariant vector, $\mathfrak{H}^{\pi(N)} \neq 0$ holds.

(iii) Let S be a compact subset of G. The *relative Kazhdan constant* for $G \geqslant N$, written as $\mathcal{K}(G, N; S)$, is defined as the supremum of κ such that (S, κ) is a relative Kazhdan pair for $G \geqslant N$. In other words,

$$\mathcal{K}(G, N; S) := \inf_{(\pi,\mathfrak{H})} \inf_{\xi \in S(\mathfrak{H})} \sup_{s \in S} \|\xi - \pi(s)\xi\|,$$

where $(\pi, \mathfrak{H})$ moves among all unitary G-representations *with* $\mathfrak{H}^{\pi(N)} = 0$.

We mention that a group G has (T) if and only if $G \geqslant G$ has relative (T). If $G \geqslant G_0 \geqslant N \geqslant N_0$, then relative (T) for $G_0 \geqslant N$ implies that for $G \geqslant N_0$ (in particular, by Corollary 2.1.14, for any group G and any compact subgroup N, $G \geqslant N$ has relative (T)). The next lemma can be shown in a similar way to one in Lemma 2.1.7.

Lemma 2.1.16. *For a group pair $N \leqslant G$, the following are equivalent:*

(i) *The pair $G \geqslant N$ has relative* (T).

(ii) *There exists a compact subset $S \subseteq G$ such that $\mathcal{K}(G, N; S) > 0$.*

In the following lemma, a priori the assumption of the *normality* of the subgroup N is necessary.

Lemma 2.1.17. *Let G be a group and $N \trianglelefteq G$ be a* normal *subgroup. Suppose $G \trianglerighteq N$ has relative* (T) *and $S \subseteq G$ be a compact subset such that $\mathcal{K} = \mathcal{K}(G, N; S) > 0$. Let $\epsilon > 0$. Then for any unitary G-representation $(\pi, \mathfrak{H})$, if $\xi \in \mathfrak{H}$ is (S, ϵ)-invariant, then there exists a $\pi(N)$-invariant vector ξ_0 such that*

$$\|\xi - \xi_0\| < \mathcal{K}^{-1}\epsilon\|\xi\|.$$

In particular, there is an inequality:

$$\text{for any } h \in N,\ \|\xi - \pi(h)\xi\| < 2\mathcal{K}^{-1}\epsilon\|\xi\|.$$

Proof. We imitate the proof of Lemma 2.1.12. The point here is the decomposition $\mathfrak{H} = \mathfrak{H}^{\pi(N)} \oplus (\mathfrak{H}_{\pi(N)})^{\perp}$ is a decomposition as *G-representation spaces*, provided N is normal. This follows from an equality

$$\pi(h)\pi(g)\xi = \pi(g)\pi(g^{-1}hg)\xi = \pi(g)\xi$$

for any $h \in N$, $g \in G$, and any $\xi \in \mathfrak{H}^{\pi(N)}$. Also, observe that the orthogonal projection $P \in \mathbb{B}(\mathfrak{H})$ onto $\mathfrak{H}^{\pi(N)}$ has the operator norm ≤ 1, namely, for any $\eta = \eta_0 + \eta_1$ with $\eta_1 = P\eta$, $\|\eta_1\| \leq \|\eta\|$ holds (this is trivial). Decompose ξ as $\xi = \xi_0 + \xi_1$, where $\xi_0 = (I - P)\xi \in \mathfrak{H}^{\pi(N)}$ and $\xi_1 = P\xi \in (\mathfrak{H}_{\pi(N)})^{\perp}$. Then we have

$$\begin{aligned} \epsilon\|\xi\| &> \sup_{s\in S} \|\xi - \pi(s)\xi\| \\ &\geq \sup_{s\in S} \|P(\xi - \pi(s)\xi)\| = \sup_{s\in S} \|\xi_1 - \pi(s)\xi_1\| \end{aligned}$$

(here P and $\pi(G)$ commute because $\mathfrak{H}^{\pi(N)}$ is $\pi(G)$-invariant), and

$$\sup_{s\in S} \|\xi_1 - \pi(s)\xi_1\| \geq \mathcal{K}(G, N; S)\|\xi_1\|.$$

These inequalities end our proof. □

Remark 2.1.18. This direct proof of Lemma 2.1.17 deeply relies on the assumption of N being normal. However with the aid of the Delorme–Guichardet theorem (see Section 2.4 and Theorem 2.4.13), it is possible to obtain a similar result for general cases (more precisely, we use the fact that relative property (T) is equivalent to relative property (FH) (Theorem 2.4.15), and that for the definition of relative (FH) there is no need to consider whether the subgroup is normal). For detailed arguments, we refer readers to a paper of P. Jolissaint [Jol].

One of the most important examples of group pairs with relative (T) is the pair $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$. More precisely, these are identified with

$$\left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{SL}_2(\mathbb{Z}),\ v \in \mathbb{Z}^2 \right\} \quad \trianglerighteq \quad \left\{ \left(\begin{array}{c|c} I_2 & v \\ \hline 0 & 1 \end{array} \right) : v \in \mathbb{Z}^2 \right\}$$

(see Chapter 0).

We also note that $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ itself does *not* have property (T). For the proofs of having relative (T), and not having (T), see respectively Subsection 2.6.1 (and Subsection 4.2.1 for a quantitative proof), and Subsection 2.5.3.

2.2 Induction and permanence properties

In this section, we collect some consequences and permanence properties concerning (T). In particular, we see that having (T) implies compact generation and compact abelianization, and that (T) is inherited to lattices. For the proof of the last assertion, inductions of unitary representation play an important role.

2.2.1 Compact generation and compact abelianization

The following proposition was one of motivations of Kazhdan to introduce property (T).

Theorem 2.2.1. *Property* (T) *implies compact generation. In particular, a discrete Kazhdan group is finitely generated.*

Proof. Suppose G is a Kazhdan group. Let I be the set of all open and compactly generated subgroups of G. Observe that $G = \bigcup_{H \in I} H$ because every element in G has a compact neighborhood (we always assume local compactness) and because the subgroup generated by a set containing an open (non-empty) subset is open. For every $H \in I$, since $H \leqslant G$ is open, we regard G/H as a discrete set. Set $\lambda_{G/h} \colon G \to U(\ell^2(G/H))$ the quasi-regular representation, namely, $\lambda_{G/H}(g)(f(xH)) := f(g^{-1}xH)$. Set $\delta_H \in \ell^2(G/H)$ the Dirac function at the point H. Let $\pi = \bigoplus_{H \in I} \lambda_{G/H}$ be the ℓ^2 sum of these representations.

We then claim $\pi \succeq 1_G$. Indeed, for a compact subset S of G, let $K \in I$ be the group generated by a relatively compact open neighborhood of S. Then $\sup_{s \in S} \|\delta_K - \pi(s)\delta_K\| = 0$, where we regard $\delta_K \in \ell^2(G/K)$ as an element of $\bigoplus_{H \in I} \ell^2(G/H)$.

Through property (T) for G, there exists a non-zero $\pi(G)$-invariant vector $\xi = \bigoplus_{H \in I} \xi_H$, and in particular there exist $H \in I$ and $\xi_H \in \ell^2(G/H)$ such that $\xi_H \neq 0$ and is $\lambda_{G/H}(G)$-invariant. This implies G/H is finite, and therefore G itself must be compactly generated. □

Thanks to Theorem 2.2.1, a group G has property (T) if and only if for some compact *generating* set $S_0 \subseteq G$, $\mathcal{K}(G; S_0) > 0$ holds. By Lemma 2.1.3, in this case if a compact subset $S \subseteq G$ is a *generating* set, then $\mathcal{K}(G; S) > 0$ holds. The following proposition can be seen as a converse of this.

Proposition 2.2.2. *Suppose a group G is Kazhdan. Let $S \subseteq G$ be a compact subset with a non-empty interior. Then if $\mathcal{K}(G; S) > 0$, then S generates G.* (*More precisely, $S \cup S^{-1}$ generates G.*)

In particular, every finite subset S of a discrete Kazhdan group G satisfying $\mathcal{K}(G; S) > 0$ is a generating set.

Proof. Let H be the group generated by S. By assumption, H is an open subgroup of G. Hence one can consider the quasi-regular representation $\lambda_{G/H} \colon G \to U(\ell^2(G/H))$. The Dirac function δ_H at H in $\ell^2(G/H)$ is $\lambda_{G/H}(H)$-invariant, and Lemma 2.1.12 shows this is in fact $\lambda_{G/H}(G)$-invariant. Therefore $G = H$. □

Remark 2.2.3. In Proposition 2.2.2, the assumption of S having a non-empty interior cannot be omitted. Indeed, Shalom shows in [Sha4], the following finite subset S of $G = \mathrm{SL}_3(\mathbb{R})$ satisfies $\mathcal{K}(G; S) > 0$:

$$\left\{ E_{1,2}(2) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_{2,1}(2) = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

(recall the definition of $E_{i,j}(r)$ in Chapter 0).

Remark 2.2.4. Kazhdan [Kaz] asked whether a discrete Kazhdan group is necessarily finitely presented. The answer is negative, and the first counterexample was provided by Margulis [Mar2]. Recall from Chapter 0 we use the symbol q for a positive power of a prime number, and $\mathbb{F}_q$ denotes the finite field of order q. It is well-known that the field $\mathbb{K} = \mathbb{F}_q((x))$ of Laurent series is a local field with respect to the absolute value $|\sum_{i=m}^{\infty} a_i x^i| := e^{-m}$ with $a_m \neq 0$. The subring $\mathbb{F}_q[x^{-1}]$, which is isomorphic to $\mathbb{F}_q[x]$ as discrete rings, is an ring of integers of the field $\mathbb{K}$. Therefore, $\mathrm{SL}_3(\mathbb{F}_q[x^{-1}])$, which is isomorphic to $\mathrm{SL}_3(\mathbb{F}_q[x])$ as discrete groups. is a lattice in $\mathrm{SL}_3(\mathbb{K})$. It shall be shown respectively in Subsection 2.6.2; and Subsection 2.2.2 that for any local field $\mathbb{K}$, $\mathrm{SL}_{m\geq 3}(\mathbb{K})$ has (T); and (T) passes to lattices. Therefore, the discrete group $\mathrm{SL}_3(\mathbb{F}_q[x])$ has (T). However, H. Behr shown in [Beh] that $\mathrm{SL}_3(\mathbb{F}_q[x])$ is *not* finitely presented.

Another example along this line is a universal lattice $\mathrm{SL}_3(\mathbb{Z}[x])$, but the proof of property (T) for this group is much involved, as we mentioned in Chapter 1.

Concerning with this direction, we see a theorem of Shalom [Sha2] in Subsection 5.2.3.

Next we will see property (T) passes to group quotients.

Lemma 2.2.5. *Let G_1, G_2 be groups, and $\phi\colon G_1 \to G_2$ be a continuous homomorphism with dense range. If G_1 has* (T)*, then so does G_2.*

In particular, property (T) *passes to group quotients.*

Proof. This proof is very easy and uses a pull-back argument. Suppose (S_1, κ) is a Kazhdan pair for G_1 with S_1 compact. Then one can easily check that $(\phi(S_1), \kappa)$ is a Kazhdan pair for G_2 (with $\phi(S_1)$ compact), by composing ϕ and a unitary G_2-representation. □

Corollary 2.2.6. *A Kazhdan group has compact abelianization. More precisely, for any Kazhdan group G, its abelianization $G/\overline{[G,G]}$ is compact. Here $[G,G]$ denotes the commutator subgroup of G, namely, the* (*normal*) *subgroup generated by the set of single commutators $\{[g,h] : g, h \in G\}$* (*see Chapter* 0)*.*

Proof. As we mentioned in Example 2.1.6, any noncompact abelian group cannot have property (T). The group quotient $G/\overline{[G,G]}$ is by definition abelian, and hence Lemma 2.2.5 ends the proof. □

Finally, we see the permanence property under group extensions:

Proposition 2.2.7. *For a group G and a normal subgroup $N \trianglelefteq G$, the following are equivalent:*

(*i*) *The group G has* (T)*.*

(*ii*) *The group G/N has* (T)*, and the pair $G \trianglerighteq N$ has relative* (T)*.*

In particular, property (T) *is stable under group extensions.*

Proof. Condition (*i*) implies condition (*ii*) is trivial, from Lemma 2.2.5.

To show the converse, firstly, we recall the following: *If S' is a compact subset of G/N, then there exists a compact subset S of G such that $\phi(S) = S'$, where $\phi\colon G \twoheadrightarrow G/N$ is the canonical projection.* Note that here we need not the assumption of normality of N, but the assumption of G being locally compact is necessary.

Hence by definition, there exist a relative Kazhdan pair (S_1, κ_1) for $G \trianglerighteq N$, a Kazhdan pair (S', κ_2) for G/N, and a compact subset S_2 of G such that S_1, S' are compact and $\phi(S_2) = S'$. We claim $(S, \kappa) := (S_1 \cup S_2, \min_{i=1,2} \kappa_i/2)$ is a Kazhdan pair for G.

Indeed, let $(\pi, \mathfrak{H})$ be a unitary G-representation with a unit (S, κ)-invariant vector. Then by applying Lemma 2.1.17 with the pair being $G \trianglerighteq N$ and $\epsilon = \kappa$, we have

$$\|P\xi\| < \mathcal{K}(G, N; S)^{-1}\kappa \leq \kappa_1^{-1}\kappa \leq 1/2.$$

Here $P \in \mathbb{B}(\mathfrak{H})$ is the orthogonal projection onto $(\mathfrak{H}_{\pi(N)})^{\perp}$. (Recall that since $G \trianglerighteq N$, this projection P commutes with $\pi(G)$.) Set $Q = I - P$ the orthogonal projection onto $\mathfrak{H}^{\pi(N)}$. Then $\|Q\xi\| \geq 1/2$, and

$$\begin{aligned}\sup_{s \in S} \|Q\xi - \pi(s)Q\xi\| &= \sup_{s \in S} \|Q(\xi - \pi(s)\xi)\| \\ \leq \sup_{s \in S} \|\xi - \pi(s)\xi\| &< \kappa \leq \kappa_2/2 \leq \kappa_2 \|Q\xi\|.\end{aligned}$$

Hence $Q\xi$ is (Q, κ_2)-invariant. Now note that the restriction of π to $\mathfrak{H}^{\pi(N)}$ factors through a unitary representation of G/N. From the fact that (Q', κ_2) is a Kazhdan pair for G/N, a standard pull-back argument then shows there must exists a non-zero $\pi(G)$-invariant vector. This verifies our claim, and therefore G has (T).

The final assertion follows from the trivial observation that if N has (T), then for any $\Gamma \geqslant N$, the pair $\Gamma \geqslant N$ has relative (T). $\square$

Remark 2.2.8. As in Proposition 2.2.7, one can*not* expect there that N itself is Kazhdan. One example is $\mathrm{SL}_m(\mathbb{R}) \ltimes \mathbb{R}^m$ with $m \geq 3$. Although this group has (T) (see Subsection 2.6.2), a normal subgroup $\mathbb{R}^m$ is not Kazhdan.

2.2.2 Heredity to lattices

Firstly, recall the definition of lattices: a subgroup Γ in a group G (recall we always assume local compactness and σ-compactness of groups and closeness of subgroups) is called a *lattice* if it is discrete and carries a *finite invariant* regular Borel measure. If moreover G/Γ is compact, then we say the lattice Γ is *cocompact* (or, *uniform*). The main result in this subsection is that property (T) passes to lattices. To see this, we need some preliminaries on inductions of unitary representations.

Definition 2.2.9. Let G be a group, $H \leqslant G$ be a subgroup. Suppose G/H carries an invariant measure μ from a left Haar measure of G. Then for a unitary representaion $(\sigma, \mathfrak{K})$, we defined the *induced representation* of G by σ, written as $\mathrm{Ind}_H^G \sigma$, as follows:

- The representation space $\mathfrak{H}_\mu$ is the (Hilbert) space of all measurable mappings $\xi\colon G \to \mathfrak{K}$ such that

  (*i*) For any $h \in H$ and almost all $x \in G$, $\xi(xh) = \sigma(h^{-1})\xi(x)$.

  (*ii*) The norm $\|\xi\|$ is finite. Here the inner product on $\mathfrak{H}_\mu$ is defined by

  $$\langle \xi_1 | \xi_2 \rangle := \int_{G/H} \langle \xi_1(x) | \xi_2(x) \rangle d\mu(xH).$$

  Here $\langle \cdot | \cdot \rangle$ in the right hand side is the inner product on $\mathfrak{K}$.

- The unitary representation $\mathrm{Ind}_H^G \sigma$ on $\mathfrak{H}_\mu$ is defined by:

  $$\mathrm{Ind}_H^G \sigma(g)(\xi(x)) := \xi(g^{-1}x).$$

For instance, with the assumptions in Definition 2.2.9, $\mathrm{Ind}_H^G 1_H$ is the quasi-regular representation $\lambda_{G/H}$ on $L^2(G/H, \mu)$. that means, $\lambda_{G/H}(g)(\xi(x)) :=\xi(g^{-1}x)$

We note that here we assume G/H carries an invariant measure (and also σ-compactness of groups, as mentioned above), and that in general setting the definition of induced representation is much more involved. We refer to Appendix E of [BHV] for this topic.

We need the following proposition on containment of trivial representation:

Lemma 2.2.10. *Under the assumptions of Definition* 2.2.9*, let σ be a unitary H-representation. Then the following are equivalent:*

(*i*) *The induced representation* $\mathrm{Ind}_H^G \sigma$ *contains* 1_G*.*

(*ii*) *The representation σ contains* 1_H*.*

Proof. In this setting, the proof is very easy. First suppose condition (*ii*) and take a non-zero invariant vector $\eta \in \mathfrak{K}$. Then $\xi\colon G \to \mathfrak{K}$ defined as $\xi(x) = \eta$ (constant function) is in $\mathfrak{H}_\mu$ by assumption, and is a non-zero $\mathrm{Ind}_H^G \sigma(G)$-invariant vector.

Hence condition (i) holds. Conversely, suppose condition (i). Take a non-zero $\mathrm{Ind}_H^G\sigma(G)$-invariant vector $\xi \in \mathfrak{H}_\mu$. Then from G-invariance, for any $x \in G$ $\xi(x) = \xi(e)$ holds, and $\xi(e) \in \mathfrak{K}$ is a non-zero $\sigma(H)$-invariant vector. Hence condition (ii) holds. $\square$

Note that in Lemma 2.2.10, without the assuption that G/H carries a finite invariant measure, it is true that condition (i) is equivalent to G/H carrying a finite invariant measure and having condition (ii). The proof of this is much harder, see Theorem E.3.1 in [BHV].

Next we need general definitions of weak containments among unitary representations. Recall for a unitary representation $(\pi, \mathfrak{H})$ of a group G, a function $G \to \mathbb{C}$ of the form $g \mapsto \langle \pi(g)\xi|\eta\rangle$ (for some ξ and η in $\mathfrak{H}$) is called a *matrix coefficient* of π. If in addition we can take $\xi = \eta$, this f is called a *diagonal matrix coefficient* of π.

Definition 2.2.11. Let G be a group and $(\pi, \mathfrak{H})$ and $(\sigma, \mathfrak{K})$ be two unitary representations of G. We say σ *weakly contains* π, wite as $\pi \preceq \sigma$, if every diagonal matrix coeffient of π can be approximated, uniformly on compact subsets of G, by finite sums of diagonal matrix coefficients of ρ. More precisely, if the following holds true: for every $\xi \in \mathfrak{H}$, every compact subset $S \subseteq G$, and $\epsilon > 0$, there exist $n \in \mathbb{N}$ and $\eta_1, \dots, \eta_n \in \mathfrak{K}$ such that

$$\sup_{s\in S}\left|\langle \pi(s)\xi|\xi\rangle - \sum_{i=1}^{n}\langle \sigma(s)\eta_i|\eta_i\rangle\right| < \epsilon.$$

Remark 2.2.12. (i) The following three observations are easy: firstly, in Definition 2.2.11, one only has to check the condition under the assumption on ξ that $\|\xi\| = 1$. Secondly, $\pi \subseteq \sigma$ implies $\pi \preceq \sigma$. Thirdly, the weak containment relation is transitive, namely, for unitary representations π_1, π_2, π_3 of a group, if $\pi_1 \preceq \pi_2$ and $\pi_2 \preceq \pi_3$, then $\pi_1 \preceq \pi_3$.

(ii) Recall that in Definiiton 2.1.1, we introduce the symbol $1_G \preceq \pi$ for the existence of almost invariant vectors for π. Here we see this definition is equivalent to the weak containment of trivial representation in the sense of Definition 2.2.11. Indeed, firstly observe for any unit vectors η_1, η_2 in a Hilbert space $\mathfrak{H}$,

$$\|\eta_1 - \eta_2\|^2 = 2\mathrm{Re}(1 - \langle \eta_1|\eta_2\rangle).$$

Let $(\pi, \mathfrak{H})$ be a unitary representation of a group G. Suppose π has almost invariant vectors. Then for any compact subset $S \subseteq G$ and $\epsilon > 0$, there exists a unit (S, ϵ)-invariant vector η for π by assumption. Then by letting $\eta_1 = \pi(s)\eta$ $(s \in S)$ and $\eta_2 = \eta$ in the equality above, we have

$$\sup_{s\in S}\mathrm{Re}(1 - \langle \pi(s)\eta|\eta\rangle) < \epsilon^2/2.$$

By observing $\langle \pi(s)\eta|\eta\rangle$ sits on the unit disc $\{|z| \leq 1\}$ of $\mathbb{C}$, we conclude

$$\sup_{s\in S} |1 - \langle \pi(s)\eta|\eta\rangle| < \epsilon.$$

This means π weakly contains 1_G. The converse is also straightforward.

The following fact is the key to proof of heredity for property (T) to lattices. Note that Theorem 2.2.13 holds without the assumption of G/H carrying a finite invariant measure.

Theorem 2.2.13. (*Continuity of inductions*) *With the assumptions in Definition* 2.2.9*, let* $(\sigma, \mathfrak{K}_\sigma)$*,* $(\tau, \mathfrak{K}_\tau)$ *be two unitary* H*-representations. If* $\sigma \preceq \tau$*, then* $\mathrm{Ind}_H^G\sigma \preceq \mathrm{Ind}_H^G\tau$ *holds.*

For the proof, see Theorem F.3.5 in [BHV].

Theorem 2.2.14. *Let* G *be a group and* $\Gamma \subseteq G$ *be a lattice. Then if* G *has* (T)*, then so does* Γ*. In particular, property* (T) *for discrete groups is inherited to finite index subgroups.*

Proof. Suppose σ is a unitary Γ-representation with $1_\Gamma \preceq \sigma$. Then by Theorem 2.2.13, $\mathrm{Ind}_\Gamma^G 1_\Gamma \preceq \mathrm{Ind}_\Gamma^G\sigma$. Since G/Γ carries a finite invariant measure, $\mathrm{Ind}_\Gamma^G 1_\Gamma \supseteq 1_G$. From item (i) of Remark 2.2.12, we have

$$1_G \preceq \mathrm{Ind}_\Gamma^G\sigma.$$

Through property (T) for G, this implies

$$1_G \subseteq \mathrm{Ind}_\Gamma^G\sigma.$$

Finally, by Lemma 2.2.10, we have $1_\Gamma \subseteq \sigma$. Therefore, Γ has property (T). □

Remark 2.2.15. (i) The relative version of the theorem is also ture. Namely, the following holds: *"let* G *be a group,* $H \leqslant G$ *be a subgroup, and* $\Gamma \leqslant G$ *be a lattice. Then if* $G \geqslant H$ *has relative* (T)*, then* $(G \cap \Gamma) \leqslant (H \cap \Gamma)$ *has relative* (T)*."* For the proof, simply imitate the proof of Theorem 2.2.14.

(ii) In fact the following hold true: *"let* G *be a group and* $H \subseteq G$ *be a subgroup* (*we do not assume discreteness*) *with* G/H *carrying a finite invariant Borel regular measure. Then* G *has* (T) *if and only if* H *has* (T)*."* For the proof of this, see Theorem 1.7.1 in [BHV].

2.3 Property (FH)

In this section, we shortly introduce *property* (FH), which happens to be equivalent to property (T), in relation to first group cohomology.

2.3.1 Affine isometric actions and first group cohomology

Definition 2.3.1. An *affine isometric action* α of a group G on a Hilbert space $\mathfrak{H}$ is a group action of G of the form:

$$\text{for any } g \in G \text{ and for any } \xi \in \mathfrak{H}, \quad \alpha(g) \cdot \xi = \pi(g)\xi + c(g).$$

Here π is a unitary representation of G on H, and $c \colon G \to \mathfrak{H}$ be a map.

We call π the *linear part* of α, and c the *transition part* of α.

Recall from Chapter 0 we always assume (affine isometric) actions on Banach spaces are strongly continuous. In Definition 2.3.1, this corresponds to the assumptions that π is strongly continuous and c is continuous.

We note that in Definition 2.3.1, there is a constraint on the transition part $c \colon G \to \mathfrak{H}$ as follows. The condition of α being a group action means the following: for any $g, h \in G$ and any $\xi \in \mathfrak{H}$,

$$\alpha(gh) \cdot \xi = \alpha(g) \cdot (\alpha(h) \cdot \xi)$$

holds. This is equivalent to the following constraint of the transition part c (and π being a group representation):

$$\text{for any } g, h \in G, \;\; c(gh) = c(g) + \pi(g)c(h).$$

This is a (1-)cocycle relation, called the *cocycle identity*, with unitary G-coefficient $(\pi, \mathfrak{H})$. We shall regard an affine isometric action α (of a group G on a Hilbert space H) as a "trivial" affine isometric action if α has a global fixed point, namely, if there exists a vector $\eta \in \mathfrak{H}$ such that

$$\text{for all } g \in G, \;\; \alpha(g) \cdot \eta = \eta.$$

In the view of Definition 2.3.1, this is equivalent to the existence of $\eta \in \mathfrak{H}$ such that

$$\text{for all } g \in G, \;\; c(g) = \eta - \pi(g)\eta,$$

where π and c are respectively the linear part and the transition part of α. The condition above on c is a (1-) coboundary relation with unitary G-coefficient $(\pi, \mathfrak{H})$ (we mention that here we allow the case of $\eta = 0$. Do not confuse with linear representation cases, in which we consider *non-zero* invariant vectors).

To sum up, we have come up with the following definitions and proposition:

Definition 2.3.2. Let G be a group, and $(\pi, \mathfrak{H})$ be a unitary G-representation.

(i) A continuous map $c \colon G \to \mathfrak{H}$ is called a *π-1-cocycle* (or shortly, a *π-cocycle*) if the following holds:

$$\text{For any } g, h \in G, \;\; c(gh) = c(g) + \pi(g)c(h).$$

This equality is called the *cocycle identity*.

(*ii*) A continuous map $c\colon G \to \mathfrak{H}$ is called a *π-1-coboundary* (or shortly, a *π-coboundary*) if the following holds:

$$\text{There exists } \xi \in \mathfrak{H} \text{ such that for any } g \in G,\ \ c(g) = \xi - \pi(g)\xi.$$

(*iii*) The space $Z^1_{\mathrm{c}}(G;\pi,\mathfrak{H})$ (or shortly, $Z^1_{\mathrm{c}}(G;\pi)$) denotes the vector space of all π-cocycles. The space $B^1_{\mathrm{c}}(G;\pi,\mathfrak{H})$ (or shortly, $B^1_{\mathrm{c}}(G;\pi)$) denotes the vector space of all π-coboundaries, which is a subspace of $Z^1_{\mathrm{c}}(G;\pi)$. The quotient vector space

$$H^1_{\mathrm{c}}(G;\pi,\mathfrak{H}) := Z^1_{\mathrm{c}}(G;\pi)/B^1_{\mathrm{c}}(G;\pi)$$

(or shortly, $H^1_{\mathrm{c}}(G;\pi)$) is called the *first cohomology group* with π-coefficient.

Proposition 2.3.3. *Let G be a group. Then for any Hilbert space $\mathfrak{H}$, there is a one-to-one transposependonce* (*as sets*) *between affine isometric actions α on $\mathfrak{H}$ and pairs (π, c) of unitary G-representation π on $\mathfrak{H}$ and π-cocycles c. The correspondence is:*

$$\alpha(g)\cdot\xi = \pi(g)\xi + c(g) \quad (g \in G,\ \xi \in \mathfrak{H}).$$

For any unitary G-representation $(\pi, \mathfrak{H})$, there is a one-to-one correspondence (*as sets*) *between affine isometric actions, up to conjugation by a translation, with linear part π; and elements in $H^1_{\mathrm{c}}(G;\pi)$. In particular, $H^1_{\mathrm{c}}(G;\pi) = 0$ if and only if every affine isometric G-action with linear part π has a global fixed point.*

We mention that $H^1(G;\pi)$ is defined as the first cohomology with π-coefficient of a group G which is viewed as a discerete group. Here we allow non σ-compact (hence uncountable) discrete groups. From this point of view, we use the symbol $H^1(G;\pi)$ instead of $H^1_{\mathrm{c}}(G;\pi)$ for discrete groups, such as $\mathrm{SL}_m(\mathbb{Z})$ and universal lattices $\mathrm{SL}_m(\mathbb{Z}[x_1,\ldots,x_k])$.

2.3.2 Definition and relative (FH)

Definition 2.3.4. Let G be a group. We say G has *property* (FH) if for any unitary G-representation, $H^1_{\mathrm{c}}(G;\pi) = 0$ holds. Equivalently, if every affine isometric G-action on a Hilbert space has a G-fixed point.

The terminology (FH) named after *Serre's property* (FA) (see Subsection 2.5.2), and means "Fixed point property on Hilbert spaces." We also define a relative version of (FH), similarly to the case of property (T).

Definition 2.3.5. Let G be a group and $H \leqslant G$ be a subgroup. We say a pair $G \geqslant H$ has *relative property* (FH) if for any unitary G-representation π and for every π-cocycle c, the restriction of c on H is a coboundary (with $\pi\mid_H$-coefficient). Equivalently, if every affine isometric G-action on any Hilbert space has an H-fixed point.

Note that there is another formulation of relative property (FH) for $G \geqslant H$: for any unitary G-representation π, the restriction map

$$\mathrm{rest}\colon H^1_{\mathrm{c}}(G;\pi) \to H^1_{\mathrm{c}}(H;\pi\mid_H)$$

is a zero-map. However, this formulation is not as powerful as the cohomological formulation of property (FH).

The following lemma, which is based on Lemma 2.1.13, is of importance because this gives powerful interpretations of (FH) and relative (FH).

Lemma 2.3.6. *Let G be a group and α be an affine isometric G-action on a Hilbert space. Then the following are all equivalent:*

(i) *The action α has a G-fixed point.*

(ii) *Any G-orbit is bounded.*

(iii) *Some G-orbit is bounded.*

(iv) *The cocycle c is bounded, where c is the transition part of α.*

Corollary 2.3.7. *Let G be a group and $H \leqslant G$ be a subgroup.*

(i) *The group G has* (FH) *if and only if every affine isometric G-action on any Hilbert space has a bounded* (G-)*orbit. These are also equivalent to the condition that for any unitary G-represatation, every π-cocycle is bounded.*

(ii) *The pair $G \geqslant H$ has relative* (FH) *if and only if every affine isometric G-action on any Hilbert space has a bounded H-orbit. These are also equivalent to the condition that for any unitary G-represatation, every π-cocycle is bounded on H.*

We note that G has (FH) if and only if $G \geqslant G$ has relative (FH). Hence in Corollary 2.3.7, item (i) follows directly from item (ii).

Proof. (Lemma 2.3.6) Condition (i) implies condition (ii) is easy, and it is trivial that conditions (ii), (iii), (iv) are equivalent (note that the range of the cocycle c is the G-orbit of the origin). Hence it is enough to show condition (iii) implies condition (i). Suppose there exists a bounded $\alpha(G)$-orbit X. By Lemma 2.1.13, there exists the Chebyshev center η of X. Because η is unique and α is affine isometric, this η is a global fixed point. $\square$

By Corollary 2.3.7, every compact group has (FH) (it is also verified more directly as follows. Let G be a compact group with μ the probability Haar measure. Then for any affine isometric G-action α, $\int_G \alpha(g)\cdot\xi d\mu$ is a global fixed point, where ξ is any vector).

2.3.3 Induction of 1-cocycles

As we have mentioned several times, property (FH) is equivalent to property (T), shall see Section 2.4. Nevertheless, we see some permanence properties of (FH). Specially, we consider inductions of affine isometric actions (or of cocycles).

The following lemma is proven in a pull-back argument, similar to the case of Lemma 2.2.5.

Lemma 2.3.8. *Let G_1, G_2 be groups, and $\phi\colon G_1 \to G_2$ be a continuous homomorphism with dense range. If G_1 has* (FH)*, then so does G_2.*

In particular, property (FH) *passes to group quotients.*

This implies that a group with (FH) has compact abelianization (recall we always assume σ-compactness).

The next lemma is trivial in the view of Corollary 2.3.7.

Lemma 2.3.9. *Property* (FH) *is stable under group extensions.*

Our goal in this subsection is show that property (FH) is inherited to *cocompact* lattices, without the use of the Delorme–Guichardet theorem. For the proof, we need inductions of (1-)cocycles, which is introduced in [Sha3].

First, we recall the definition of Borel fundamental domains, and the following basic fact (for the proof, we refer to Proposition B.2.4 in [BHV]).

Definition 2.3.10. Let G be a group and $H \leqslant G$ be a subgroup. A *Borel fundamental domain* for H is a Borel subset $\mathcal{D} \subseteq G$ such that $G = \bigsqcup_{h\in H} \mathcal{D}h$ (this symbol means a disjoint union).

Proposition 2.3.11. *Let G be a group and $\Gamma \leqslant G$ be a discrete subgroup.*

- (i) *There exists a Borel fundamental domain for Γ.*
- (ii) *If Γ is a lattice, then every Borel fundamental domain for Γ has finite Haar measure. If Γ is moreover cocompact, then for every Borel fundamental domain $\mathcal{D}$ satisfies the following: for any compact subset $S \subseteq G$, the set $\{\gamma \in \Gamma : \mathcal{D}\gamma \cap S \neq \emptyset\}$ is finite.*

In this subsection, henceforth, we let G be a group, $\Gamma \leqslant G$ be a lattice, and $\mathcal{D}$ be a Borel fundamental domain for Γ; we let μ be a Haar measure of G with $\mu(\mathcal{D}) = 1$; and we identify $\mathcal{D}$ with G/Γ and regard $\mathcal{D}$ as a (left) G-space. We define a map $\beta\colon G \times \mathcal{D} \to \Gamma$ by the following rule:

$$\beta(g, x) = \gamma \text{ if and only if } g^{-1}x\gamma \in \mathcal{D}.$$

Lemma 2.3.12. *With the setting above, the map β is a* Borel cocycle*. That means, β satisfies the following equality:*

$$\textit{for any } g, h \in G \textit{ and any } x \in \mathcal{D}, \quad \beta(gh, x) = \beta(g, x)\beta(h, g^{-1}x).$$

Proof. Recall we identify $\mathcal{D}$ with G/Γ and endow $\mathcal{D}$ with the associated left G-action. Therefore, in the equality in the lemma, "$g^{-1}x \in \mathcal{D}$" means $g^{-1}x \cdot \beta(g,x) \in \mathcal{D}$. After observing this, one directly obtains the conclusion through the definition of β. $\square$

With the preparation above, firstly, we shall reformulate induced representations in terms of the Borel cocycle β. Let $(\sigma, \mathfrak{K})$ be a unitary Γ-representation. Then $\mathrm{Ind}_\Gamma^G \sigma$ coincides with the following G-representation π:

- The representation space of π is $L^2(\mathcal{D}, \mathfrak{K})$, equipped with a natural inner product: for $\xi_i \in L^2(\mathcal{D}, \mathfrak{K})$ $(i = 1, 2)$,
$$\langle \xi_1 | \xi_2 \rangle := \int_{\mathcal{D}} \langle \xi_1(x) | \xi_2(x) \rangle d\mu(x).$$
Here in the right hand side of the equality, $\langle \cdot | \cdot \rangle$ is the inner product of $\mathfrak{K}$.

- The representation is defined by the equality: for any $g \in G$, $\xi \in L^2(\mathcal{D}, \mathfrak{K})$ and $x \in \mathcal{D}$,
$$\pi(g)\xi(x) := \sigma(\beta(g,x))\xi(g^{-1} \cdot x).$$

Note that this π becomes a group representation because β is a Borel cocycle (Lemma 2.3.12). Thus we identify $\mathrm{Ind}_\Gamma^G \sigma$ with the representation π above on $\mathfrak{H}_\sigma := L^2(\mathcal{D}, \mathfrak{K})$.

Secondly, we proceed to the definition of induced cocycles. For the well-definedness of induced cocycles, we need some restriction on lattices.

Definition 2.3.13. A lattice Γ in a group G is said to be 2-*integrable* if either of the following two conditions is satified:

(1) the lattice Γ is cocompact;

(2) the lattice Γ is finitely generated, and for some (equivalently any) symmetric finite generating set S of Γ, there exists a Borel fundamental domain $\mathcal{D} \subseteq G$ such that
$$\text{for any } \ g \in G, \quad \int_D l_S(\beta(g,x))^2 d\mu(x) < \infty.$$
Here $l_S \colon \Gamma \to \mathbb{Z}_{\geq 0}$ denotes the *word length* on Γ with respect to S. Namely, for $\gamma \in \Gamma$, $l_S(\gamma)$ is the smallest number $n \geq 0$ such that $\gamma \in S^n$.

Remark 2.3.14. We note that for a finitely generated group Γ, for every pair of (symmetric) finite generating sets S_1 and S_2, l_{S_1} and l_{S_2} are *bi-Lipschitz equivalent*. That means, for every pair S_1 and S_2, there exist a constant $C > 1$ such that for any $\gamma \in \Gamma$, $C^{-1} \cdot l_{S_2}(\gamma) \leq l_{S_1}(\gamma) \leq C \cdot l_{S_2}(\gamma)$ holds. Therefore, the inequality in condition (2) in Definition 2.3.13 does not depend on the choice of S.

The following proposition is the reason why one needs 2-integrability for lattices here:

Proposition 2.3.15. *Suppose a lattice* Γ *in a gorup* G *is* 2*-integrable* (*with a Borel fundamental domain* $\mathcal{D}$). *Let* $(\sigma, \mathfrak{K})$ *be a unitary* Γ*-representation and* c *be a* σ-(1-)*cocycle. Set a map* $\tilde{c}$ *defined by the following equality:*

$$\tilde{c}(g)(x) := c(\beta(g,x)) \quad (g \in G,\ x \in \mathcal{D}).$$

Then this $\tilde{c}$ *ranges into* $\mathfrak{H}_\sigma = L^2(\mathcal{D}, \mathfrak{K})$, *and is an* $\mathrm{Ind}_\Gamma^G \sigma$*-cocycle.*

Proof. Firstly, we observe that this $\tilde{c}$ satisfies the formal cocycle identity, namely,

$$\textit{for any } g,h \in G,\ \ \tilde{c}(gh) = \tilde{c}(g) + \mathrm{Ind}_\Gamma^G \sigma(g)\tilde{c}(h).$$

This is because by Lemma 2.3.12, for any $g, h \in G$ and $x \in \mathcal{D}$,

$$\begin{aligned}\tilde{c}(gh)(x) &= c(\beta(gh,x)) = c(\beta(g,x)\beta(h,g^{-1}x)) \\ &= c(\beta(g,x)) + \sigma(\beta(g,x))c(\beta(h,g^{-1}x)) = \tilde{c}(g)(x) + \mathrm{Ind}_\Gamma^G \sigma(g)\tilde{c}(h).\end{aligned}$$

Here we also use the fact that c is a σ-cocycle.

Secondly, we verify the square integrability of $\tilde{c}$, namely,

$$\text{for any } \ g \in G,\ \ \int_{\mathcal{D}} \|\tilde{c}(g)(x)\|^2 d\mu(x) < \infty.$$

Here is the main part of the proof. We firstly treat the case of that Γ is cocompact. Then by item (*ii*) of Lemma 2.3.13, for every $g \in G$, $\beta(g,x)$ takes only finitely many values, and hence we have the conclusion. We next deal with the case of that Γ satisfies condition (2) in Definition 2.3.13. Take a finite generating set S of Γ, and set $M < \infty$ as $\sup_{s \in S} \|c(s)\|$. Since $\tilde{c}$ satisfies the formal cocycle condition, for any $g \in G$ and $x \in \mathcal{D}$

$$\|c(\beta(g,x))\| \leq M \cdot l_S(\beta(g,x))$$

holds. Thus the conclusion is confirmed from the inequality in condition (2) of Definition 2.3.13.

Finally, we see the continuity of $\tilde{c}$, but this is almost by definition (observe that $\tilde{c}$ is a measurable map). □

Theorem 2.3.16. *Suppose* G *be a group and* Γ *is a* 2-integrable *lattice in* Γ. *Then if* G *has* (FH), *then so does* Γ.

Proof. Take any unitary Γ-representation $(\sigma, \mathfrak{K})$ and any σ-cocycle c. By the assumption of 2-integrability of Γ, by employing Proposition 2.3.15, we obtain the induced cocycle $\tilde{c}$. By property (FH) for G, this $\tilde{c}$ is a coboundary, namely, there exists $\xi \in L^2(\mathcal{D}, \mathcal{K})$ such that for any $g \in G$, $\tilde{c}(g) = \xi - \mathrm{Ind}_\Gamma^G \sigma(g)\xi$. Therefore, by letting $\eta = \xi(e) \in \mathfrak{K}$, we have

$$\text{for any } \ \gamma \in \Gamma,\ \ c(\gamma) = \eta - \sigma(\gamma)\eta.$$

This means c is a σ-coboundary. Hence Γ has (FH). □

Remark 2.3.17. The relative version of the theorem is also true. Namely, the following holds: *"let G be a group, $H \leqslant G$ be a subgroup, and $\Gamma \leqslant G$ be a 2-integrable lattice. Then if $G \geqslant H$ has relative* (FH)*, then $(G \cap \Gamma) \geqslant (H \cap \Gamma)$ has relative* (FH)*."*

2.4 Delorme–Guichardet theorem

In this section, we prove the Delorme–Guichardet theorem, which states property (T) is equivalent to property (FH). For the proof, we study on positive definite functions and conditionally negative definite functions on groups.

2.4.1 Positive definite functions

Definition 2.4.1. (i) Let X be a topological space. A *positive definite kernel* on X is a continuous function $\Phi\colon X \times X \to \mathbb{C}$ such that the following holds: for any $n \in \mathbb{N}$, any complex numbers $c_1, \ldots, c_n$, and any $x_1, \ldots, x_n \in X$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \Phi(x_i, x_j) \geq 0.$$

Here $\overline{c_j}$ means the complex conjugation of c_j.

(ii) Let G be a group. A *positive definite function* on G is a continuous function $\phi\colon G \to \mathbb{C}$ such that the kernel defined by

$$(g, h) \mapsto \phi(h^{-1}g)$$

is positive definite.

Note that the inequality in item (i) is equivalent to that the $n \times n$ matrix $[\Phi(x_i, x_j)]_{i,j}$ is positive definite.

Example 2.4.2. The following are typical examples:

(i) Let $f\colon X \to \mathfrak{H}$ be a continuous map from a topological space X to a Hilbert space $\mathfrak{H}$. Then a kernel

$$X \times X \to \mathbb{C};\ (x, y) \mapsto \langle f(x) | f(y) \rangle$$

is positive definite.

(ii) Let $\pi\colon G \to U(\mathfrak{H})$ be a unitary representation of a group G. Then a function

$$G \to \mathbb{C};\ g \mapsto \langle \pi(g)\xi | \xi \rangle$$

is positive definite. Here $\xi \in \mathfrak{H}$ is any vector.

Indeed, for item (ii), it follows from the equality that for any $n \in \mathbb{N}$, any $c_1, \ldots, c_n \in \mathbb{C}$, and any $g_1, \ldots, g_n \in G$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \langle \pi(g_i)\xi | \pi(g_j)\xi \rangle = 2 \left\| \sum_{i=1}^{n} c_i (\pi(g_i)\xi - \xi) \right\|^2 .$$

The following theorem, so-called the *GNS construction* (named after Gelfand–Naimark–Segal) states that Example 2.4.2 is universal.

Theorem 2.4.3. (*GNS construction for positive definite kernels*)

(i) *Let X be a topological space. Then for any positive definite kernel Φ on X, there exist a Hilbert space $\mathfrak{H}$ and a continuous map $f\colon X \to \mathfrak{H}$ such that for any $x, y \in X$, $\Phi(x, y) = \langle f(x) | f(y) \rangle$. Moreover, if one takes $\mathfrak{H}$ such that the linear span of $\{f(x) : x \in X\}$ is dense in $\mathfrak{H}$, then the pair $(\mathfrak{H}, f)$ is unique up to canonical isomorphism.*

(ii) *Let G be a group. Then for any positive definite function ϕ on G, there exists a tripleHilbert $(\pi, \mathfrak{H}, \xi)$ of a unitary G-representation $(\pi, \mathfrak{H})$ and a vector $\xi \in \mathfrak{H}$ such that for any $g \in G$, $\phi(g) = \langle \pi(g)\xi | \xi \rangle$ holds. Moreover, if one takes $\mathfrak{H}$ such that the linear span of $\{\pi(g)\xi : g \in G\}$ is dense in $\mathfrak{H}$ (equivalently, ξ is a cyclic vector for $(\pi, \mathfrak{H})$), then the triple $(\pi, \mathfrak{H}, \xi)$ is unique up to canonical isomorphism.*

Note that in item (ii), by uniqueness we write $(\pi, \mathfrak{H}, \xi)$ as $(\pi_\phi, \mathfrak{H}_\phi, \xi_\phi)$ (, where ξ_ϕ is a cyclic vector,) and call it the *GNS-triple* for ϕ. For the proof of Theorem 2.4.3, see Theorem C.1.4 and Theorem C.4.10 in [BHV].

It is clear that for a topological space X, positive definite kernels on X are closed under positive linear combinations, and under a pointwise limit (inside continuous kernels on X). The next lemma, which is a corollary of Theorem 2.4.3, states that they are also closed under (pointwise) multiplications.

Lemma 2.4.4. *Let X be a topological space. If Φ_1 and Φ_2 are positive definite kernels on X, then so is $\Phi_1\Phi_2$ (here the multiplication is pointwise). In particular, if ϕ_1 and ϕ_2 are positive definite functions on a group G, then so is $\phi_1\phi_2$.*

Proof. By Theorem 2.4.3, there exist corresponding $(f_1, \mathfrak{H}_1)$ and $(f_2, \mathfrak{H}_2)$. Consider the tensor product Hilbert space $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ and define $f\colon X \to \mathfrak{H}$ by $f(x) = f_1(x) \otimes f_2(x)$. Then $(f, \mathfrak{H})$ realizes $\Phi_1\Phi_2$. □

2.4.2 Conditionally negative definite functions

Definition 2.4.5. (i) Let X be a topological space. A *conditionally negative definite kernel* on X is a continuous function $\Psi\colon X \times X \to \mathbb{R}$ such that the following three conditions are satisfied:

(a) For any $x \in X$, $\Psi(x,x) = 0$.

(b) For any $x, y \in X$, $\Psi(x,y) = \Psi(y,x)$.

(c) For any $n \in \mathbb{N}$, any *real* numbers $c_1, \ldots, c_n$ *with* $\sum_{i=1}^n c_i = 0$, and any $x_1, \ldots, x_n \in X$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \Psi(x_i, x_j) \le 0.$$

(ii) Let G be a group. A *negative definite function* on G is a continuous function $\psi\colon G \to \mathbb{R}$ such that the kernel defined by

$$(g,h) \mapsto \psi(h^{-1}g)$$

is conditionally negative definite.

The following are typical examples:

Example 2.4.6. (i) Let $f\colon X \to \mathfrak{H}$ be a continuous map from a topological space X to a Hilbert space $\mathfrak{H}$. Then, a kernel

$$X \times X \to \mathbb{R};\ (x,y) \mapsto \|f(x) - f(y)\|^2$$

is conditionally negative definite.

(ii) Let α be a affine isometric action of a group G on a Hilbert space $\mathfrak{H}$. Then for any $\xi \in \mathfrak{H}$, a function

$$G \to \mathbb{R};\ g \mapsto \|\alpha(g)\cdot\xi - \xi\|^2$$

is conditionally negative definite. In particular, for any unitary G-representation $(\pi, \mathfrak{H})$ and any π-cocycle c, a function

$$G \to \mathbb{R};\ g \mapsto \|c(g)\|^2$$

is conditionally negative definite (note that by cocycle identity, $c(e) = 0$. This follows from the observation that $c(g)$ coincides with $\alpha(g)\cdot 0$ where α is the affine isometric action associated with (π, c), see Proposition 2.3.3).

Indeed, for item (ii), it follows from the equality that for any $n \in \mathbb{N}$, any $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum_{i=1}^n c_i = 0$, and any $g_1, \ldots, g_n \in G$,

$$\sum_{i=1}^n \sum_{j=1}^n c_i c_j \|\alpha(g_i)\cdot\xi - \alpha(g_j)\cdot\xi\|^2 = -2\left\|\sum_{i=1}^n c_i\alpha(g_i)\cdot\xi\right\|^2.$$

We note that there is a GNS construction theorem for conditionally negative definite kernels, and that states item (i) of Example 2.4.6 is universal. (See Theorem C.2.3 in [BHV].) We note that item (ii) of Example 2.4.6 is *not* universal, because what one needs are continuous G-action on a topological space X, and conditionally negative definite kernel on X which is G-invariant.

Remark 2.4.7. We also make a remark that in the view of conditionally negative kernels and functions, it is more natural to regard Hilbert spaces as *real* Hilbert paces. Also compare with the Mazur–Ulam theorem states that any (surjective) isometry on a *real* Banach space is linear. This implies that any isometric group action on a *real* Hilbert space is automatically affine.

2.4.3 Schoenberg's theorem

The following theorem, due to Schoenberg [Sch], relates positive definite functions to conditionally negative kernels, and vice versa.

Theorem 2.4.8. (*Schoenberg's theorem*) *Let X be a topological space, and $\Psi\colon X \times X \to \mathbb{R}$ be a continuous kernel on X such that for any $x \in X$ $\Psi(x,x) = 0$ and for any $x, y \in X$ $\Psi(x,y) = \Psi(y,x)$. Then the following are equivalent:*

(i) The kernel Ψ is conditionally negative definite.

(ii) For any $t \geq 0$, $e^{-t\Psi}$ is positive definite.

As a corollary, through Example 2.4.6 we obtain the following corollary:

Corollary 2.4.9. *Let G be a group and π be a unitary G-representation. Let c be a π-cocycle. Then for any $t \geq 0$, the function*

$$G \to \mathbb{R}; \quad g \mapsto \exp(-t\|c(g)\|^2)$$

is positive definite.

This corollary is a key to proving Delorme's part of the Delorme–Guichardet theorem, namely, property (T) implies property (FH). Roughly speaking, the importance of Corollary 2.4.9 is the following: property (FH) concerns affine isometric actions, and apparently this may have much information than property (T) has (because affine isometric actions have not only linear parts but also cocycle parts, and a priori property (T) does not seem to give any information for cocycle parts). However, thanks to Corollary 2.4.9, in the view of Theorem 2.4.3, one can extract some information on linear (unitary) representations from that of cocycles.

Since Corollary 2.4.9 is needed for our purpose, we only give a proof of this. For the proof of Theorem 2.4.8, see for instance Theorem C.3.2 in [BHV].

Proof. (Corollary 2.4.9) We show the case of $t = 1$ for simplicity. Set $\phi(g) = \exp(-\|c(g)\|^2)$. In the view of Remark 2.4.7, we may assume the Hilbert space is real. Then for $g, h \in G$,

$$\begin{aligned}
\phi(h^{-1}g) = \exp(-\|c(h^{-1}g)\|^2) &= \exp(-\|c(h^{-1}) + \pi(h^{-1})c(g)\|^2) \\
&= \exp(-\|\pi(h^{-1})(-c(h) + c(g))\|^2) = \exp(-\|c(g) - c(h)\|^2) \\
&= \exp(-\|c(g)\|^2)\exp(-\|c(h)\|^2)\exp(2\langle c(g)|c(h)\rangle).
\end{aligned}$$

Here we use $c(h^{-1}) = -\pi(h^{-1})c(h)$, which directly follows from the cocycle identity. By the equality

$$\exp(2\langle c(g)|c(h)\rangle) = \sum_{n\geq 0} \frac{(2\langle c(g)|c(h)\rangle)^n}{n!},$$

the kernel on G; $(g,h) \mapsto \exp(2\langle c(g)|c(h)\rangle)$ is a positive definite kernel. Also, the kernel on G; $(g,h) \mapsto \exp(-\|c(g)\|^2)\exp(-\|c(h)\|^2)$ is a positive definite kernel, because this can be seen as a map $(g,h) \mapsto \langle \exp(-\|c(g)\|^2)|\exp(-\|c(h)\|^2)\rangle_{\mathbb{R}}$ (here we see $\mathbb{R}$ as a 1-dimensional Hilbert space). Therefore, the kernel on G; $(g,h) \mapsto \exp(-\|c(h^{-1}g)\|^2)$ is positive definite by Lemma 2.4.4. This means the function $G \ni g \mapsto \exp(-\|c(g)\|^2)$ is a positive definite function on G, as desired. $\square$

We state the following definition and lemma, which is needed in Subsection 3.3.1.

Definition 2.4.10. A continuous function $F\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called a *Bernstein function* if there exist a positive measure ν on Borel subsets of $\mathbb{R}_{>0}$ and $C \geq 0$ such that for any $t > 0$

$$F(t) = Ct + \int_0^{+\infty} (1 - e^{-tx})d\nu(x)$$

holds. Here ν satisfies the following two conditions:

(i) For any $\epsilon > 0$, $\nu([\epsilon, \infty)) < \infty$.

(ii) The inequality $\int_0^1 x d\nu(x) < \infty$ holds.

Lemma 2.4.11. *Let Ψ be a conditionally negative definite kernel on a topological space X, and F be a Bernstein function. Then $F \circ \Psi$ is conditionally negative definite.*

Note that conditionally negative definite kernels on a topological space X are closed under positive linear combinations, and under a pointwise limit (inside continuous kernels on X).

Proof. (Lemma 2.4.11) By Shoenberg's theorem (Theorem 2.4.8), for every $t > 0$ $e^{-t\Psi}$ is positive definite, and hence $1 - e^{-t\Psi}$ is negative definite (in particular, conditionally negative definite). Therefore conclusion follows from the observation above. Note that conditions on the measure ν in Definition 2.4.10 is needed for the well-definedness of $F \circ \Psi$, namely, $F \circ \Psi(x,y) < \infty$ for any $x, y \in X$ and $F \circ \Psi$ being continuous. $\square$

Remark 2.4.12. In Corollary 2.4.9, in the view of Theorem 2.4.3, there must exist the GNS triple for the positive definite function $g \mapsto \exp(-t\|c(g)\|^2)$. One can explicitly construct this triple by taking the (*full*) *Fock space*, as in below:

From the view point of Remark 2.4.7, suppose the representation $(\pi, \mathfrak{H})$ is real. Define $\mathrm{Exp}(\mathfrak{H}) := \bigoplus_{n\geq 0} \mathfrak{H}^{\otimes^n}$, where $\mathfrak{H}^{\otimes^0} := \mathbb{C}$. Also, define a map (among sets)

$$\mathfrak{H} \hookrightarrow \mathrm{Exp}(\mathfrak{H});\ \xi \mapsto \mathrm{Exp}(\xi) := 1 \oplus \xi \oplus \frac{\xi^{\otimes^2}}{\sqrt{2!}} \oplus \frac{\xi^{\otimes^3}}{\sqrt{3!}} \oplus \cdots .$$

For any $s \geq 0$, define a map (among sets)

$$\phi^s \colon \mathfrak{H} \hookrightarrow \mathrm{Exp}(\mathfrak{H});\ \xi \mapsto \phi^s(\xi) := \exp\left(-\frac{s^2}{2}\|\xi\|^2\right)\mathrm{Exp}(s\xi).$$

Then we have for any $s \geq 0$ and any $\xi, \eta \in \mathfrak{H}$,

$$\langle \phi^s(\xi) | \phi^s(\eta)\rangle = \exp\left(-\frac{s^2}{2}\|\xi - \eta\|^2\right).$$

In particular, this ϕ^s maps $\mathfrak{H}$ into the unit sphere $S(\mathrm{Exp}(\mathfrak{H}))$.)

Now let α be the affine isometirc G-action on $\mathfrak{H}$ associated with (π, c). For $s \geq 0$, denote by $\mathfrak{K}^s$ the closure of linear span of $\{\phi^s(\xi) : \xi \in \mathfrak{H}\}$ in $\mathrm{Exp}(\mathfrak{H})$. We define unitary G-representation π^s_α on $\mathfrak{K}^s$ by the following equality: for any $\lambda \in \mathbb{C}$ and any $\xi \in \mathfrak{H}$,

$$\pi^s_\alpha(g)(\lambda\phi^s(\xi)) := \lambda\phi^s(\alpha(g)\cdot\xi).$$

This is a priori densely defined, and can be extended to the closure $\mathfrak{K}^s$. Indeed, for a vector $\sum_i \lambda_i\phi^s(\xi_i) \in \mathfrak{K}^s$,

$$\begin{aligned}
\left\|\pi^s_\alpha(g)(\sum_i \lambda_i\phi^s(\xi_i))\right\|^2 &= \left\|\sum_i \lambda_i\phi^s(\alpha(g)\cdot\xi_i)\right\|^2 = \sum_i \lambda_i\overline{\lambda_j}\langle\phi^s(\alpha(g)\cdot\xi_i)|\phi^s(\alpha(g)\cdot\xi_j)\rangle \\
&= \sum_{i,j} \lambda_i\overline{\lambda_j}\exp(-s^2/2\cdot\|\alpha(g)\cdot\xi_i - \alpha(g)\cdot\xi_j\|^2) \\
&= \sum_{i,j} \lambda_i\overline{\lambda_j}\exp(-s^2/2\cdot\|\xi_i - \xi_j\|^2) \\
&= \left\|\sum_i \lambda_i\phi^s(\xi_i)\right\|^2.
\end{aligned}$$

Therefore, π^s_α is a unitary representation on $\mathfrak{K}^s$, with the equality

$$\langle \pi^s_\alpha(g)\phi^s(0)|\phi^s(0)\rangle = \exp\left(-\frac{s^2}{2}\|c(g)\|^2\right).$$

Thus we obtain the GNS triple $(\pi^s_\alpha, \mathfrak{K}^s, \phi^s(0))$ for the positive definite function $g \mapsto \exp(-t\|c(g)\|^2)$, where $s = \sqrt{2t}$.

It is worth noting that as seen in above, it is usually quite difficult (or, almost impossible) to observe the change of the corresponding GNS triples explicitly when we take an operation to positive definite (or conditionally negative definite) functions. A significant merit of considering positive definite functions (not unitary representations themselves) is that one can study properties on the whole unitary representations on a group, by means of analysis and certain operations on positive definite functions, without dealing with respective representation spaces.

2.4.4 Proof of Delorme–Guichardet theorem

Now we state the Deloreme–Guichardet theorem:

Theorem 2.4.13. (*Delorme* [Del], *Guichardet* [Gui]) *For a group G, the following are equivalent:*

(i) *The group has property* (T).

(ii) *The group has property* (FH).

More precisely, P. Delorme proved condition (i) implies condition (ii), and A. Guichardet proved condition (ii) implies condition (i).

Proof. Firstly, we will show that Guichardet's implication: "property (FH) implies property (T)" ((ii) $\Rightarrow$ (i)), which is less involved. Here recall that we always assume groups are σ-compact. Take a unitary representation $(\pi, \mathfrak{H})$ with $\pi \not\supseteq 1_G$. The point here is the space of π-cocycles $Z^1(G;\pi)$ can be endowed with a Fréchet space structure, and that the subspace of π-coboundaries $B^1(G;\pi)$ is usually *not* closed. Indeed, for the former part, since G is σ-compact, we can define a separating sequence of seminorms (m_S), indexed by a countable family of compact subsets $S \subseteq G$ which covers G, by

$$m_S \colon Z^1(G;\pi) \to \mathbb{R}_{\geq 0};\ c \mapsto \sup_{s\in S} \|c(s)\|.$$

For the latter part, we claim if $B^1(G;\pi)$ is closed in $Z^1(G;\pi)$, then $\pi \not\succeq 1_G$ (, namely, π cannot have almost invariant vectors). To show the claim, note that $B^1(G;\pi)$ is the image of the bounded linear map

$$\tau \colon \mathfrak{H} \to Z^1(G;\rho);\ \xi \mapsto (\tau(\xi))(g) = \xi - \pi(g)\xi,$$

which is injective by the assumption that $\pi \not\supseteq 1_G$. Hence if $B^1(G;\pi)$ is closed, by the open mapping theorem $\tau \colon \mathfrak{H} \to B^1(G;\rho)$ must have a bounded inverse. This implies that there exist a constant $M > 0$ and a compact subset $S \subseteq G$ such that for any $\xi \in \mathfrak{H}$,

$$\|\xi\| \leq M\|\tau(\xi)\|_S = M \cdot \sup_{s\in S} \|\xi - \pi(s)\xi\|.$$

Therefore $\pi \not\succeq 1_G$ follows.

To finish the proof of Guichardet's implication, take any unitary representation π of a group G with (FH) which satisfies $\pi \not\supseteq 1_G$. Then by (FH) the π-cohomology group vanishes, and in particular $B^1(G;\pi) \subseteq Z^1(G;\pi)$ must be closed with respect to the topology in the paragraph above. Through the argument above, this forces π to satisfy $\pi \not\succeq 1_G$. Therefore, for a unitary G representation π, $\pi \succeq 1_G$ implies $\pi \supseteq 1_G$. This means G has (T), as desired.

Finally, we proceed to the proof of Delorme's implication: "property (T) implies property (FH)" ($(i) \Rightarrow (ii)$). Take a contraposition, and suppose G does not have (FH). Then there exist a unitary G-representation σ and a σ-cocycle c which is *not* a coboundary. By Corolary 2.4.9 and Theorem 2.4.3, for any $t \geq 0$, there exists the GNS-triple $(\pi_t, \mathfrak{K}_t, \xi_t)$ associated with the positive definite function $G \ni g \mapsto \exp(-t\|c(g)\|^2)$. For explicit construction, $\pi_t = \pi^{\sqrt{2t}}$ as in Remark 2.4.12. Set

$$\pi = \bigoplus_{n=1}^{\infty} \pi_{1/n}.$$

First, we cliam $\pi \not\supseteq 1_G$. This follows from the fact that for each $n \geq 1$, $\pi_{1/n} \not\supseteq 1_G$ (otherwise the affine isometric action associated with (σ, c) must have a fixed point and contradiction occurs). Next, we claim $\pi \succeq 1_G$. Indeed, $\exp(-1/n \cdot \|c(s)\|^2) \to 1$ as $n \to \infty$ uniformly on every compact subset $S \subseteq G$, and this implies for each compact subset $S \subseteq G$ and every $\epsilon > 0$, there exists sufficiently large n such that $\xi_{1/n}$ is (S, ϵ)-invariant (note that for any $t \geq 0$, ξ_t is a unit vector, because $\exp(-t \cdot \|c(0)\|^2) = 1$). Hence π has almost invariant vectors.

Therefore, the group G does not have (T). This argument ends our proof of Delorme's implication.

$\square$

Remark 2.4.14. Here are remarks on the Delorme–Guichardet theorem:

(i) There is a relative version of the Delorme–Guichardet theorem, as follows:

Theorem 2.4.15. *Let G be a group and $H \leqslant G$ be a subgroup. Then the following are equivalent:*

(i) *The pair $G \geqslant H$ has relative* (T).

(ii) *The pair $G \geqslant H$ has relative* (FH).

The proof goes along a similar way to that of Theorem 2.4.13.

(ii) In the proof of Guichardet's implication (FH) $\Rightarrow$ (T), the following is in fact hold: "*for any group G and any unitary representation π with $\pi \not\supseteq 1_G$, the space $B^1(G;\pi)$ is closed in $Z^1(G;\pi)$ with the Fréchet topology (, namely, the topology of uniform convergence on compact subsets) if and only if $\pi \not\succ 1_G$.*" In the proof we have shown "only if" direction, and "if" direction is much easier. In general, it is much more convenient to consider the closure $\overline{B}^1(G;\pi)$ instead of treating $B^1(G;\pi)$. This yields the important concept of *reduced cohomology*. For more details and discussions, see Chapter 5.

(*iii*) In the proof of Guichardet's implication, we do not change the representation space $(\pi, \mathfrak{H})$. However note that in the proof of Delorme's implication (T) $\Rightarrow$ (FH), we *do* change the representation space. One explanation is that we use Schoenberg's theorem; and the other, more explicit explanation is we replace with the full Fock space. Therefore, for a fixed family of unitary representations $(\pi_i, \mathfrak{H}_i)_{i\in I}$ of a group G, condition (a) in below always implies condition (b); but the converse is *no longer* true:

(a) For any $i \in I$, $H^1(G;\pi_i) = 0$.

(b) For any $i \in I$, if $\pi_i \succeq 1_G$, then $\pi_i \supseteq 1_G$.

For instance, consider the family $\{\lambda_G\}$, namely, the single representation λ_G (the left regular representation on $L^2(G)$). Then condition (b) is satisfied for any noncompact nonamenable group (for amenability, see Subsection 2.5.2). However, F_2, which denotes the free group of rank 2, does *not* satisfy condition (a), although it is a infinite nonamenable group and hence satisfies condition (b).

Therefore, it is worth noting Delorme's implication holds true because there is a theory of positive definite and conditionally negative definite functions. This means, firstly, this holds true thanks to a special property for *unitary* representations; and secondly, this holds true because we consider the family of *all* unitary representations on arbitrary Hilbert spaces of a group. Compare with Subsection 3.2.3.

(*iv*) For Guichardet's implication, we need the assumption of σ-compactness of groups. In fact, there exists a *uncountable* discrete group with (FH), see for instance a paper of Y. de Cornulier [dCo2]. Note that for the proof of Theorem 2.2.1, σ-compactness is not needed. Hence every (locally compact) (T) groups are compactly generated. Therefore, Guichardet's implication is *no longer* true for non-σ-compact groups.

2.5 The Haagerup property – as a strong negation

In this section, we briefly treat the *Haagerup property*, which is equivalent to *Gromov's a-T-menability*, for groups. These properties can be regarded as strong negation of Kazhdan's property (T). A main reference in this section a book of Cherix–Cowling–Jolissaint–Julg–Valette [CCJJV].

2.5.1 Two definitions

Kazhdan's property (T) (, which is equivalent to property (FH),) represents strong rigidity of groups (property (T) for groups means difficulty in finding almost invariant vectors; and property (FH) represents first cohomology vanishing with unitary coefficients). The Haagerup property [Haa] and Gromov's a-T-menability [Gro3] are repsectively strong negation of these, and represent strong non-rigidity.

Definition 2.5.1. Let G be a group and $(\pi, \mathfrak{H})$ be a unitary G-representation.

(i) The representation π is called a C_0-*representation* if every matrix coefficient

$$G \ni g \mapsto \langle \pi(g)\xi|\eta\rangle \in \mathbb{C} \quad (\xi, \eta \in \mathfrak{H})$$

vanishes at infinity. Here a continuous function $f\colon X \to \mathbb{C}$ on a topologocal space is said to *vanish at infinity* if for any $\epsilon > 0$, the set $\{x \in X : |f(x)| \geq \epsilon\} \subseteq X$ is compact.

(ii) A π-cocycle $c\colon G \to \mathfrak{H}$ is said to be *metrically proper* (or shortly, *proper*) if for any $M > 0$, the set $\{g \in G : \|c(g)\| \leq M\} \subseteq G$ is compact. If this condition is satisfied, we write as "$\|c(g)\| \to \infty$ $(g \to \infty)$." We say an affine isometric action of G on a Hilbert space is (*metrically*) *proper* if the cocycle part is proper. In general, let a discrete group Γ acts on a metric space (X, d) $\alpha\colon \Gamma \curvearrowright X$ by isometries. Then we say the action α is (*metrically*) *proper* if for some (equivalently, any) $x \in X$, the following holds true: for any $M \geq 0$, the set $\{g \in G : d(g \cdot x, x) \leq M\} \subseteq G$ is finite.

Note that in item (i), we only have to check the condition for every diagonal matrix coefficients.

Example 2.5.2. An example of C_0-representation is λ_G for any group G. Indeed, if G is compact, this is trivial. If G is non-compact, we use the fact that $C_c(G)$, the space of continuous functions with compact support, is dense in $L^2(G)$. For details, see Proposition C.4.6 in [BHV].

For the proof of the following theorem, which states the Haagerup property is equivalent to Gromov's a-T-menability, one employs Schoenberg's theorem (in a similar way to the proof of Theorem 2.4.13). For details, see Theorem 2.1.1 in [CCJJV].

Theorem 2.5.3. *For a group G, the following are equivalent:*

(i) *The group G has the* Haagerup property. *That means, there exists a C_0-unitary representation π of G which satisfies $\pi \succeq 1_G$.*

(ii) *The group G is* a-T-menable *in the sense of Gromov. That means, there exists a proper cocycle into a unitary representation.*

The following proposition indicate precise meaning that these properties are strong negation of property (T).

Proposition 2.5.4. *Let $G \geqslant H$, Λ be groups. Suppose $G \geqslant H$ has relative* (T)*, and Λ has the Haagerup property. Then for every continuous homomorphism $\phi\colon G \to \Lambda$, the image of H is relatively compact in Λ.*

In particular, every continuous homomorphism from a Kazhdan group into an a-T-menable group has relatively compact image.

Proof. One proof is the following: by (relative version of) Lemma 2.2.5, $\overline{\phi(G)} \geq \overline{\phi(H)}$ has relative (T). Therefore there exists a C_0-unitary representation $(\pi, \mathfrak{H})$ of Λ, which satisfies $\mathfrak{H}^{\pi(\overline{\phi(H)})} \neq 0$. By the definition of C_0-representations, this forces $\overline{\phi(H)}$ to be compact.

Here is another proof in terms of different characterization: by (relative version of) Lemma 2.3.8, $\overline{\phi(G)} \geq \overline{\phi(H)}$ has relative (FH). Therefore there exists a proper cocycle c into a unitary representation, which is bounded on $\overline{\phi(H)}$. By the definition of properness, this forces $\overline{\phi(H)}$ to be compact. □

Remark 2.5.5. Here are remarks on the Haagerup properties.

(*i*) One deep implication of a group having the Haagerup property is the following theorem of Higson–Kasparov [HiKa] and J.-L. Tu [Tu]: "*a group with the Haagerup property fulfills the Baum–Connes conjecture* [BCH] (*with coefficients*)." In contrary, Kazhdan's property (T) can be seen as a *main obstruction* in establishing (surjectivity side of) the Baum–Connes conjecture (with coefficients). In this thesis, we will not go further into this topic.

(*ii*) In the view of Proposition 2.5.4, it is natural to ask the following question: "*if a group G does not have the Haagerup property, then does it follow G contains noncompact subgroup H such that $G \geqslant H$ has relative* (T)*?*" This question was answered by de Cornulier [dCo1] negatively. He shown, for instance, $\mathrm{SO}_3(\mathbb{Z}[2^{1/3}]) \ltimes \mathbb{Z}[2^{1/3}]^3$ is a counter example. In [dCo3], de Cornulier extended relative (T) and relative (FH) for a pair of a locally compact group and a closed *subset*.

(*iii*) The Haagerup property is stable under subgroups, direct products, and increasing unions of open subgroups. This is *not* stable under group quotients. Indeed, free groups have the Haagerup property, as we will see in Subsection 2.5.3.

(*iv*) By theory of induction (see Subsection 2.2.2), the following holds true: if a lattice Γ in a group G has the Haagerup property, then so does G. In particular, every virtually free group has the Haagerup property. Here a group G is said to be *virtually free* if G contains a free group with finite index. In fact, the following more general fact is known: if a subgroup H of G which is *co-Følner* has the Haagerup property, then so does G. For the definition of co-Følner property and the proof, see Chapter 6 of [CCJJV].

2.5.2 Amenable groups

Amenablity of groups has a significant role in geometric and analytic group theory, and has thousands of formulations. For comprehensive treatment for this subject, we refer readers to a book of A. T. Paterson [Pat] and Appendix G in [BHV]. Before stating definitions of amenability, we briefly explain on means.

Definition 2.5.6. Let X be a set, $\mathcal{B}$ be a σ-algebra of X, and μ be a measure on $(X, \mathcal{B})$.

(i) A *mean* m on $\mathcal{B}$ is a finitelly additive probability measure on $\mathcal{B}$. Namely, a nonnegative valued function $\mathcal{B} \to \mathbb{R}_{\geq 0}$ that satisfies the following:

(1) $m(X) = 1$

(2) for any pair wise disjoint finitely many sets $A_1, \ldots, A_n \in \mathcal{B}$, $m(\bigsqcup_{i=1}^n A_i) = \sum_{i=1}^n m(A_i)$.

If moreover X is endowed with an action of a group G which leaves $\mathcal{B}$ invariant, then a mean m is said to be (*left*) *G-invariant* if for any $g \in G$ and $A \in \mathcal{B}$, $m(g \cdot A) = m(A)$ holds.

(ii) Let E be a closed subspace of $L^\infty(X, \mathcal{B}, \mu)$ which contains $1 = \chi_X$ and is stable under complex conjugation. A *mean* M on E is a linear functional $E \mapsto \mathbb{C}$ that satisfies the following:

(1) $M(1) = 1$.

(2) M is positive, namely, for any nonnegative valued function $\phi \in E \subseteq L^\infty(X, \mathcal{B}, \mu)$, $M(\phi) \geq 0$.

If in addition X is endowed with an action of a group G which leaves $\mathcal{B}$ invariant, consider the left G-action on $L^\infty(X, \mathcal{B}, \mu)$ by $\phi(x) \mapsto g \cdot \phi(x) = \phi(g^{-1} \cdot x)$. If moreover E is invariant under this G-action then a mean m is said to be (*left*) *G-invariant* if for any $g \in G$ and $\phi \in E$, $M(g \cdot \phi) = M(\phi)$ holds.

It can be shown there is the following one-to-one correspondence:

$$\left\{ \begin{array}{c} \text{means } m \text{ on } \mathcal{B} \\ \text{which are absolute continuous with respect to } \mu \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{means } M \\ \text{on } L^\infty(X, \mathcal{B}, \mu) \end{array} \right\}.$$

Here m is said to be *absolutely continuous* with respect to μ, if $m(A) = 0$ whenever $A \in \mathcal{B}$ staisfies $\mu(A) = 0$. In this correspondence, the map from the right hand side to left hand side is easy: $m(A) := M(\chi_A)$. We note that for the construction of the inverse map, one needs much more care. Thanks to this correspondence, henceforth we will not distinguish these two concepts of means. Let $\ell^\infty(G)$ denote the ℓ^∞ space for G and for this construction, we temporarily regard G as a (possible uncountable) discrete group. Let $UCB(G)$ denote the closed subspace of $\ell^\infty(G)$ of all functions ϕ which are uniformly left continuous, equivalently, the mapping $G \ni g \mapsto g \cdot \phi \in \ell^\infty$ is continuous. Note that $UCB(G)$ is invariant under the G-action on $\ell^\infty(G)$.

Here we extract some equivalent formulations of amenability. We note that in this theorem, we do not need σ-compactness assumption.

Theorem 2.5.7. *For a group G, the following are all equivalent. If G satisfies either of these conditions, G is said to be* amenable.

(*i*) (*Existence of invariant mean* 1) *G has a* (*left*) *invariant mean on the class of Borel subsets of G.*

(*ii*) (*Existence of invariant mean* 2) *G has a* (*left*) *invariant mean on $UCB(G)$.*

(*iii*) (*Fixed point property on convex compact subsets*) *any continuous affine action of G on a non-empty convex compact subset of a locally convex topological vector space has a G-fixed point.*

(*iv*) (*Følner's condition*) *there exists a Følner net. Nemely, for every compavt subset $S \subseteq G$ and every $\epsilon > 0$, there exists a Borel subset D of G with $0 < \mu(D) < \infty$ such that*
$$\sup_{s \in D} \frac{\mu(sD \triangle D)}{\mu(D)}.$$
Here μ is a left Haar measure.

(*v*) (*The Hulanicki–Reiter condition*) *$\lambda_G \succeq 1_G$ holds, where λ_G is the left regular representtaion on $L^2(G)$.*

Amenability is stable under subgroups, group quotients, extensions, and inductive limits. Also amenability is extended to a group from a lattice (in general, from a *co-amenable* subgroup. See [Pat]). As we mentioned in Example 2.1.6, any abelian group is amenable. Therefore, all solvable groups are amenable. Also, any finite generated group *with subexponential growth* is amenable. Here we say a finitely generated group has *subexponential growth* if for some (equivalently, any: see Remark 2.3.14) finite generating set $S \subseteq G$, the equality

$$\text{for any } t > 0, \quad \lim_{n \to \infty} \frac{|\{g \in G : l_S(g) \leq n\}|}{e^{tn}} = 0.$$

An interesting and famous example of groups with subexponential growth is the *Grigorchuk group* (this group has *intermediate growth*). For details, see the paper of R. I. Grigorchuk [Gri].

Also, we note that the free group F_2 of rank 2 is *not* amenable (because for instance, this group admits a paradoxical decomposition). Since any subgroup of an amenable group is amenable, this implies the following: any group which contains a subgroup isomorphic to F_2 is *not* amenable. An example of nonamenable groups which does not contain F_2 is an (infinite) free Burnside group.

As a corollary of Theorem 2.5.3 and Theorem 2.5.7, we have the following:

Lemma 2.5.8. *Any amenable group has the Haagerup property.*

2.5.3 Other examples

In this subsection, we show that the free group F_n $(2 \leq n \leq \infty)$ and the special linear group $\mathrm{SL}_2(\mathbb{Z})$ have the Haagerup property.

Definition 2.5.9. Let G be a finitely generated group, and S be a symmetric finite generated set. Then the *Cayley graph* of G associated with S, written as $\mathrm{Cay}(G;S)$, is defined as follows:

- The vertex set V is G, as a set.
- The edge set E is the set of all oriented pair $e = (v, vs)$, where $v \in V(= G)$ and $s \in S$ and $vs \in V$ means the group multiplication.

Since S is symmetric, $(v, vs) \in E$ implies $(vs, v) \in E$. Hence we also identify $\mathrm{Cay}(G;S)$ with non-oriented graph. If S does not contain e_G, $\mathrm{Cay}(G;S)$, as a non-oriented graph, is a connected $|S|$-regular graph without self loops or multiple edges.

We endow $\mathrm{Cay}(G;S)$ with a metric space structure by the shortest path metric d (with setting each edge length$= 1$). Also, we equip $\mathrm{Cay}(G;S)$ with a left G-action (, namely, $G \ni g$ sends $V \ni v \mapsto gv \in V$). By the definition of the edge set E, this action on $\mathrm{Cay}(G;S)$ is isometric, and (metrically) proper in the sense of (ii) in Definition 2.5.1.

A basic example is the following: let F_2 denote the free group of rank 2, and let $a, b \in F_2$ be free generators. Then, for a generating set $S = \{a^{\pm}, b^{\pm}\}$, the Cayley graph of F_2 $T := (V, E) = \mathrm{Cay}(F_2; S)$ is a 4-regular tree, as a non-oriented graph.

We will show that F_2 has the Haagerup property. The key here is, with the setting in the paragraph above, F_2 acts on a (4-regular) tree T properly by isometries. From this point, we regard the edge set E of T as the set of all *oriented* edges. Set a (real) Hilbert space $\mathfrak{H}$ as the subspace of all vectors ξ in $\ell^2(E, \mathbb{R})$ with the following condiiton:

$$\text{for any edge } (v, vs) \in E,\ \ \xi((vs, v)) = -\xi((v, vs)).$$

The isometric action of F_2 on T involves a natural action on E, and this induces an orthogonal F_2-representation on $\mathfrak{H}$. We write this representation as π, namely, for any $g \in F_2$, $\xi \in \mathfrak{H}$, and any $(v, vs) \in E$,

$$\pi(g)\xi((v, vs)) = \xi((g^{-1}v, g^{-1}vs)).$$

Next, observe that since T is a tree, for any distinct $v, w \in V$, there exists a *unique* (shortest) path from v to w. We denote by $[v, w]$ this path. Set $z\colon V \times V \to \mathfrak{H}$ by the following equality: for any $v, w \in V$ and any $e \in E$,

$$z(v,w)(e) := \begin{cases} 0 & \text{if } v = w \text{ or } e \text{ is not on } [v,w]; \\ 1 & \text{if } e \text{ is on } [v,w] \text{ and points from } v \text{ to } w; \\ -1 & \text{if } e \text{ is on } [v,w] \text{ and points from } w \text{ to } v. \end{cases}$$

Note that then this z fulfills the following two conditions:

(1) For any $g \in F_2$ and $v, w \in V$, $\pi(g)z(v, w) = z(gv, gw)$.

(2) For any $u, v, w \in V$, $z(u, v) + z(v, w) = z(u, w)$.

Indeed, for item (2), observe the fact that for distinct $u, v, w \in V$, there exists a unique point which lies on $[u, v]$, $[v, w]$, and $[u, w]$.

Finally, fix a vertex $v \in V$, and set

$$c\colon F_2 \to \mathfrak{H};\ c(g) = z(gv, v).$$

Then from item (1) and (2), it is easy to show that c is a π-cocycle. Moreover, this c is a proper cocycle because the action $F_2 \curvearrowright T$ is proper. More precisely, for any $g \in F_2$,

$$\|c(g)\| = d(gv, v)^{1/2} = (l_S(g))^{1/2}.$$

Therefore, F_2 is a-T-menable, and equivalently has the Haagerup property.

In fact, in this case, one can prove in more direct way. However, in above argument, we also obtain the following. Here we say a graph is *locally finite* if for every vertex, the valence is finite.

Theorem 2.5.10. *Let Γ be a discrete group, and T be a locally finite tree. Suppose Γ acts on T properly by isometries. Then G has the Haagerup property.*

Corollary 2.5.11. *For any $2 \leq n \leq \infty$, the free group F_n of rank n has the Haagerup property. Here F_∞ denotes the free group of countably many generators, viewed as a discrete group. The special linear group $\mathrm{SL}_2(\mathbb{Z})$ has the Haagerup property.*

Note that $F_1 \cong \mathbb{Z}$ is amenable.

Proof. (Corollary 2.5.11) We have seen F_n has the Haagerup property for $1 \leq n < \infty$ in the argument above. Also, F_2 contains a subgroup isomorphic to F_∞ and hence F_∞ has the Haagerup property.

In the case of $G = \mathrm{SL}_2(\mathbb{Z})$, note a famous fact that G contains F_2 with finite index. Indeed, consider the following subgroup of G

$$\left\langle E_{1,2}(2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ E_{2,1}(2) = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle.$$

Then it is well-known that this subgroup is a free group of rank 2, and that it has index 12 (for the proof of the former fact, we use so-called a ping-pong lemma. Compare with the proof of Proposition 2.6.3 below). Therefore through item (iv) of Remark 2.5.5, we obtain the conclusion. □

Note that in fact $\mathrm{SL}_2(\mathbb{Z})$ itself can act on a locally finite tree properly by isometries. This follows from the decomposition of $\mathrm{SL}_2(\mathbb{Z})$ as an amalgamated free product of finite groups:

$$\mathrm{SL}_2(\mathbb{Z}) \cong (\mathbb{Z}/4\mathbb{Z}) \star_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z}),$$

and the Bass–Serre theory. For details on trees, we refer to a book [Ser] by J.-P. Serre.

Other examples of groups with the Haagerup properties are $\mathrm{SO}_{m,1}$, $\mathrm{SU}_{m,1}$ $(m \geq 1)$; Coxeter groups; Baumslag–Solitar groups; Thompson groups (such as Thompson's F); and groups acting property on CAT(0) cubical complexes (by isometries).

Remark 2.5.12. Here are remarks on groups acting on trees, and extensions of a-T-menable groups. For the former topic, see [Ser] for comprehensive treatment.

(i) For a connected graph V with countable vertices (viewed as a metric space with the shortest path distance), one can endow $\mathrm{Aut}(V)$ with the topology of pointwise convergence and view $\mathrm{Aut}(V)$ as a topological group (if V is locally finite, then $\mathrm{Aut}(V)$ is locally compact with respect to this topology). Thus one can consider continuous action of a topological group on trees.

(ii) In the opposite of a group acting properly on trees, Serre defined property (FA) as follows:

Definition 2.5.13. A group G is said to have *property* (FA) of Serre if any continuous G-action on a tree (by isometries) has either a globally fixed vertex or a globally fixed nonoriented edge.

This definition is equivalent to the condition that for any continuous G-action on a tree, some (equivalently any) G-orbit is bounded. We use the symbol (FA) instead of the original one FA, in order to emphasize this property represents some *rigidity* aspect.

Along a similar line to a proof of Theorem 2.5.10, one can show the following theorem of Y. Watatani [Wat].

Theorem 2.5.14. *If a group has* (FH) (*, equivalently,* (T))*, then it has* (FA)*.*

Note that the converse of this theorem is *not* true. Indeed, it is known that every Coxeter group has (FA). However as we stated in above, it has the Haagerup property and hence it cannot have (T) unless it is finite.

Also, it is worth noting that property (FA) does *not* pass to subgroups of finite index. A famous and surprising example is the following: $\mathrm{Out}(F_3)$, the outer automorphism group of the free group of rank 3, is known to have (FA) [Bog], [CuVo2] (this is true for any $n \geq 4$ finite). Nevertheless, J. McCool [McC] found a finite index subgroup of $\mathrm{Out}(F_3)$ which maps onto $\mathbb{Z}$, hence not having (FA). Note that if a (discrete) group G has (T), then any finite index subgroup has (T) (Theorem 2.2.14) and hence has (FA) by Theorem 2.5.14. This shows, for instance, any finite index subgroup of $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ has (FA), because $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ has (T) (we will see this in Subsection 2.6.2).

(*iii*) We note that the Haagerup property is *not* stable under group extensions. Indeed, we see $\mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{Z}^2$ have the Haagerup property. However, as we stated in Subsection 2.1.3, the semidirect product $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ has an infinite subgroup $\mathbb{Z}^2$ with relative (T) (we will show this in Subsection 2.6.1). In the view of item (*iv*) of Remark 2.5.5, it is true that the Haagerup property is stable under group extensions by *amenable* group quotients. A recent notable result of de Cornulier–Stalder–Valette [CSV] states the Haagerup property is stable under *wreath products*.

2.6 Examples of Kazhdan groups

In this section, we see some typical examples of Kazhdan groups. The main goal in this section is to show *totally higher rank lattices*, defined in Chapter 0, has property (T). We shall examine the basic case of this, more precisely, the case of $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$, which can be seen as a lattice in $\mathrm{SL}_{m\geq 3}(\mathbb{R})$.

Recall from Chapter 0 that we use the symbol $\mathbb{K}$ for local fields.

2.6.1 $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \trianglerighteq \mathbb{R}^2$ has relative (T)

The following theorem is the essential in proving property (T) for totally higher rank lattices:

Theorem 2.6.1. *For any* (*local field*) $\mathbb{K}$*, the pair* $\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2 \trianglerighteq \mathbb{K}^2$ *has relative* (T)*.*

Recall as in Chapter 0, the symbol above means these groups are identified with

$$\left\{\left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array}\right) : W \in \mathrm{SL}_2(\mathbb{K}),\, v \in \mathbb{K}^2\right\} \quad \trianglerighteq \quad \left\{\left(\begin{array}{c|c} I_2 & v \\ \hline 0 & 1 \end{array}\right) : v \in \mathbb{K}^2\right\}.$$

Since $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ is a lattice in $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, by item (i) of Remark 2.2.15 we obtain the following corollary:

Corollary 2.6.2. *The pair* $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$ *has relative* (T).

For the proof of Theorem 2.6.1, we need spectral theory for the abelian (additive) group $\mathbb{K}^2$. We briely recall the unitary dual $\hat{\mathbb{K}}^2$ of $\mathbb{K}^2$ is (non-canonically) identified with $\mathbb{K}^2$ as follows: fix a unitary character χ of $\mathbb{K}$ other than the unit character. The mapping

$$\mathbb{K}^2 \to \hat{\mathbb{K}}^2; \quad x \mapsto \chi_x, \quad \chi_x(y) = \chi(\langle x, y\rangle)$$

is a topological group isomorphism. Here for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{K}^2$, $\langle x, y\rangle = x_1y_1 + x_2y_2$. Under this identification, the dual action of a matrix $h \in \mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}^2$ corresponds to the inverse transpose of the standard action on $\mathbb{K}^2$, namely, $\mathbb{K}^2 \ni x \mapsto {}^t h^{-1}x$. We also recall from Subsection 2.5.2 for the definitions of means.

Next, we prove the following proposition:

Proposition 2.6.3. *If m is an* $\mathrm{SL}_2(\mathbb{K})$*-invariant mean on the class of Borel subsets of* $\mathbb{K}^2$*, then m is the Dirac measure at* 0.

Proof. Let $|\cdot|$ be an absolute value which defines the topology of $\mathbb{K}$. Set

$$\Omega := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 \setminus \{0\} : |y| \geq |x|\right\}.$$

We claim that $m(\Omega) = 0$. Indeed, choose a sequence $(\lambda_n)_{n\geq 1}$ in $\mathbb{K}$ with $|\lambda_{n+1}| \geq |\lambda_n| + 2$ for all n and $|\lambda_1| \geq 2$, and set $\Omega_n = h_n\Omega$, where

$$h_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{K}).$$

Then by $\mathrm{SL}_2(\mathbb{K})$-invariance of m, for any $n \geq 1$ $m(\Omega_n) = m(\Omega)$. Also, since

$$\Omega_n \subseteq \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{K}^2 \setminus \{0\} : \frac{|x|}{|\lambda_n| + 1} \leq |y| \leq \frac{|x|}{|\lambda_n| - 1}\right\},$$

and $0 < |\lambda_n| + 1 < |\lambda_m| - 1$ for any $1 \leq n < m$, the sequence $(\Omega_n)_{n\geq 1}$ is pairwise disjoint. Therefore by finite additivity, for any $n \geq 1$, $n \cdot m(\Omega) \leq m(\mathbb{K}^2) = 1$. This shows our claim.

Also, there exists $h \in \mathrm{SL}_2(\mathbb{K})$ such that $h\Omega = \mathbb{K}^2 \setminus (\Omega \cup \{0\})$ Explicitly, for instance,

$$h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore $m(\{0\}) = 1$, and this proves the proposition. $\square$

Proof. (Theorem 2.6.1) Set $H = \mathrm{SL}_2(\mathbb{K})$, $N = \mathbb{K}^2$, and $G = H \ltimes N$. Suppose $G \trianglerighteq N$ does not have relative (T). Then there exists a unitary representation $(\pi, \mathfrak{H})$ of G such that $\pi\mid_N \not\supseteq 1_N$ and $\pi \succeq 1_G$. In particular, there exists a unitary G-representation $(\pi, \mathfrak{H})$ such that the following two conditions hold:

(a) The space $\mathfrak{H}^{\pi(N)} = 0$.

(b) The representation $\pi\mid_H$ has almost invariant vectors.

By spectral theorem for the abelian group N, the representation $\pi\mid_N : N \to U(\mathfrak{H})$ yields the corresponding spectral measure

$$E \colon \mathcal{B}(\hat{N}) \to \mathbb{B}(\mathfrak{H}),$$

where $\mathcal{B}(\hat{N})$ denotes the σ-algebra of all Borel subsets of $\hat{N}$. Namely, E satisfies the following: for any $r \in N$ and $\xi \in \mathfrak{H}$,

$$\langle \pi(r)\xi|\xi\rangle = \int_{\hat{N}} \chi(r) d\langle E(\chi)\xi|\xi\rangle.$$

By the uniqueness of the spectral measure and $\pi(h)\pi(r)\pi(h^{-1}) = \pi(hrh^{-1})$ ($h \in H, r \in N$), we have the following:

$$\text{for any } h \in H \text{ and any } \hat{C} \in \mathcal{B}(\hat{N}), \quad \pi(h)E(\hat{C})\pi(h^{-1}) = E(h \cdot \hat{C}).$$

Here h acts on $\hat{N}$ by $h\chi(x) = \chi(h^{-1}x)$. From this point, we identify $\hat{N}$ with N by the correspondence explained in this subsection. Then the equality above means:

$$\text{for any } h \in H \text{ and any } C \in \mathcal{B}(N), \ \ \pi(h)E(C)\pi(h^{-1}) = E({}^t h^{-1} C).$$

Here on the left hand side, the action is given by the matrix multiplication.

Note that H is compactly generated, and fix a comcact generating set $S \subseteq H$ for H. By condition (b) for any $n \in \mathbb{N}$, there exists a unit $(S, 1/n)$-invariant vector ξ_n. Thus we obtain a sequence of unit vectors $(\xi_n)_{n\in\mathbb{N}}$ such that

$$\text{for any } h \in H, \ \ \lim_n \|\xi_n - \pi(h)\xi_n\| = 0$$

(see Corollary 2.1.14). Now consider the product space $K = \Pi_{T\in\mathbb{B}(\mathfrak{H})} D_T$, where for each $T \in \mathbb{B}(\mathfrak{H})$, D_T is the closed disc in $\mathbb{C}$ of radius $\|T\|$. We endow K with the product topology, and then Tychnoff's theorem shows K is compact. Since

$(\langle T\xi_n|\xi_n\rangle)_{T\in\mathbb{B}(\mathfrak{H})}$ is in K, there exists a sub*net* $(\xi_j)_{j\in J}$ such that for all $T\in\mathbb{B}(\mathfrak{H})$ the limit

$$\lim_{j\in J}\langle T\xi_j|\xi_j\rangle$$

exists. We define a map $M\colon \mathbb{B}(\mathfrak{H})\to\mathbb{C}$ by setting $M(T)$ as the limit above. Then M is a positive linear functional, sending I to 1, and $\pi(H)$-bi invariant, that means

$$\text{for any } h\in H \text{ and } T\in\mathbb{B}(\mathfrak{H}),\ \ M(\pi(h)T)=M(T\pi(h))=M(T).$$

Indeed, it is straightforward to see M is a positive linear functional and sends I to 1. For $\pi(H)$-bi invariance, it follows from the following inequalities: for all $j\in J$,

$$\begin{aligned}|\langle\pi(h)T\xi_j|\xi_j\rangle-\langle T\xi_j|\xi_j\rangle| &= |\langle T\xi_j|\pi(h^{-1})\xi_j-\xi_j\rangle|\le\|T\|\|\pi(h^{-1})\xi_j-\xi_j\|,\\ |\langle T\pi(h)\xi_j|\xi_j\rangle-\langle T\xi_j|\xi_j\rangle| &= |\langle T(\pi(h)\xi_j-\xi_j)|\xi_j\rangle|\le\|T\|\|\pi(h)\xi_j-\xi_j\|.\end{aligned}$$

Finally, we obtain a mean m on $N=\mathbb{K}^2$ by

$$\mathcal{B}(N)\ni C\mapsto m(C):=M(E(C))\in\mathbb{R}_{\geq 0}$$

(recall that (orthogonal) projections are positive operators). By $\pi(H)$-invariance of M and the equality $\pi(h)E(C)\pi(h^{-1})= E({}^t h^{-1}C)$, it follows this m is invariant under the inverse transpose multiplication of $H=\mathrm{SL}_2(\mathbb{K})$. This means m is an $\mathrm{SL}_2(\mathbb{K})$-invariant mean. By Proposition 2.6.3, this m must coincide with the Dirac measure at $0\in\mathbb{K}^2$. This in particular means $E(\{0\})\neq 0$. However this contradicts condition (a) because $E(\{0\})$ is a projection onto $\mathfrak{H}^{\pi(N)}$. □

Remark 2.6.4. (i) In fact, we prove the following:

> **Theorem 2.6.5.** *Let $\mathbb{K}$ be any local field. Let $H=\mathrm{SL}_2(\mathbb{K})$, $N=\mathbb{K}^2$, and $G=H\ltimes N$. Then the pair $H\ltimes N\trianglerighteq N$ has* strong relative property (T)*, in the sense of Bader–Furman–Gelander–Monod* [BFGM]*. That means, for any unitary G-representation $(\pi,\mathfrak{H})$, whenever $\pi\mid_H\succeq 1_H$, $\pi\mid_N\supseteq 1_N$ follows.*
>
> This observation is a key to the proof of property $(\mathrm{F}_{\mathcal{L}_p})$ for totally higher rank lattice in [BFGM]. For details, we will see in Subsection 3.3.3.
>
> We warn that this strong relative property (T) does *not* pass to lattices. Indeed, tha pair $G=\mathrm{SL}_2(\mathbb{Z})\ltimes\mathbb{Z}^2\trianglerighteq\mathbb{Z}^2=N$ does *not* have this property. For the proof of this observation, consider the unitary representation $(\pi,\ell^2(N))$ induced by affine action of G on N: $(W,v_1)\cdot v_2=Wv_2+v_1$ $(W\in H=\mathrm{SL}_2(\mathbb{Z})$ and $v_1,v_2\in N$. Then $\pi\mid_H\succeq 1_H$: in fact, $\pi\mid_H\supseteq 1_H$ (the Dirac function δ_0 on the origin of N is $\pi(H)$-invariant). Nevertheless, trivially $\pi\mid_N\not\supseteq 1_N$.

(ii) By Theorem 2.6.1 (and Corollary 2.6.2), the following is easily deduced:

Proposition 2.6.6. *Let $m \geq 3$. Then for any local field $\mathbb{K}$, the pair* $\mathrm{SL}_m(\mathbb{K}) \ltimes \mathbb{K}^m \trianglerighteq \mathbb{K}^m$ *has relative* (T). *The pair* $\mathrm{SL}_m(\mathbb{Z}) \ltimes \mathbb{Z}^m \trianglerighteq \mathbb{Z}^m$ *has relative* (T).

Proof. Set $G = \mathrm{SL}_m(\mathbb{K}) \ltimes \mathbb{K}^m$ and $N = \mathbb{K}^m$. Let π be any unitary G-representation, and c be any π-cocycle. Two points here are the following: firstly, by Theorem 2.4.15, the pair $\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2 \trianglerighteq \mathbb{K}^2$ has relative (FH). Secondly, G contains an isomorphic copy of $\mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2$ with $\mathbb{K}^2$ part included in N: for instance,

$$
\begin{aligned}
G &= \left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{SL}_m(\mathbb{K}),\, v \in \mathbb{K}^m \right\} \\
&\geqslant \left\{ \left(\begin{array}{cc|c} W' & 0 & v' \\ 0 & I_{m-2} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) : W' \in \mathrm{SL}_2(\mathbb{K}),\, v' \in \mathbb{K}^2 \right\} \\
&\trianglerighteq \left\{ \left(\begin{array}{cc|c} I_2 & 0 & v' \\ 0 & I_{m-2} & 0 \\ \hline 0 & 0 & 1 \end{array} \right) : v' \in \mathbb{K}^2 \right\} \cong \mathbb{K}^2 \leqslant N.
\end{aligned}
$$

For any $1 \leq i < j \leq m$, set $N_{i,j}$ as an isomorphic copy of $\mathbb{K}^2$ in N which consits of all elements in N whose all but i-th and j-th entries are zero. Here we identify N with the additive group of column vectors of size m. Thanks to the two observations above, we have that c is bounded on each $N_{i,j}$. Note that N is boundedly generated by $(N_{i,j})_{i,j}$, namely, there exists global $M \geq 1$ such that any $r \in N$ can be written as a product of at most M elements in $\bigcup_{i,j} N_{i,j}$ (one can take M above with $M \leq m/2+1$). Since by cocycle identity, for any $g_1, g_2 \in G$

$$\|c(gh)\| \leq \|c(g)\| + \|c(h)\|,$$

we conclude that c is bounded on N.

By Corollary 2.3.7, this means the pair $G \leqslant N$ has relative (FH). Again by Theorem 2.4.15, $G \leqslant N$ has relative (T). The latter assertion in the proposition follows from the same argument. □

2.6.2 $\mathrm{SL}_m(\mathbb{R})$ and $\mathrm{SL}_m(\mathbb{Z})$ have (T) for $m \geq 3$

We represent how to deduce property (T) for $\mathrm{SL}_{m\geq 3}(\mathbb{K})$ from Theorem 2.6.1. One way is to appeal to the Howe–Moore ergordicity theorem [HoMo], although in this case it seems we use too deep theorem for the task.

Theorem 2.6.7. (*Howe–Moore's theorem*) *Let G be a group of the form $\Pi_{i=1}^m \mathbf{G}_i(k_i)$, where k_i are local fields, $\mathbf{G}_i(k_i)$ are k_i-points of Zariski connected simple k_i-algebraic groups. Let π be a unitary G-representation such that for each $G_i = \mathbf{G}_i(k_i)$ $\pi\mid_{G_i} \not\supseteq$*

1_{G_i}. *Then* π *is a* C_0 *representation, namely, for any vector* ξ *and* η, *the matrix coefficient*

$$G \ni g \mapsto \langle \pi(g)\xi | \eta \rangle \in \mathbb{C}$$

is a C_0*-function* (*recall item* (*i*) *of Definition* 2.5.1).

In particular, every Zariski connected simple algebraic group G *over a local field enjoys the* Howe–Moore property *in the sense of* [CCLTV]. *Namely, any unitary* G*-representation with* $\pi \not\supseteq 1_G$ *is* C_0.

The proof of this theorem is involved. For instance, see the original paper [HoMo], or Chapter 2 of a book of R. J. Zimmer [Zim]. For studies on the Howe–Moore property, we refer to [CCLTV]. The following corollary is surprising, and of importance.

Corollary 2.6.8. *Let* G *be a Zariski connected simple algebraic group* G *over a local field. Then for any unitary* G*-representation* $(\pi, \mathfrak{H})$, *there is an equality:*

$$\text{for any non-compact subgroup } H \leqslant G, \;\; \mathfrak{H}^{\pi(H)} = \mathfrak{H}^{\pi(G)}.$$

Proof. The inclusion $\mathfrak{H}^{\pi(H)} \supseteq \mathfrak{H}^{\pi(G)}$ is trivial. For the inverse inclusion, suppose there exists a unit vector $\xi \in \mathfrak{H}^{\pi(H)} \setminus \mathfrak{H}^{\pi(G)}$. Then thanks to the Howe–Moore property, $g \mapsto \langle \pi(g)\xi|\xi\rangle$ is C_0. This contradicts the observation that the function above takes value 1 on H. $\square$

Theorem 2.6.9. *For any local field* $\mathbb{K}$ *and* $m \geq 3$, $\mathrm{SL}_m(\mathbb{K})$ *has property* (T).

Compare with the fact that $\mathrm{SL}_2(\mathbb{K})$ has the Haagerup property (this follows from the group above contains F_2 as a lattice).

Proof. Set $G = \mathrm{SL}_m(\mathbb{K})$. Take any unitary G-representation π with $\pi \succeq 1_G$. Suppose $\pi \not\supseteq 1_G$. Firstly observe G contains an isomorphic copy of $H \ltimes N$, where $H = \mathrm{SL}_2(\mathbb{K})$ and $N = \mathbb{K}^2$. Then by relative (T) for $H \ltimes N \trianglerighteq N$ (Theorem 2.6.1), $\pi\mid_N \supseteq 1_N$. (Note that $\pi \succeq 1_G$ implies $\pi\mid_{H\ltimes N} \succeq 1_{H\ltimes N}$.) Secondly, note that G and π satisfies the assumption of the Howe–Moore theorem. By Corollary 2.6.8, $\pi\mid_N \supseteq 1_N$ implies $\pi \supseteq 1_G$ (because N is noncompact). This is a contradiction. Therefore G has (T). $\square$

Theorem 2.6.10. *For* $m \geq 3$, $\mathrm{SL}_m(\mathbb{Z})$ *has property* (T).

Corollary 2.6.11. *Let* $\mathbb{K}$ *be a local field and* $m \geq 3$. *Then* $\mathrm{SL}_m(\mathbb{K}) \ltimes \mathbb{K}^m \trianglerighteq \mathbb{K}^m$ *has property* (T). *Also* $\mathrm{SL}_m(\mathbb{Z}) \ltimes \mathbb{K}^m \trianglerighteq \mathbb{Z}^m$ *has* (T).

Proof. This follows from Theorem 2.6.9, Propsotion 2.6.6 and Proposition 2.2.7. It is also not difficult to deduce directly from Theorem 2.6.9 in a similar argument to that in Proposition 2.6.6. $\square$

2.6.3 General totally higher rank lattices

Firstly, we consider the case of $\mathrm{Sp}_{2m}(\mathbb{K})$. We recall the definition. In this subsection, let A be a commutative ring (recall from Chapter 0 we always assume a ring is associative and has unit). As mentioned in Chapter 0, in this thesis, we use the following alternating matrix:

$$J_m := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

The *symplectic group* over A, written as $\mathrm{Sp}_{2m}(A)$, is defined as the multiplicative group of symplectic matrices in matrix ring $\mathrm{M}_{2m}(A)$ associated with the alternating matrix J_m, namely,

$$\mathrm{Sp}_{2m}(A) = \{g \in \mathrm{M}_{2m}(A) : {}^t g J_m g = J_m\}.$$

For $m \geq m_0 \geq 2$, by $\mathrm{SL}_{m_0}(A) \leqslant \mathrm{Sp}_{2m}(A)$ (or, $\mathrm{SL}_{m_0}(A) \hookrightarrow \mathrm{Sp}_{2m}(A)$) we mean the inclusion is realized in the following way:

$$\left\{ \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & I_{m-m_0} & 0 & 0 \\ 0 & 0 & {}^t W^{-1} & 0 \\ 0 & 0 & 0 & I_{m-m_0} \end{pmatrix} : W \in \mathrm{SL}_{m_0}(A) \right\} \leqslant \mathrm{Sp}_{2m}(A).$$

Also, we denote by $S^{m*}(A^m)$ the additive group of all symmetric matrices in $\mathrm{M}_m(A)$. Finally, by $\mathrm{SL}_m(A) \ltimes S^{m*}(A^m) \trianglerighteq S^{m*}(A^m)$, we identify these groups respectively with

$$\left\{ (W, v) := \left(\begin{array}{c|c} W & v \\ \hline 0 & {}^t W^{-1} \end{array} \right) : W \in \mathrm{SL}_m(A),\ v \in S^{m*}(A^m) \right\}$$
$$\trianglerighteq \left\{ \left(\begin{array}{c|c} I_m & v \\ \hline 0 & I_m \end{array} \right) : v \in S^{m*}(A^m) \right\}.$$

Thus the action of $\mathrm{SL}_m(A)$ on $S^{m*}(A^m)$ is:

$$(W, 0)(I_m, v)(W^{-1}, 0) = (I_m, W v {}^t W) \quad (W \in \mathrm{SL}_m(A), v \in S^{m*}(A^m)).$$

Note that these groups are included in $\mathrm{Sp}_{2m}(A)$. The following theorem is a symplectic version of relarive (T) result, which is parallel to Theorem 2.6.1:

Theorem 2.6.12. *For a local field $\mathbb{K}$, the pair $\mathrm{SL}_2(\mathbb{K}) \ltimes S^{2*}(\mathbb{K}^2) \trianglerighteq S^{2*}(\mathbb{K}^2)$ has relative* (T).

The proof is basically the same as that of Theorem 2.6.1, which is based on the spectral theory for the abelian group $S^{2*}(\mathbb{K}^2)$. However in this case it is more complicated. See Section 1.5 in [BHV]. In fact, this pair has strong relative property (T), see item (i) of Remark 2.6.4.

Corollary 2.6.13. *The pair* $\mathrm{SL}_2(\mathbb{Z}) \ltimes S^{2*}(\mathbb{Z}^2) \trianglerighteq S^{2*}(\mathbb{Z}^2)$ *has relative* (T).

The following theorem can be deduced from Theorem 2.6.12 and Corollary 2.6.8:

Theorem 2.6.14. *Let* $\mathbb{K}$ *be a local field and* $m \geq 2$. *Then* $\mathrm{Sp}_{2m}(\mathbb{K})$ *and* $\mathrm{Sp}_{2m}(\mathbb{Z})$ *have* (T).

We note that the case of SL_3 and Sp_4 are the basis of property (T) for totally higher rank groups. For more precise meaning, see below.

The following theorem is the main goal in this chapter. Recall from Chapter 0 that we define a *totally higher rank* (*algebraic*) *group* as a group of the following form: $G = \Pi_{i=1}^{m}\mathbf{G}_i(k_i)$, where k_i are local fields, $\mathbf{G}_i(k_i)$ are k_i-points of Zariski connected simple k_i-algebraic groups (with finite center), and each simple factor $\mathbf{G}_i(k_i)$ has (local) rank ≥ 2. We define *totally higher rank lattice* as a lattice in a totally higher rank group. Note that the original paper of Kazhdan [Kaz], the assumption of each local rank ≥ 3 was needed. This result is extended to the case of each local rank ≥ 2 by Delaroche–Kirillov, Vaserstein, and Wang.

Theorem 2.6.15. (*based on Kazhdan* [Kaz]) *Any totally higher rank group and totally higher rank lattice have* (T).

Proof. We only treat the case of a connected simple algebraic group $\mathbf{G}$ over a local field $\mathbb{K}$ of rank ≥ 2. Then the following is well-known:

Lemma 2.6.16. *For the group* $\mathbf{G}$ *in the setting above, there exists a almost* $\mathbb{K}$*-simple algebraic group* $\mathbf{H}$ *over* $\mathbb{K}$ *whose simply connected covering is isomorphic over* $\mathbb{K}$ *to either* SL_3 *or* Sp_4

This lemma follows from the classification of root systems. For details, see Chapter I (1.6.2) in [Mar2].

Let $\mathbf{H}$ be as in Lemma 2.6.16. Let $\tilde{\mathbf{H}}$ be its simply connected covering, and $\phi\colon \tilde{\mathbf{H}} \to \mathbf{H}$ be the canonical homomorphism. Set $H = \mathbf{H}(\mathbb{K})$. Since $\tilde{\mathbf{H}}(\mathbb{K})$ has (T), so does $\phi(\tilde{\mathbf{H}}(\mathbb{K}))$. Therefore H has (T), because $\phi(\tilde{\mathbf{H}}(\mathbb{K}))$ is a (normal) subgroup of finite index. Corollary 2.6.8 ends our proof. □

Corollary 2.6.17. *Every totally higher rank lattice is finitely generated, has finite abelianization, and has property* (FA) (*recall Definition* 2.5.13).

Proof. Follows from Theorem 2.6.15, Theorem 2.2.1, Corollary 2.2.6, and Theorem 2.5.14 □

2.6.4 Other examples

As we stated in Subsection 2.5.3, $\mathrm{SO}_{m,1}$ and $\mathrm{SU}_{m,1}$ have the Haagerup property. For rank 1 groups, B. Kostant shown the following result:

Theorem 2.6.18. (*Kostant* [Kos]) *For* $m \geq 2$, $\mathrm{Sp}_{m,1}$ *has* (T).

Kostant also shown that $F_{4(-20)}$ has (T). For the proof of these results, see Section 3.3 in [BHV] for geometric proof.

As some discrete generalization of rank 1 groups, there is a well-known conception of *hyperbolic groups*, in the sense of Gromov [Gro2].

Definition 2.6.19. Let $\delta \geq 0$.

(i) A geodesic metric space (X, d) is said to be δ-*hyperbolic* if for any geodesic triangle $\triangle\alpha\beta\gamma$, the following three conditions are satisfied:

$$\alpha \subseteq N_\delta(\beta) \cup N_\delta(\gamma);$$
$$\beta \subseteq N_\delta(\gamma) \cup N_\delta(\alpha);$$
$$\gamma \subseteq N_\delta(\alpha) \cup N_\delta(\beta).$$

Here for $S \subseteq X$, $N_\delta(S)$ is the δ-neighborhood of S, namely, the set of all points $x \in X$ such that $\inf_{s \in S} d(x, s) \leq \delta$ holds. The space X is said to be *hyperbolic* if there exists $\delta \geq 0$ such that X is δ-hyperbolic.

(ii) A finitely generated group G is said to be (*word*) *hyperbolic* if there exists a finite generating set S and $\delta \geq 0$ such that the Cayley graph $\mathrm{Cay}(G; S)$ (recall definition 2.5.9) is δ-hyperbolic.

Let G be a finitely generated group. It is known that whether $\mathrm{Cay}(G; S)$ is a hyperbolic metric space does *not* depend on the choice of finite generating sets. Therefore, we can consider hyperbolicity of finitely generated groups as a property of groups. One typical example of hyperbolic groups is a finitely generated free group, because a tree is 0-hyperbolic.

Therefore it might seem to be reasonable to ask whether all hyperbolic groups have the Haagerup property. However it is *far from* being true: for instance, any cocompact lattice in $\mathrm{Sp}_{n,1}$ is a hyperbolic group, and by Theorem 2.6.18 this has property (T). In fact, there are plenty of (in some sense, "generic") examples of (infinite) hyperbolic groups with (T). The main tool for producing such examples is initiated by Gromov [Gro4], and is to consider so-called *random groups*. Here we briefly explain one model of random groups, so-called *density model*.

Definition 2.6.20. (*Density model*) Let F_n be the free group with $n \geq 2$ free generators $a_1, \ldots, a_n$. For any integet l, let $S_l \subseteq F_n$ be the set of redued words in those generators (, namely, words with each letter being in $\{a_1^{\pm}, \ldots, a_n^{\pm}\}$ without cancellation occurring) of length l.

Let $0 \leq d \leq 1$.

(i) A *random set of relators at density* d *at length* l is a $(2n-1)^{dl}$-tuple of elements of S_l, randomly picked among all elements of S_l uniformly and independently.

(*ii*) A *random group at density d at length l* is the group G presented by $\langle a_1, \ldots a_n | R \rangle$, where R is a random set of relators at density d at length l. Namely, G is a group quotient of F_n by the normal closure of R in F_n.

(*iii*) For a property for groups, the property is said to *occur with overwhelming probability at density d* if the probability of G having the property tends to 1 as $l \to \infty$.

On random groups, we refer to papers of Gromov [Gro4], [Gro6]; and surveys of Y. Ollivier [Oll] and L. Silberman [Sil].

We collect results on property (T) and the Haagerup property for random groups, which are respectively due to Gromov, and Olliver–Wise.

Theorem 2.6.21. (*Gromov* [Gro4]*; Olliver–Wise*[OlWi]) *Let* $0 \leq d \leq 1$.

(*i*) *If $d > 1/2$, then with overwhelming probability a random group at density d is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$. In particular, is finite.*

If $d < 1/2$, then with overwhelming probability a random group at density d is (*infinite*) *torsion free hyperbolic group.*

(*ii*) *If $d > 1/3$, then with overwhelming probability a random group at density d has* (T).

(*iii*) *If $d < 1/6$, then with overwhelming probability a random group at density d has the Haagerup property.*

This theorem in particular implies there exists uncountably many (non-isomorphic) Kazhdan groups. We mention that it is also known if $d < 1/5$, then with overwhelming probability a random group at density d does *not* have (T). Little is known on what happens at critical points, or at intermediate phase.

We note that A. Żuk [Zuk] found a powerful local criterion for (finitely generated) groups having (T), and that he shown in another model of random groups (so-called *triangular model*) with overwhelming probability either a random group at density d is free ($d < 1/3$); or a random group at density d has (T) ($d > 1/3$).

Another example of Kazhdan groups are some class of Kac–Moody groups. We refer to papers of Dymara–Januszkiewicz [DyJa]; Ballmann–Swiatkowski [BaSw2]; and Cartwright–Mlotkowski–Steger [CMS]. We note that the last two papers above examine groups acting on $\tilde{A}_2$ buildings. Also we refer to Chapter 5 of [BHV].

2.7 Expander graphs – as an application

In this section, we briefly explain expander graphs, which can be seen as one important application of property (T). In this section, we regard edges of a graph as a nonoriented object, unless otherwise stated, and assume all graphs are connected.

Definition 2.7.1. Let X is a finite graph, with the vertex set V and the edge set E. Suppose X is d-regular for some $d \in \mathbb{N}$, and X does not have multiple edges.

(a) The *expansion constant* (or, the *isoperimetric constant*) of X is the following real number:

$$h(X) := \inf\left\{\frac{|\partial S|}{|S|} : \emptyset \neq S \subseteq V,\ |S| \leq \frac{1}{2}|V|\right\}.$$

Here for $S \subseteq V$, $\partial S \subseteq E$ is the set of all edges which connect a vertex in S and a vertex in $V \setminus S$.

(b) The *spectral gap* of X, written as $\lambda_1(X)$, is the smallest positive eigenvalue of the Laplacian operator $\Delta_X$associated with X. Here the Δ_X is defined as the matrix $\in \mathrm{M}_{|V|}(\mathbb{C})$ of the form

$$\Delta_X = d - A,$$

where A is the adjacency matrix, namely for $v, w \in V$, $A(v, w)$ is 1 if $(v, w) \in E$ and 0 otherwise.

Remark 2.7.2. The original definition of Δ_X is

$$\Delta_X = \frac{1}{2}\partial^*\partial \in \mathbb{B}(\ell^2(V)).$$

Here $\partial\colon \ell^2(V) \to \ell^2(E_\pm)$ means the differential operator,

$$\partial f((v, w)) := f(w) - f(v) \quad (f \in \ell^2(V)).$$

Here the symbol $E_\pm$ means the set of edges with orientation.

Since X is connected, the space of constant function in $\ell^2(V)$ coincides with the eigenspace of Δ_X associated with the eigenvalue 0. Therefore, for $\lambda > 0$, $\lambda_1(X) \geq \lambda$ is equivalent to the following condition (recall Δ_X is a positive operator):

$$\text{for any } \xi \in \ell^2(V) \text{ with } \xi \perp 1,\ \ \lambda\|\xi\|^2 \leq \langle \Delta_X \xi | \xi \rangle$$

(this argument is called a Rayreigh quotient argument).

Definition 2.7.3. A sequence $(X_n)_{n\geq 1}$ of finite graphs X_n, with vertex sets V_n and edges sets E_n without multiple edges is called a *family of expanders* if the following conditions are satisfied:

(i) There exists $d \in \mathbb{N}$ such that for all n, X_n is d-regular.

(ii) The number $|V_n|$ increases to infinity as $n \to \infty$.

(iii) Either of the following two equivalent conditions holds:

(*a*) (*expander condition*) there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, $h(X_n) \geq \epsilon$;

(*b*) (*spectral expander condition*) there exists $\lambda > 0$ such that for any $n \in \mathbb{N}$, $\lambda_1(X_n) \geq \lambda$.

Note that the equivalence between item (*a*) and item (*b*) in Definition 2.7.3 follows from the following inequality:

$$\text{for any } d\text{-regular graph,} \quad \frac{\lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d\lambda_1(X)}.$$

For the proof of the inequality above and details of expanders, we refer to a survey of Hoory–Linial–Wigderson [HLW]. From the expander condition, a family of expanders has the following property, which is of extreme importance in network theory: the diameter of $X_{n\geq 1}$ is of polylog order with respect to $|V_{n\geq 1}|$.

With the aid of random graphs, it was known a family of expanders exists, but this method is not constructive. The first explicit construction is due to Margulis [Mar1], and he employs property (T). Before seeing this, we need some definitions.

Definition 2.7.4. Let G be a discrete group.

(*i*) The group G is said to be *residually finite* if for any $g \in G \setminus \{e_G\}$, there exists a finite group quotient $p\colon G \twoheadrightarrow H$ such that $p(g) \neq e_H$. Equivalently, if there exists a sequence of normal subgroups $G_n \trianglelefteq G$ of finite index such that $\bigcap_n G_n = \{e_G\}$.

(*ii*) The group G is called a *linear group* if it is finitely generated and if it has a faithful linear representation $G \to \mathrm{GL}_m(K)$ for some $m \geq 2$ and K a field.

In other words, by *linear groups* we mean a finitely generated (*not necessarily* closed or discrete) subgroup in a group of the form $\mathrm{GL}_m(K)$.

For instance, it is easy to see $\mathrm{SL}_{m\geq 2}(\mathbb{Z})$ is residually finite. The following theorem of A. I. Malcev is well-known:

Theorem 2.7.5. (*Malcev*) *Every linear group is residually finite.*

For the proof, see for instance Theorem 6.4.12 in a book of Brown–Ozawa [BrOz].

Recall the definition of the Kazhdan constant $\mathcal{K}(G;S)$ in Definition 2.1.10 and the Cayley graph $\mathrm{Cay}(G;S)$ inDefinition 2.5.9.

Theorem 2.7.6. *Suppose G is a discrete group with* (T) *and is residually finite. Then one can construct a family of expanders from finite group quotients of G. More precisely, the following holds true:*

Let S be a symmetric finite generating set without containing e_G, and $\mathcal{K} = \mathcal{K}(G;S)$ be the Kazhdan constant. Let $(G_n)_{n\geq 1}$ be a sequence of finite index normal subgroups of G with $|G/G_n|$ increasing to infinity as $n \to \infty$ and with $p_n\colon G \twoheadrightarrow G/G_n$

being injective on S. Consider the sequence of Cayley graphs $X_n = \mathrm{Cay}(p_n(G); p_n(S))$. Then $(X_n)_{n\geq 1}$ is a family of expanders, with satisfying

$$\lambda_1(X_n) \geq \mathcal{K}^2.$$

Note that in fact what is needed here is relative (T) instead of (T). Margulis' explicit construction uses relative (T) for the pair $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$. However in that case we deal with graphs with multiple edges, and we need some modification (because the diffrential operator cannot be defined in this case).

Proof. Existence of such $(G_n)_n$ is from residual finiteness. Fix n and set $H = G/G_n$ (a finite group), set $X = X_n$, and identify $p_n(S)$ with S. Consider the right regular representation $\pi \colon H \to \ell^2(H)$. By assumption, we can choose subset $S_0 \subseteq S$ such that $S = S_0 \sqcup S_0^{-1}$. Now consider the positive operator D on $\ell^2(V)$ defined by

$$D = \sum_{s\in S_0} (1 - \pi(s))^*(1 - \pi(s)).$$

Then D coincides with Δ_X. Indeed

$$D = 2|S_0| - \sum_{s\in S_0} (\pi(s) + \pi(s^{-1})) = |S| - \sum_{s\in S} \pi(s),$$

and this equals Δ_X by definition.

Note that $\mathcal{K}(H; S_0) = \mathcal{K}(H; S)$ and $\mathcal{K}(H; S) \geq \mathcal{K}(G; S) = \mathcal{K}$ hold (these follow from the definition of Kazhdan constant). Therefore we have the following estimate: for any $\xi \in (\ell^2(V)_{\pi(G)})^\perp$ (note that this orthogonal complement is identical to $\{\xi \in \ell^2(V) : \xi \perp 1\}$),

$$\begin{aligned}
\|\xi\|^2 &\leq (\mathcal{K}(H; S_0)^{-1} \sup_{s\in S_0} \|\xi - \pi(s)\xi\|)^2 \\
&\leq \mathcal{K}^{-2} \sup_{s\in S_0} \|(1 - \pi(s))\xi\|^2 \\
&\leq \mathcal{K}^{-2} \sum_{s\in S_0} \|(1 - \pi(s))\xi\|^2 = \mathcal{K}^{-2}\langle D\xi|\xi\rangle.
\end{aligned}$$

By Remark 2.7.2, we obtain the conclusion. □

Therefore, estimation of Kazhdan constants has significant meaning in particular for theory of expander graphs. Note that our proof of property (T) for $\mathrm{SL}_3(\mathbb{Z})$ does *not* give such estimation. We shall see some estimation in Subsection 4.2.3.

Finally, we note an important role of expander graphs on coarse geometry as below. The concept of uniform embeddability into a Hilbert space is paid attention in a paper of Gromov [Gro5].

Definition 2.7.7. Let (X, d_X) and (Y, d_Y) be a metric space, and G be a finitely generated group.

(*i*) Let $f\colon X \to Y$ be a (not necessarily continuous) map. We say f is a *uniformly embedding* into Y, if there exist two monotone increasing functions $\rho_{\pm}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that the following two conditions hold:

(*a*) For any $x_1, x_2 \in X$, $\rho_-(d_X(x_1,x_2)) \leq d_Y(f(x_1), f(x_2)) \leq \rho_+(d_X(x_1,x_2))$.

(*b*) $\lim_{t\to\infty} \rho_-(t) = \infty$.

We say X is *uniformly embeddable* into Y if there exists a uniform embedding $f\colon X \to Y$.

(*ii*) We say G is *uniformly embeddable* into Y if there exists a uniform embedding $f\colon \mathrm{Cay}(G;\mathrm{S}) \to Y$ for some (equivalently any) symmetric finite generating set S of G.

(*iii*) We say X is *with bounded geometry* if for any $r > 0$ there exist $N > 0$ such that for any $x \in X$, $|B(x,r)| \leq N$. Here $B(x,r)$ is the closed ball with center x and radius r.

We refer to a book of J. Roe [Roe] for comprehensive treatment on coarse geometry. In coarse geometry, we consider a finitely generated group G as a metric space $\mathrm{Cay}(G;\mathrm{S})$. Usually the symbol $|G|$ is used for an associated metric space with G (since for any pair of symmetric finite generating sets, the word lengths associated to those are bi-Lipschitz equivalent, $|G|$ is uniquely determined as a coarse equivalence class). However we will not use this symbol in this thesis, because we use $|\cdot|$ for the number of finite sets. Note that G, viewed as a metric space, is with bounded geometry.

The following celebrated theorem of G. Yu [Yu1] and Kasparov–Yu [KaYu] suggests importance of uniform embeddability into Hilbert spaces and into uniformly convex Banach spaces. Although we will not go into details around the statement, we state the theorem. For the definition of uniform convexity, see Subsection 3.1.1.

Theorem 2.7.8. (*Yu* [Yu1]*; Kasparov–Yu* [KaYu]) *Suppose X is a metric space with bounded geometry. If X is uniformly embeddable into a Hilbert space, then X satisfies the coarse Baum–Connes conjecture. In particular, if a finitely generated group G is uniformly embeddable into a Hilbert space, then G, as a group, satisfies the injectivity side of the Baum–Connes conjecture and satisfies the Novikov conjecture.*

In above, the same statements hold true if a Hilbert space is replaced with any uniform convex Banach space.

For a family of expanders $(X_n)_n$, we regard it as a metric space as follows: for any $v \in X_n$ and $w \in X_m$, we define the distance d by $d(v,w) := d_n(v,w)$ if $n = m$; and $d(v,w) := \mathrm{diam}(X_n) + \mathrm{diam}(X_m) + 1$ otherwise. Here d_n is the graph metric on X_n and $\mathrm{diam}(X)$ is the diameter of the graph X (, namely, the maximum of distances of two vertices).

Lemma 2.7.9. *Any family of expander, viewed as a single metric space in the manner above, is* not *uniformly embeddable into a Hilbert space.*

Proof. Take any family of expanders $(X_n)_n = ((V_n, E_n)_n)$. Let d be the regularity of each X_n. Suppose there exists a uniform embedding f into a Hilbert space ℓ^2. Restrict f on each X_n, and by translation of the origin respectively we get a sequence of uniform embeddings $f_n \colon X_n \to \ell^2$ such that the following holds:

(a) For any n, $\sum_{v\in V_n} f_n(v) = 0$.

(b) There exists a constant C (independent of n) such that $\|f_n(v)-f_n(w)\| \geq C\|v-w\|$ whenever $(v,w) \in E_n$.

(c) For any $r > 0$ there exists a constant M_r (independent of n) such that for any n, at most M_r points of V_n are mapped by f_n into any ball of radius r in ℓ^2.

Indeed, for item (b), set $C = \rho_+(1)$; and for item (c), set $M_r = \rho_-^{-1}(r)$, in the definition of a uniform embedding. We use the definition of spectral expanders, and take $\lambda > 0$ such that for all n $\lambda_1(X_n) \geq \lambda$. Then in the view of Remark 2.7.2, for n the following holds:

$$\text{for any } \xi \in \ell^2(V_n) \text{ with } \xi \perp 1, \quad \lambda \sum_{v\in V_n} |\xi(v)|^2 \leq \sum_{(v,w)\in E_{n,\pm}} |\xi(v) - \xi(w)|^2.$$

Here $E_{n,\pm}$ means the set of all oriented edges. By summing the inequality above up associated with the various components of the vector-valued function f_n, we obtain the following:

$$\lambda \sum_{v\in V_n} \|f_n(v)\|^2 \leq \sum_{(v,w)\in E_{n,\pm}} \|f_n(v) - f_n(w)\|^2.$$

Here we use item (a). By using item (b) and noticing $|E_{n,\pm}| = d|V_n|$, we have

$$\sum_{v\in V_n} \|f_n(v)\|^2 \leq \lambda^{-1} d C^2 |V_n|.$$

In particular, for all $v \in V_n$, $\|f_n(v)\| \leq C'|V_n|^{1/2}$ holds, where C' is a universal constant, independent of n and v.

As $n \to \infty$, this contradicts item (c) because $|V_n|^{-1/2} \to 0$. Therefore, no uniform embeddings exist. □

The following observation implies that a family of expanders represents strong rigidity in coarse setting. This is an analogue to that property (T) represents strong rigidity in group setting. In comparison, we state the following surprising theorem of Guentner–Higson–Weinberger. Note that there is a concept of *property* A of Yu [Yu1] for metric spaces (and for finitely generated groups), and that property A implies uniform embeddability into a Hilbert space (a slogan is the following: *uniform embeddability is coarse version of the Haagerup property; and property* A *is coarse version of amenability.* Compare with Lemma 2.5.8).

Theorem 2.7.10. (*Guentner–Higson–Weinberger* [GHW]) *Every linear group G has property* A. *In particular, G admits uniformly embedding into a Hilbert space.*

Therefore although $\mathrm{SL}_3(\mathbb{Z})$ has (T), it satisfies the injectivety part of the Baum–Connes conjecture and thus satisfies the Novikov conjecture. We note that the surjectivity part of the Baum–Connes conjecture for $\mathrm{SL}_3(\mathbb{Z})$ is a big open problem, and (T) for this group is one of obstructions in accessing the surjectivity side. We also note that in a recent work of Guentner–Tessera–Yu [GTY], they show all linear groups have *finite geometric complexity*, which is stronger than having property A.

Chapter 3

Property (T_B) and (F_B)

In this chapter, we treat property (T_B) and property (F_B), which are respectively variants of property (T) and property (FH), introduced by a paper of Bader–Furman–Gelander–Monod [BFGM]. We see in general (F_B) is a stronger property than (T_B). We will examine the case of $B = \mathcal{L}_p$, the class of all L^p spaces (for fixed $p \in (1,\infty)$), on details and see the proof in [BFGM] of the fact any totally higher rank lattice enjoys ($\mathrm{F}_{\mathcal{L}_p}$) for all p. Basic reference in this chapter is the original paper of Bader–Furman–Gelander–Monod [BFGM]. Also, we refer to a book of Baniyamini–Lindenstrauss [BeLi] for preliminaries on Banach spaces.

3.1 Preliminaries on Banach spaces

In this section, we recall basic facts on uniform convexity, uniform smoothness, and superreflexivity of Banach spaces. Also, we see some useful facts in [BFGM] on uniformly bounded representations on a superreflexive Banach space, which are employed in this thesis.

3.1.1 Uniform convexity, uniform smoothness, and superreflexivity

Definition 3.1.1. Let E be a Banach space (recall from Chapter 0, $B(E)$ denotes the closed unit ball at the origin in E).

(*i*) The space E is said to be *uniformly convex* (or *uc*) if for all $0 < \epsilon < 2$, $d_{\|\cdot\|}(\epsilon) > 0$ holds. Here for $0 < \epsilon < 2$, we define

$$d_{\|\cdot\|}(\epsilon) = \inf\left\{1 - \frac{\|\xi + \eta\|}{2} : \xi, \eta \in B(E), \text{ and } \|\xi - \eta\| \geq \epsilon\right\}.$$

(*ii*) The space E is said to be *uniformly smooth* (or *us*) if $\lim_{\tau \to 0} r_{\|\cdot\|}(\tau)/\tau = 0$ holds. Here for $\tau > 0$, we define

$$r_{\|\cdot\|}(\tau) = \sup \left\{ \frac{\|\xi + \eta\| + \|\xi - \eta\|}{2} - 1 : \|\xi\| \leq 1, \, \|\eta\| \leq \tau \right\}.$$

(*iii*) The space E is said to be *ucus* if E is uc and us.

(*iv*) The space E is said to be *superreflexive* if it admits a compatible ucus norm.

We call d and r the *modulus of convexity* and that of *smoothness* respectively.

We refer to Appendix A of [BeLi] for details on ucus Banach spaces.

Example 3.1.2. (*i*) Any Hilbert space $\mathcal{H}$ is ucus. Indeed, by the parallelogram law,

$$d_{\|\cdot\|_{\mathcal{H}}}(\epsilon) = 1 - \sqrt{1 - (\epsilon/2)^2}$$

and

$$r_{\|\cdot\|_{\mathcal{H}}}(\tau) = \sqrt{1 + \tau^2} - 1.$$

(*ii*) It is a theorem that any L^p space is ucus if $1 < p < \infty$. However L^1 spaces and L^∞ spaces (infinite dimensional) are not superreflexive (the latter fact follows from the fact that superreflexivity implies reflexivity, which is also nontrivial in this definition of superreflexivity).

The following lemma is due to Lindenstrauss, and gives a new characterization of uniform smoothness:

Lemma 3.1.3. *Let $(B, \|\cdot\|)$ be a Banach space. Then for any $\tau > 0$,*

$$r_{\|\cdot\|}(\tau) = \sup_{0 < \epsilon < 2} \left\{ \frac{\epsilon\tau}{2} - d_{\|\cdot\|_*}(\epsilon) \right\}.$$

In particular, B is us if and only if B^* is uc.

The *duality mapping*, defined below, plays a key role in this chapter. Recall from Chapter 0 we use the symbol $\langle \cdot, \cdot \rangle$ for the duality $B \times B^* \to \mathbb{C}$. We mention that the usual order is $B^* \times B \to \mathbb{C}$, but that for our purpose the order above is more convenient.

Definition 3.1.4. (*Duality mapping*) Let B be a us Banach space. For $\xi \in S(B)$, we define $\xi^* \in S(B^*)$ as the unique element in $S(B^*)$ such that $\langle \xi, \xi^* \rangle = 1$ holds. We call this map $S(B) \to S(B^*); \xi \mapsto \xi^*$ the *duality mapping*.

Note that although we can extend duality mapping to a map $B \to B^*$ by real multiplication, the resulting map is in general *not* conjugate linear. Namely, $(\xi + \eta)^*$ does *not* coincide with $\xi^* + \eta^*$, unless B is a Hilbert space. Therefore, we restrict the duality mapping on the unit sphere.

One merit of restricting the duality mapping on the unit sphere is that then it is uniformly continuous. The following proposition is Proposition A.5 in [BeLi], but we note that there seems to be a misprint in the original version. Here we present a corrected one.

Lemma 3.1.5. *Let B be us. Suppose $0 < \kappa < 2$. Then for any $\xi, \eta \in S(B)$ with $\|\xi - \eta\| \leq \kappa$, there is an inequality*

$$\|\xi^* - \eta^*\|_* \leq 2 \cdot \frac{r_{\|\cdot\|}(2\kappa)}{\kappa}.$$

In particular, the duality mapping $S(B) \to S(B^*)$ is uniformly continuous.

We deal with a (uniformly bounded) representation on a (superreflexive) Banach space. Here we see a representation yields a corresponding representation on the dual Banach space.

Definition 3.1.6. Let ρ be a representation of a group G on a Banach space B. Then we define the *contragredient representation* $\rho^\dagger$ of G on B^* as follows:

$$\text{for any } g \in G,\ \phi \in B^* \text{ and } \xi \in B,\ \langle \xi, \rho^\dagger(g)\phi \rangle = \langle \rho(g^{-1})\xi, \phi \rangle.$$

Lemma 3.1.7. *Let ρ be a representation of a group G on a Banach space B.*

(i) For any $g \in G$, $\|\rho(g^{-1})\| = \|\rho^\dagger(g)\|$. Here the norms are operator norms with repsect to respectively $(B, \|\cdot\|_B)$ and $(B^*, \|\cdot\|_{B^*})$.

(ii) Suppose B is us and ρ is isometric. Then there is an equality

$$\text{for any } \xi \in S(B) \text{ and any } g \in G,\quad (\rho(g)\xi)^* = \rho^\dagger(g)\xi^*.$$

Here $\xi \mapsto \xi^*$ denotes the duality mapping on the unit sphere.

Proof. Item (i) is a direct consequence of the Hahn–Banach theorem. Item (ii) holds true because $\|\rho^\dagger(g)\xi^*\|_{B^*} = \|\xi^*\|_{B^*} = 1$ and $\langle \rho(g)\xi, \rho^\dagger(g)\xi^* \rangle = \langle \xi, \xi^* \rangle = 1$. □

Finally, we state the following lemma, which plays a basic role in theory of fixed point properties:

Lemma 3.1.8. *Let X be a* bounded *subset in a uc Banach space B. Then there exists a* unique *closed ball with the minimum radius which includes X. We define the* Chebyshev center *of X as the center of this ball.*

Proof. Firstly, we show existence. Set $r = \inf_{\eta \in B} \sup_{\xi \in X} \|\xi - \eta\|$. Since X is bounded, $r < \infty$. For any $t > r$, set

$$C_t = \{\eta \in B : \sup_{\xi \in X} \|\xi - \eta\| \leq t\} \neq \emptyset.$$

Then C_t is a closed and bounded convex subset. Now we employ the fact that a uc Banach space is reflexive. Therefore, each C_t is weak*-compact, and clearly $(C_t)_{t>r}$ satisfies the finite intersection property. Thus the set

$$C = \bigcap_{t>r} C_t$$

is nonempty, and an element in C attains the minimum r.

Secondly, we show uniqueness, but this directly follows form uniform convexity of B. □

In the view of item (i) of Example 3.1.2, Lemma 2.1.13 is a special case of Lemma 3.1.8.

3.1.2 Changing norms and space decomposition

In the next section, we mainly consider an isometric (linear) representation on a superreflexive Banach space. By Definition 3.1.1, this a priori means that this representation can be seen as a uniformly bounded representation on a ucus Banach space, in the following sense.

Definition 3.1.9. A representation ρ of Γ on B is said to be *uniformly bounded* if $|\rho| := \sup_{g \in \Gamma} \|\rho(g)\|_{\mathbb{B}(B)} < +\infty$.

One observation of Bader–Furman–Gelander–Monod states that in fact one can take a norm in above such that the representation is *isometric*.

Proposition 3.1.10. (*Proposition* 2.3 [BFGM]) *Let ρ be a uniformly bounded representation ρ of a group G on a superreflexive Banach space B. Then there exists a compatible norm on B with respect to which B is ucus and ρ is isometric.*

Proof. By the definition of superreflexivity we may assume B is ucus. Put $M = |\rho|$.

Firstly, we show there exists a compatible *uc* norm on B with respect to which ρ is isometric. Indeed, for an original norm $\|\cdot\|_B$, set a new norm $\|\cdot\|$ by

$$\|\xi\| := \sup_{g \in G} \|\rho(g)\xi\|_B.$$

Then by assumption this is compatible with $\|\cdot\|_B$, and $\rho(G)$-invariant. We show the uniform convexity. Let $\alpha > 0$ and $\xi, \eta \in B$ with $\|\xi\| \leq 1, \|\eta\| \leq 1$ and $\|(\xi + \eta)/2\| > 1 - \alpha$. Then there exists $g \in G$ such that

$$\left\| \frac{\rho(g)\xi + \rho(g)\eta}{2} \right\|_B > 1 - \alpha \quad \text{while} \quad \|\rho(g)\xi\|_B \leq \|\xi\| \leq 1 \text{ and } \|\rho(g)\eta\|_B \leq \|\eta\| \leq 1.$$

By the definition of the modulus of convexity of $\|\cdot\|_B$, we have

$$\alpha \geq d_{\|\cdot\|_B}(\|\rho(g)\xi - \rho(g)\eta\|) \geq d_{\|\cdot\|_B}\left(\frac{\|\xi-\eta\|}{M}\right).$$

This exactly shows

$$\text{for any } \epsilon > 0,\ \ d_{\|\cdot\|}(\epsilon) \geq d_{\|\cdot\|_B}(\epsilon/M) > 0.$$

This shows $\|\cdot\|$ is also uc.

Now let $N(B)$ be the set of all compatible norms on B equipped with the metric

$$d(\|\cdot\|_1, \|\cdot\|_2) := \log(\text{the norm ratio}).$$

Here the *norm ratio* between two compatible norms is defined by

$$\max\{\sup_{x\neq 0} \|x\|_1/\|x\|_2, \sup_{x\neq 0} \|x\|_2/\|x\|_1\}.$$

Note that $N(B)$ becomes a complete metric space. Let $N(G)^{\rho(G)}$ stands for the closed subspace of norms on B invariant under $\rho(G)$. Define the subset $N_{\mathrm{uc}}(V)^{\rho(G)}$ as the set of all uc $\rho(G)$-invariant norms. Then this is given by the following countable intersection:

$$N_{\mathrm{uc}}(V)^{\rho(G)} = \bigcap_{n=1}^{\infty} O_n, \quad \text{where } \ O_N := \{\|\cdot\| \in N(V)^{\rho(G)} : d_{\|\cdot\|}(1/n) > 0\}.$$

(Recall $d_{\|\cdot\|}$ denotes the modulus of convexity.) Observe the sets O_n are open. Also, we claim $N_{\mathrm{uc}}(V)^{\rho(G)}$ is dense in $N(B)^{\rho(G)}$. Indeed, let $\|\cdot\|_0$ be an element in $N_{\mathrm{uc}}(V)^{\rho(G)}$ (we see in the first paragraph of this proof that this set is non-empty.) Then any $\|\cdot\| \in N(V)^{\rho(G)}$ can be viewed as a limit of uniformly convex norms $\|\cdot\| + \epsilon\|\cdot\|_0$ as $\epsilon \to 0$.

By duality between $N_{\mathrm{uc}}(B^*)^{\rho^\dagger(G)}$ and $N_{\mathrm{us}}(B)^{\rho(G)}$ (see Lemma 3.1.3 and Definition 3.1.6), we also show $N_{\mathrm{uc}}(V)^{\rho(G)}$ is an intersection of open subsets and is dense. Here $N_{\mathrm{us}}(B)^{\rho(G)}$ denotes the set of all uniformly smooth $\rho(G)$-invariant norms in $N(B)$.

Therefore the set $N_{\mathrm{uc}}(B)^{\rho(G)} \cap N_{\mathrm{us}}(B)^{\rho(G)}$ is an intersection of open dense subsets in $N(B)^{\rho(G)}$. Since $N(B)^{\rho(G)}$ is a complete metric space, the Baire category theorem applies. Thus $N_{\mathrm{uc}}(B)^{\rho(G)} \cap N_{\mathrm{us}}(B)^{\rho(G)}$ is also dense in $N(B)^{\rho(G)}$, and in particular is not empty. This ends our proof. $\square$

Thanks to this proposition, as long as treating uniformly bounded representation on a superreflexive Banach space, we may assume the representation is isometric and the Banach space is ucus.

Finally, we need a construction of natural complement of subspaces, associated with an isometric representation ρ of a group G on a Banach space B. If a representation $(\pi, \mathfrak{H})$ is unitary, then the canonical orthogonal complement does the job, and there is a decomposition of $\mathfrak{H}$ as $\pi(G)$-spaces

$$\mathfrak{H} = \mathfrak{H}^{\pi(G)} \oplus \mathfrak{H}^{\perp}_{\pi(G)}.$$

In general case, it is also possible to find a canonical candidate of complement. Since we need the relative version, we define in the following setting:

Definition 3.1.11. ([BFGM]) Let G be a group, $N \trianglelefteq G$ be a *normal* subgroup. Let ρ be a uniformly bounded G-representation on a Banach space B. Then we define a closed subspace $B'_{\rho(N)}$ as the annihilator of $(B^*)^{\rho^\dagger(N)}$. Here $(B^*)^{\rho^\dagger(N)}$ means the subspace of all $\rho^\dagger(N)$-invariant vectors in B^*.

Since N is normal, $B'_{\rho(N)}$ is $\rho(G)$-invariant space.

In general, it is not necessarily true that $B = B^{\rho(N)} \oplus B'_{\rho(N)}$. However, Bader–Furman–Gelander–Monod show this holds true if B is superreflexive (note that for the $B'_{\rho(N)}$ is independent of the choice of compatible norms on B).

Proposition 3.1.12. (*Proposition* 2.6 *and Proposition* 2.10, [BFGM]) *Let G be a group, $N \trianglelefteq G$ be a normal subgroup. Let ρ be a uniformly bounded G-representation on a superreflexive Banach space B. Then there is a decomposition of B as $\rho(G)$-spaces*

$$B = B^{\rho(N)} \oplus B'_{\rho(N)}.$$

Moreover, let any $\xi = \xi_0 + \xi_1$ where $\xi \in B$, $\xi_0 \in B^{\rho(N)}$ and $\xi_1 \in B'_{\rho(N)}$. Then for any uc compatible norm $\|\cdot\|'$ on B, there are inequalities

$$\|\xi_0\|' \leq \|\xi\|' \quad \text{and} \quad \|\xi_1\|' \leq 2\|\xi\|'.$$

In particular, the G-representation ρ restricted on $B'_{\rho(N)}$ is isomorphic to the G-representation ρ' on the quotient Banach space $B/B^{\rho(N)}$ induced by ρ.

Proof. With the aid of Proposition 3.1.10, take a ucus $\rho(G)$-invariant norm $\|\cdot\|$.

By item (*ii*) of Lemma 3.1.7, note that for any $\xi \in S(B^{\rho(N)})$ and any $\eta \in B'_{\rho(N)}$, $\langle \eta, \xi^* \rangle = 0$ holds. Hence,

$$1 = \langle \xi, \xi^* \rangle = \langle \xi - \eta, \xi^* \rangle \leq \|\xi - \eta\|.$$

Therefore $B^{\rho(N)} \cap B'_{\rho(N)} = \{0\}$. We then claim $B^{\rho(N)} \oplus B'_{\rho(N)}$ (, which is closed,) is dense in B. Indeed, suppose there exists $\phi \in B^* \setminus \{0\}$ which vanishes on $B^{\rho(N)}$ and on $B'_{\rho(N)}$. We may assume ϕ is a unit vector in B^*. By the latter condition, the Hahn–Banach shows $\phi \in (B^*)^{\rho^\dagger(N)}$. With noting that B is reflexive and B^* is us, we can consider the duality mapping $S(B^*) \to S(B)$. Then $\phi^* \in S(B)$ in fact lies in

$S(B^{\rho(N)})$ for the reason above. This contradicts our assumption of ϕ vanishing on $B^{\rho(N)}$. Therefore, we conclude $B = B^{\rho(N)} \oplus B'_{\rho(N)}$.

The inequalities in the proposition follow from our first argument, because at that point only uniform smoothness of the norm on B is needed. $\square$

3.2 Definitions of (T_B) and (F_B)

In this section, we define property (T_B) and property (F_B) for groups, for a given Banach space (or a class of Banach spaces). Also, we define relative property (T_B) for a group pair $G \trianglerighteq N$; and property (F_B) for a group pair $G \geqslant N$ (we will explain below why normality is needed for the definition of relative (T_B)). Finally, we see (F_B) does not necessarily imply (T_B), whereas (T_B) always implies (F_B).

3.2.1 Definitions and examples

Firstly, we define a representation having almost invariant vectors.

Definition 3.2.1. Let ρ be a representation of a group G on a Banach space.

(i) For a subset $S \subseteq G$ and $\kappa > 0$, a vector $\xi \in \mathfrak{H}$ is said to be (S,κ)-*invariant* if

$$\sup_{s \in S} \|\xi - \rho(s)\xi\| < \kappa \|\xi\|.$$

(ii) We say ρ has *almost invariant vectors* if for any compact subset $S \subseteq G$ and $\kappa > 0$, there exists an (S,κ)-invariant vectors. We write $\rho \succeq 1_G$ if ρ has almost invariant vectors.

Note that the concept of a representation having almost invariant vectors is determined independently of the choices of compatible norms on B. Also note that the symbol $\rho \succeq 1_G$ is artificial, because in general there is no analogue of diagonal matrix coefficients (note that if B is us, then the map

$$G \ni g \mapsto \langle \rho(g)\xi, \xi^* \rangle \in \mathbb{C}$$

works, where $\xi \in S(B)$).

Recall that property (T) for a group is defined by the condition "for any unitary G-representation, $\pi \not\supseteq 1_G \Rightarrow \pi \not\succeq 1_G$." However, in general setting, the straight generalization of above does not give information for the case of $\pi \supseteq 1_G$. More precisely, if we consider unitary representation, then we can restrict our representation on the orthogonal complement of the space of invariant vectors. However even in the case of considering isometric representation on L^p spaces, the canonical complement, defined in Definition 3.1.11 is a subspace of L^p space, and usually *not* realizable as an L^p space on any measure space. Therefore, the following definition is appropriate:

Definition 3.2.2. ([BFGM]) Let B be a Banach space.

(*i*) A pair $G \trianglerighteq N$ of groups is said to have *relative property* (T_B) if for any isometric representation ρ of G on B, the isometric representation ρ' on the quotient Banach space $B/B^{\rho(N)}$, naturally induced by ρ, satisfies

$$\rho' \not\succeq 1_G.$$

By Proposition 3.1.12, if B is superreflexive, then the definition above is equivalent to the following: for any isometric representation ρ of G on B, the restriction of ρ on $B'_{\rho(N)}$ (see Definition 3.1.11) does not have almost invariant vectors.

(*ii*) A group G is said to have *property* (T_B) if $G \trianglerighteq G$ has relative (T_B).

Note that here normality of the subgroup N is needed, otherwise $B^{\rho(N)}$ may not be $\rho(G)$-invariant.

We move to the definition of property (F_B). First, we define the first group cohomology with isometric Banach coefficient.

Definition 3.2.3. Let B be a Banach space. Let G be a group, and ρ be an isometric G-representation on B.

(*i*) A continuous map $c\colon G \to B$ is called a *ρ-cocycle* if the following holds:

$$\text{For any } g, h \in G,\ \ c(gh) = c(g) + \rho(g)c(h).$$

This equality is called the *cocycle identity*.

(*ii*) A continuous map $c\colon G \to B$ is called a *ρ-coboundary* if the following holds:

$$\text{There exists } \xi \in B \text{ such that for any } g \in G,\ \ c(g) = \xi - \pi(g)\xi.$$

(*iii*) The space $Z^1_{\mathrm{c}}(G;\rho,B)$ (or shortly, $Z^1_{\mathrm{c}}(G;\rho)$) denotes the vector space of all ρ-cocycles. The space $B^1_{\mathrm{c}}(G;\rho,B)$ (or shortly, $B^1_{\mathrm{c}}(G;\rho)$) denotes the vector space of all ρ-coboundaries, which is a subspace of $Z^1_{\mathrm{c}}(G;\rho)$. The quotient vector space

$$H^1_{\mathrm{c}}(G;\rho,B) := Z^1_{\mathrm{c}}(G;\rho)/B^1_{\mathrm{c}}(G;\rho)$$

(or shortly, $H^1_{\mathrm{c}}(G;\rho)$) is called the *first cohomology group* with ρ-coefficient.

(*iv*) A cocycle c is said to be *proper* if it tends to infinity as $g \to \infty$, namely, if for any real $M > 0$ the subset $\{g \in G : \|c(g)\| \leq M\}$ in G is compact.

If G is discrete, we use the symbols $Z^1(G;\rho), B^1(G;\rho), H^1(g;\rho)$ instead.

In the same way as one in Proposition 2.3.3, we have the following: there is a one-to-one correspondence (as sets) between affine isometric actions α on B and pairs (ρ, c) of isometric G-representation ρ on B and ρ-cocycles c. The correspondence is:

$$\alpha(g)\cdot\xi = \rho(g)\xi + c(g) \quad (g \in G,\ \xi \in B).$$

In the correspondence above, we call ρ the *linear part* of α, and call c the *cocycle part* (or *transition part*) of α.

Definition 3.2.4. Let B be a Banach space.

(i) A group pair $G \geqslant H$ is said to have *relative property* (F_B) if for any isometric representation ρ of G and for every ρ-cocycle c, the restriction of c on H is a coboundary (with $\rho\mid_H$ coefficient). Equivalently, if every affine isometric G-action on B has an H-fixed point.

(ii) A group $G \geqslant H$ is said to have *property* (F_B) if $G \geqslant G$ has relative (T_B). Equivalently, if for any isometric G-representation ρ on B,

$$H^1_{\mathrm{c}}(G;\rho,B) = 0.$$

By Lemma 3.1.8, there is a useful characterization of (relative) property (F_B) in the case of that B is superreflexive.

Lemma 3.2.5. *Let B a superreflexive Banach space. Then for any affine isometries action α of a group G on B, the following are all equivalent:*

(i) *The action α has a G-fixed point.*

(ii) *Any G-orbit is bounded.*

(iii) *Some G-orbit is bounded.*

(iv) *The cocycle part of α is bounded.*

In particular, a group pair $G \geqslant H$ has relative (F_B) if and only if for any isometric G-representation on B and for any ρ-cocycle c, c is bounded on H.

Note that we define these properties also for a given class of Banach spaces as we mentioned in Chapter 0. Namely, for a class of Banach space $\mathcal{C}$, property $(P_{\mathcal{C}})$ is defined as having property (P_B) for all $B \in \mathcal{C}$. Here are classes of Banach spaces of our main concern:

- $\mathcal{H}$ denotes the class of all Hilbert spaces;
- $\mathcal{L}_p$ denotes the class of all L^p spaces on any measure spaces;

- $[\mathcal{H}]$ denotes the class of all Banach spaces which admit compatible norms to those of Hilbert spaces.

Recall from Chapter 0 that *we always assume* $p \in (1, \infty)$ (and that we always assume groups are σ-compact, unless otherwise stating). Hence by Example 3.1.2, all elements in those classes are superreflexive. We will explain some reasons why we are mainly interested in superreflexive cases in the example below:

Example 3.2.6. (*i*) Property $(\mathrm{T}_{\mathcal{H}})$ is identical to property (T), and property $(\mathrm{F}_{\mathcal{H}})$ is identical to property (FH). Therefore, in the case of $B = \mathcal{H}$, (relative) property (T_B) is equivalent to (relative) property (F_B), thanks to the Delorme–Guichardet theorem (Theorem 2.4.13). In some sense, thus one can define relative property (T) for a pair $G \geqslant H$ where H is not normal.

(*ii*) Consider the closed subspace $B = L^1_0(G) \subseteq L^1(G)$ of all elements ξ in $L^1(G)$ with $\int_G \xi(x) d\mu(x) = 0$. Then whenever G is noncompact, G *fails* to have (F_B). To see this, choose an element $\eta \in L^1(G)$ with $\int_G \eta(x) d\mu(x) = 1$ and construct a λ_G-cocycle $c\colon G \to B$ by

$$c(g) = \eta - \lambda_G(g)\eta,$$

where λ_G is the left regular representation on $L^1(G)$. Note that this c ranges into B, although $\eta \notin B$. However this c is *not* a coboundary in B. Indeed, if c is coboundary in B, namely, there exists $\zeta \in B$ such that $c(g) = \zeta - \lambda_G(g)\zeta$ holds, then this the vector $\eta - \zeta$ is $\lambda_G(G)$-invariant, and hence it equals 0 because G is noncompact. This is a contradiction.

This example shows Lemma 3.2.5 is *false* for general Banach space B. Note that $L^p_0(G)$ is *no longer* closed in $L^p(G)$ for $p \in (1, \infty)$, where $L^p_0(G) := \{\xi \in L^p(G) : \int_G \xi(x) d\mu(x) = 1\}$.

(*iii*) Let G be an infinite discrete group. Then G does not have property $(\mathrm{F}_{\ell^\infty(G)})$. Indeed, let d be an unbounded left-invariant metric on G (for instance, a word length with respect to a (possibly infinite) symmetric generated set). Define a cocycle $c\colon G \to \ell^\infty(G)$ by

$$c(g)(x) = d(g, x) - d(e_G, x)$$

where ρ_G is the right regular representation on $\ell^\infty(G)$. It is straightforward to check this is indeed a cocycle. This is not a coboundary, because c is unbounded ($c(g)(e) = d(g, e)$).

(*iv*) Even strictly convex Banach space cases, there exists an example similar to above two. Haagerup–Przybyszewska [HaPr] show any group G admits a *proper* cocycle on the strictly convex Banach space $\bigoplus_{n \geq 1} L^{2n}(G)$. Here the direct sum is taken in ℓ^2-sense.

Finally, we define the relative Kazhdan constant for property (T_B).

Definition 3.2.7. ([Mim1]) Let B be a superreflexive Banach space, $G \trianglerighteq N$, and $S \subseteq G$ be a compact subset. Let ρ be an isometric G-representation on B. We define the *relative Kazhdan constant for property* (T_B) for $(G, N; S, \rho)$ by the following equality:

$$\mathcal{K}(G, N; S, \rho, B) = \inf_{\xi \in S(B_1)} \sup_{s \in S} \|\xi - \rho(s)\xi\|.$$

Here $B_1 = B'_{\rho(N)}$ as in Definition 3.1.11. We also write $\mathcal{K}(G, N; S, \rho, B)$ as $\mathcal{K}(G, N; S, \rho)$ for short.

By definition, if $G \trianglerighteq N$ has relative (T_B) and G is compactly generated, then for any ρ and any compact generating set $S \subseteq G$, $\mathcal{K}(G, N; S, \rho)$ is strictly positive.

Note that for the proof of Lemma 2.1.7, we need direct sum argument. Therefore, in general case, even if $G \trianglerighteq N$ has relative (T_B) and a compact subset $S \subseteq G$ generates G, this does *not* necessarily imply $\inf_\rho \mathcal{K}_B(G, N; S, \rho) > 0$, where ρ moves among all isometric G-representation on B. For instance, the class $[\mathcal{H}]$ is *not* stable under (at least L^2) direct sum. This is because there exists a sequence of elements $(B_n)_n$ in $[\mathcal{H}]$ with the norm ratio of between B_n and a Hilbert space tends to infinity.

However, if a class $\mathcal{C}$ of Banach space admits direct-sum operation inside $\mathcal{C}$, then in the setting above, $\inf_\rho \mathcal{K}_{\mathcal{C}}(G, N; S, \rho) > 0$ holds. Main examples are $\mathcal{L}_p$, including $\mathcal{H} = \mathcal{L}_2$, with L^p-direct sum operation. In the case of $B = [\mathcal{H}]$, if we instead consider a subclass of $[\mathcal{H}]$ with uniformly bounded norm ratio (to Hilbert space norms), then L^2-sum works.

The following lemma is a generalization of Lemma 2.1.16.

Lemma 3.2.8. ([Mim1]) *Suppose B is us, G is a compactly generated group and S is a compact generating set of G. Let $N \trianglelefteq G$. Let ρ be any isometric representation of G on B, ξ be any vector in B and set $\delta_\xi := \sup_{s \in S} \|\xi - \rho(s)\xi\|$. If a pair $G \trianglerighteq N$ has relative* (T_B)*, then there exists a $\rho(N)$-invariant vector $\xi_0 \in B$ with*

$$\|\xi - \xi_0\| \leq 2\mathcal{K}^{-1}\delta_\xi.$$

Here $\mathcal{K}$ stands for the relative Kazhdan constant $\mathcal{K}(G, N; S, \rho)$ for (T_B)*. In particular, there is an inequality*

$$\text{for any } l \in N, \ \ \|\xi - \rho(l)\xi\| \leq 4\mathcal{K}^{-1}\delta_\xi.$$

Proof. Decompose B as $B = B_0 \oplus B_1$, where $B_0 = B^{\rho(N)}$ and $B_1 = B'_{\rho(N)}$. Take the associated decomposition of ξ $\xi = \xi_0 + \xi_1$ $(\xi_0 \in B_0, \xi_1 \in B_1)$. Then

$$\xi - \rho(s)\xi = (\xi_0 - \rho(s)\xi_0) + (\xi_1 - \rho(s)\xi_1)$$

is the decomposition of $\xi - \rho(s)\xi$. By Proposition 3.1.12 and the definition of relative Kazhdan constant for (T_B), there are the following inequalities:

$$\begin{aligned}2\delta_\xi &= 2\sup_{s\in S}\|\xi-\rho(s)\xi\| \\ &\geq \sup_{s\in S}\|\xi_1-\rho(s)\xi_1\| \geq \mathcal{K}\|\xi_1\|.\end{aligned}$$

This shows $\|\xi_1\| \leq 2\mathcal{K}^{-1}\delta_\xi$, as desired.

For the latter part, observe for any $l \in N$,

$$\xi-\rho(l)\xi = (\xi_0-\rho(l)\xi_0)+(\xi_1-\rho(l)\xi_1) = \xi_1-\rho(l)\xi_1.$$

Therefore, the norm is at most $2\|\xi_1\| \leq 4\mathcal{K}^{-1}\delta_\xi$. □

3.2.2 Property $(\mathbf{F}_{\boldsymbol{B}})$ implies $(\mathbf{T}_{\boldsymbol{B}})$

Our proof of Guichardet's direction "(FH) ⇒ (T)" does not use any special property for Hilbert spaces. Thus one can extend it to a general setting, and obtain the following theorem of Bader–Furman–Gelander–Monod:

Theorem 3.2.9. ([BFGM]) *For any Banach space B, property* (F_B) *implies property* (T_B).

Note that here even superreflexivity assumption for B is unnecessary.

3.2.3 Property $(\mathbf{T}_{\boldsymbol{B}})$ does *not* implies $(\mathbf{F}_{\boldsymbol{B}})$

We note, in contrast to Theorem 3.2.9, property (T_B) does *not* imply property (F_B) in general. An extremely intriguing example is the case of $B = \mathcal{L}_p$ for $p > 2$ is sufficiently large. We will see in Subsection 3.3.3.

3.3 Property $(\mathbf{T}_{\mathcal{L}_p})$ and $(\mathbf{F}_{\mathcal{L}_p})$

In this section, we examine property $(\mathrm{T}_{\mathcal{L}_p})$ and $(\mathrm{F}_{\mathcal{L}_p})$. The main goals in this section are the following two: firstly, we show property (T) implies $(\mathrm{T}_{\mathcal{L}_p})$ (in fact, they show these are equivalent), a theorem due to Bader–Furman–Gelander–Monod; secondly, we see results of P. Pansu, Bourdon–Pajot and Yu which indicate that $(\mathrm{F}_{\mathcal{L}_p})$ is strictly stronger than (T) if $p \gg 2$.

3.3.1 Facts for L^p spaces

We collect needed facts on L^p spaces. Recall the definition of Bernstein function from Definition 2.4.10.

Lemma 3.3.1. *For $0 < \alpha < 1$, the function $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}; x \mapsto x^\alpha$ is a Bernstein function.*

Proof. It can be shown there exists some positive constant $c_\alpha > 0$ such that

$$x^\alpha = c_\alpha \int_0^\infty (1 - e^{-tx}) t^{-\alpha-1} dt.$$

□

Corollary 3.3.2. *Let p be $1 < p \leq 2$. Then for any L^p space B on a measure space, the kernel*

$$B \times B \to \mathbb{R};\ (\xi_1, \xi_2) \mapsto \|\xi_1 - \xi_2\|^p$$

is conditionally negative definite.

Proof. If $p = 2$, we see in item (i) of Example 2.4.6. For $1 < p < 2$, the conclusion follows from the case of $p = 2$, Theorem 2.4.10, and Lemma 3.3.1. □

Remark 3.3.3. Note that this fact itself is true even for $0 < p \leq 1$. Also, there is a theorem of Bretagnolle–Dacunha-Castelle–Krivine [BDK] that states the following: let $1 \leq p \leq 2$. Then for a Banach space B, the kernel defined in Corollary 3.3.2 is conditionally negative definite if and only if B is isometrically isomorphic to a closed subspace of some L^p space.

We warn that the statement of Corollary 3.3.2 is *no longer* true for $p > 2$.

Next, we need so-called the *Mazur map*. See Theorem 9.1 of [BeLi] for details.

Theorem 3.3.4. (*the Mazur map*) *Let ν be a σ-finite measure. Then for any $1 < p, q < \infty$, the* Mazur map *$M_{p,q} \colon L^p(\nu) \to L^q(\nu)$, defined by*

$$M_{p.q}(f) = \mathrm{sign}(f)|f|^{p/q},$$

is a (nonlinear) map which induces a uniformly continuous homeomorphism between the unit spheres $M_{p.q} \colon S(L^p(\nu)) \to S(L^q(\nu))$.

The statement of this theorem is true also for $p = 1$.

Finally, we need the classification theorem for linear isometries of L^p related spaces, which is called the *Banach–Lamperti theorem*. For a measure space (X, ν) and Banach space B, we let $L^p(X, B, \nu)$ denote the p-integrable B-valued function space by means of the Bochner integral (this space is needed for p-inductions. See Subsection 3.4.2).

Theorem 3.3.5. (*Banach–Lamperti*) *For any $p \neq 2$, any linear isometry U of $L^p(X, E, \nu)$ has the form*

$$Uf(x) = f(T(x))h(x) \left(\frac{dT_*\nu}{d\nu}(x) \right)^{1/p}.$$

Here T is a measurable, measure-class preserving map of (X, ν), and h is a measurable function with $|h(x)| = 1$ almost everywhere.

As a corollary of the Banach–Lamperti theorem, we obtain the following.

Corollary 3.3.6. *Let $p \neq 2$. Then for any linear isometry U on a L^p space on a measure space, the conjugation*

$$U \mapsto M_{p,2} \circ U \circ M_{2,p}$$

is a linear map. Here $M_{p,q}$ denotes the Mazur map.

Note that the Mazur map itself is nonlinear.

3.3.2 Property (T) implies $(\mathbf{T}_{\mathcal{L}_p})$

We will show the following theorem of Bader–Furman–Gelander–Monod. Recall $p \in (1, \infty)$, unless otherwise stated.

Theorem 3.3.7. ([BFGM]) *For any p,* (T) *implies* $(\mathrm{T}_{\mathcal{L}_p})$.

Remark 3.3.8. In fact, Bader–Furman–Gelander–Monod show the converse is also true. See Section 4.c in [BFGM].

Proof. We will sketch two ways in proving Theorem 3.3.7.

(1) The first way is imitate the proof of Delorme's direction (T) $\Rightarrow$ (FH). Thanks to Corollary 3.3.2, it is possible if $1 < p < 2$. Thus we obtain (T) $\Rightarrow$ $(\mathrm{F}_{\mathcal{L}_p})$ for $1 < p < 2$. (For details see Section 3.b of [BFGM].) By Theorem 3.2.9, in particular (T) $\Rightarrow$ $(\mathrm{T}_{\mathcal{L}_p})$ holds for $1 < p < 2$. Next, observe property (T_B) concerns only linear representations. Therefore with the aid of contragredient representation and duality mapping, for a superreflexive Banach space B, (T_B) is equivalent to (T_{B^*}). Therefore (T) $\Rightarrow$ $(\mathrm{T}_{\mathcal{L}_p})$ holds for any p.

(2) The second way is valid for L^p spaces on a σ-finite measure. Theorem 3.3.4 and Theorem 3.3.5 are the main tools here. It is the basic idea of Theorem C (and the previous work of Puschnigg [Pus]) for property (T_B) for noncommutative L^p spaces.

    Let $p \neq 2$. Set $B = L^p(\nu)$ and $\mathfrak{H} = L^2(\nu)$, where μ is a σ-finite measure. Suppose G does not have (T_B). Then there exists an isometric representation ρ on G such that the restriction of ρ on B'_ρ has almost invariant vectors. Set $\pi = M_{p,2} \circ \pi \circ M_{2,p}$. Then by Corollary 3.3.6, this π becomes a unitary representation $G \to U(\mathfrak{H})$. Thanks to Theorem 3.3.4 (uniform continuity), we also see that the restriction of π on $\mathfrak{H}^{\perp}_{\pi(G)}$ has almost invariant vectors. Therefore G does not have (T). This ends the proof for the case where $B = L^p(\nu)$ on a σ-finite measure.

□

Remark 3.3.9. The proof(s) of Theorem 3.3.7 in fact show the following:

Theorem 3.3.10. *For any p, relative property* (T) *implies relative property* ($\mathrm{T}_{\mathcal{L}_p}$). *More precisely, for any pair of a group and a normal* (*possibly non-proper*) *subgroup $G \trianglerighteq N$, if it has relative* (T), *then it has relative* ($\mathrm{T}_{\mathcal{L}_p}$).

This observation shall play a key role in Section 8.1.

3.3.3 Property (T) does *not* imply ($\mathrm{F}_{\mathcal{L}_p}$)

In this subsection, we see (T) does *not* imply ($\mathrm{F}_{\mathcal{L}_p}$) for $p \gg 2$. Recall from Subsection 2.6.4 we see plenty of examples of (infinite) hyperbolic groups with (T).

The following result is due to P. Pansu [Pan], deduced by means of study of L^p*-cohomology*, which is constructed from differential forms:

Theorem 3.3.11. (*Pansu* [Pan]) *For any $m \geq 2$, $\mathrm{Sp}_{m,1}$ fails to have* ($\mathrm{F}_{\mathcal{L}_p}$), *provided $p > 4m + 2$.*

Recall Kostant's theorem that $\mathrm{Sp}_{m\geq 2,1}$ has (T) (Theorem 2.6.18). Therefore, at least $p > 10$, ($\mathrm{F}_{\mathcal{L}_p}$) is strictly stronger than (T).

The following definition is not common, but it is convenient for our theory:

Definition 3.3.12. Let B be a Banach space or a class of Banach spaces. A group G is said to have the *B-Haagerup property* (or G is *a-F_B-menable*) if G admit a proper cocycle with some isometric B coefficient.

The following theorem of Cluckers–Cornulier–Louvet–Tessera–Valette is interesting.

Theorem 3.3.13. ([CCLTV]) *Any simple connected algebraic group G over a local field has the "bounded or proper" property. This means, any cocycle of G with isometric* (*superreflexive*) *Banach coefficient is either bounded or proper.*

By combining Theorem 3.3.11 and Theorem 3.3.13, we obtain the following theorem:

Theorem 3.3.14. *For any $m \geq 2$, $\mathrm{Sp}_{m,1}$ has the $\mathcal{L}_p$-Haagerup property, provided $p > 4m + 2$.*

For general (discrete) hyperbolic group, first Bourdon–Pajot [BoPa] show for each group, there exists $p \gg 2$ such that it fails to have ($\mathrm{F}_{\mathcal{L}_p}$) (with some estimate of such p). Later, Yu [Yu2] shows in fact they have the $\mathcal{L}_p$-Haagerup property for respectively large p:

Theorem 3.3.15. (*Yu* [Yu2]) *For every hyperbolic group H, there exists $p > 2$ such that H admits a proper cocycle on $B = \ell^p(H \times H)$, where the isometric representation of G on B is the diagonal translation.*

In particular, for each H there exists $p > 2$ such that H has the $\mathcal{L}_p$-Haagerup property.

We make remark that to the best knowledge of the author, there is no known example of $p > q > 2$ in which $(\mathrm{F}_{\mathcal{L}_p})$ implies $(\mathrm{F}_{\mathcal{L}_q})$.

3.4 $\mathrm{SL}_3(\mathbb{Z})$ has $(\mathbf{F}_{\mathcal{L}_p})$

In this section, we see the following result of Bader–Furman–Gelander–Monod: any totally higher rank group has $(\mathrm{F}_{\mathcal{L}_p})$ for any p. We pay a special attention for the case of $\mathrm{SL}_{m\geq 3}(\mathbb{R})$. Next we examine theory of p-induction of cocycles, which enables us to deduce $(\mathrm{F}_{\mathcal{L}_p})$ for certain lattices.

3.4.1 $\mathrm{SL}_3(\mathbb{R})$ has $(\mathbf{F}_{\mathcal{L}_p})$

In this subsection, the following is the main goal.

Theorem 3.4.1. *Let $\mathbb{K}$ be a local field. For any $m \geq 3$, $\mathrm{SL}_m(\mathbb{K})$ has $(\mathrm{F}_{\mathcal{L}_p})$ for any p. For any $m \geq 2$, $\mathrm{Sp}_{2m}(\mathbb{K})$ has $(\mathrm{F}_{\mathcal{L}_p})$ for any p.*

To show Theorem 3.4.1, we observe the following: firstly, by combining Theorem 3.3.10 and Theorem 2.6.5, we have the following result.

Theorem 3.4.2. ([BFGM]) *For any local field $\mathbb{K}$, the pair $G = \mathrm{SL}_2(\mathbb{K}) \ltimes \mathbb{K}^2 \trianglerighteq \mathbb{K}^2 = N$ has* strong relative $(\mathrm{T}_{\mathcal{L}_p})$ *in the following sense: for any isometric representation (ρ, B) with $B \in \mathcal{L}_p$, $\rho \mid_H$ on $B'_{\rho(N)}$ does not have almost invariant vectors. Here $H = \mathrm{SL}_2(\mathbb{K})$.*

Next, there is a generalization of the Howe–Moore theorem (Theorem 2.6.7). This is an unpublished result of Shalom, and stated in Section 9 in [BFGM] with proof.

Theorem 3.4.3. (*Shalom*) *Let G be a group of the form $\Pi_{i=1}^m \mathbf{G}_i(k_i)$, where k_i are local fields, $\mathbf{G}_i(k_i)$ are k_i-points of Zariski connected simple k_i-algebraic groups.*

Let B be a ucus Banach space. Let ρ be an isometric G-representation on B such that for each $G_i = \mathbf{G}_i(k_i)$ $\pi \mid_{G_i} \not\supseteq 1_{G_i}$. Then ρ is a C_0-representation, namely, for any vector $\xi \in B$ and $\phi \in B^*$, the matrix coefficient

$$G \ni g \mapsto \langle \rho(g)\xi, \phi \rangle \in \mathbb{C}$$

is a C_0-function.

Corollary 3.4.4. *Let G be a Zariski connected simple algebraic group G over a local field. Let B be a ucus Banach space. Then for any isometric G-representation ρ on B, there is an equality:*

$$\text{for any non-compact subgroup } H \leqslant G, \;\; B^{\rho(H)} = B^{\rho(G)}.$$

The difficulty in proving Theorem 3.4.1 is to deduce some form of property (F_B) from property (T_B). This direction is in general *not* true, and therefore we need some technique.

Proof. (Theorem 3.4.1) For simplicity, we only consider the case of $\mathrm{SL}_3(\mathbb{K})$. Let $G = \mathrm{SL}_3(\mathbb{K})$, and find a copy of $H = \mathrm{SL}_2(\mathbb{K})$ in the left upper corner and of $N = \mathbb{K}$ in the $(1,3)$-$(2,3)$-th unipotent part. Also set A be the multiplicative abelian group of diagonal elements, with its diagonal entries $[t, t, t^{-2}]$ $(t \in \mathbb{K}^{\times})$. Then for any isometric G-representation ρ on B, we have the following:

(1) The group H commutes with A.

(2) The restricted representation $\rho \mid_H$ on $B'_{\rho(G)}$ does not have almost invariant vectors.

Indeed, item (1) is obvious. Item (2) follows from Theorem 3.4.2 and Corollary 3.4.4 (we use $B^{\rho(N)} = B^{\rho(G)}$).

Let c be any ρ-cocycle. We project c to $B_1 = B'_{\rho(G)}$, and write it c_1. By $\rho(G)$-invariance of B_1, c_1 is also a ρ-cocycle on B_1. Then by the cocycle identity, for any $h \in H$ and $a \in A$,

$$c_1(h) + \rho(h)c_1(a) = c_1(ha) = c_1(ah) = c_1(a) + \rho(a)c_1(h).$$

Here we use item (1). Hence we have

$$c_1(a) - \rho(h)c_1(a) = c_1(h) - \rho(a)c_1(h).$$

By item (b), there exist a compact subset $S \subseteq H$ and $\kappa > 0$ such that for any $\xi \in B_1$, $\sup_{s \in S} \|\xi - \rho(s)\xi\| \geq \kappa\|\xi\|$. Set $R = \sup_{s \in S} \|c_1(s)\| < \infty$. Then from the equality above, we have,

$$\text{for any } a \in A, \;\; \kappa\|c_1(a)\| \leq \sup_{s \in S} \|c_1(s) - \rho(a)c_1(s)\| \leq 2R.$$

Therefore c_1 is bounded on A. By Lemma 3.2.5, there exists an A-fixed point for the affine isometric G-action α_1 on B_1 associated with (ρ, c_1). Note that this fixed point is unique, because by Corollary 3.4.4 B_1 has no non-zero $\rho(A)$-invariant vectors. Define $\eta \in B_1$ the unique $\alpha_1(A)$-fixed point.

Finally, set $T \leqslant G$ be the diagonal maximal split torus. Since T and A commute, any element in set $\alpha_1(T) \cdot \eta$ is $\alpha_1(A)$-fixed. Again by the uniqueness, $\alpha_1(T) \cdot \eta$

equals the single point η. This means η is $\alpha_1(T)$-fixed. Now consider the Cartan decomposition of G

$$G = KTK,$$

where K is the maximal compact subgroup. Through this decomposition, it follows from the argument above that any $\alpha_1(G)$ orbit is bounded. Since G has trivial abelianization (in general, finite abelianization follows from (T)), any affine isometric action on $B_0 = B^{\rho(G)}$ with linear part $\rho(= 1_G$ on $B_0)$ has bounded orbit. By combining these two boundedness, we obtain the boundedness of the affine isometric G-action on B with linear part ρ and cocycle part c. By Lemma 3.2.5, this means c is a coboundary. Therefore, we establish $(\mathrm{F}_{\mathcal{L}_p})$ for G. □

Note that the decomposition $B = B_0 \oplus B_1$ in this proof is essential for the argument.

In a similar manner to the proof above, we have the following theorem of Bader–Furman–Gelander–Monod (for more details, see Section 5.c in [BFGM]):

Theorem 3.4.5. ([BFGM]) *Any totally higher rank algebraic group has* $(\mathrm{F}_{\mathcal{L}_p})$ *for all* p.

3.4.2 p-inductions

We explain p-induction of cocycles in Section 8 of [BFGM]. Recall the induction of cocycles in unitary setting in Subsection 2.3.3. Let $\Gamma \leqslant G$ be a lattice, and $\mathcal{D}$ be a Borel fundamental domain for Γ. We identify $\mathcal{D}$ with G/Γ. Recall the definition of the Borel cocycle $\beta\colon G \times \mathcal{D} \to \Gamma$:

$$\beta(g, x) = \gamma \text{ if and only if } g^{-1}x\gamma \in \mathcal{D}.$$

Definition 3.4.6. A lattice Γ in a group G is said to be *p-integrable* if either of the following two conditions is satisfied:

(1) the lattice Γ is cocompact;

(2) the lattice Γ is finitely generated, and for some (equivalently any) symmetric finite generating set S of Γ, there exists a Borel fundamental domain $\mathcal{D} \subseteq G$ such that

$$\text{for any } \ g \in G, \quad \int_D l_S(\beta(g, x))^p d\mu(x) < \infty.$$

Here $l_S\colon \Gamma \to \mathbb{Z}_{\geq 0}$ denotes the word length on Γ with respect to S.

It is shown in §2 in a paper of Shalom [Sha3] that higher rank lattices, in the sense in Chapter 0 is p-integrable.

Suppose the lattice $\Gamma \leqslant G$ is p-integrable. Take a cocycle on a Banach space B, then consider the space $E = L^p(\mathcal{D}, B)$ of all p-integrable vector valued functions by means of the Bochner integral. A result of Figiel–Pisier implies the following:

Lemma 3.4.7. *If B is ucus, then so is E.*

Finally, define an induced cocycle $\tilde{c}$ of c on E as follows:

$$\tilde{c}(g)(x) := c(\beta(g,x)) \quad (g \in G,\ x \in \mathcal{D}).$$

The p-integrability condition ensures the well-definedness (namely, $\tilde{c} \in E$). With the aid of these theories, the following theorem is deduced in a similar manner to one in Theorem 2.3.16.

Proposition 3.4.8. ([BFGM]) *Keep the notation of this subsection. Suppose Γ is p-integrable. Then if G has (F_E), then Γ has (F_B).*

Thus finally, we obtain the following theorem of Bader–Furman–Gelander–Monod:

Theorem 3.4.9. ([BFGM]) *Any totally higher rank lattice has $(\mathrm{F}_{\mathcal{L}_p})$ for all p. In particular, $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ has $(\mathrm{F}_{\mathcal{L}_p})$ for all p.*

Proof. This follows from Theorem 3.4.5 and Proposition 3.4.8 because the class $\mathcal{L}_p$ is stable under p-inductions with the same p. $\square$

3.5 Property $(\mathrm{T}_{[\mathcal{H}]})$ and $(\mathrm{F}_{[\mathcal{H}]})$

Recall the class $[\mathcal{H}]$ consists of all Banach spaces B admitting compatible norms to those of Hilbert spaces. Therefore, by considering compatible Hilbert norms on B, we have the following (see also the proof of Proposition 3.1.10):

- Property $(\mathrm{T}_{[\mathcal{H}]})$ for a group G is equivalent to having the following condition: for any *uniformly bounded* representation ρ of G on a Hilbert space, $\rho \succeq 1_G$ implies $\rho \supseteq 1_G$.

- Property $(\mathrm{F}_{[\mathcal{H}]})$ for a group G is equivalent to having the following condition: for any *uniformly bounded* representation ρ on a Hilbert space, every ρ-cocycle is a coboundary (equivalently, is bounded). Also, this is equivalent to the following: any affine *uniformly bi-Lipschitz* action ρ of G on a Hilbert space has a G-fixed point (equivalently, has bounded orbits).

For relative versions, similar results hold.

These properties are respectively called $(\overline{T}_{\mathcal{H}})$ and $(\overline{F}_{\mathcal{H}})$. It seems to be an open problem whether $(\mathrm{T}_{[\mathcal{H}]})$ implies $(\mathrm{F}_{[\mathcal{H}]})$.

The following is an unpublished result of Shalom (see item (3) of Remark 1.7 in [BFGM]):

Theorem 3.5.1. (*Shalom*)

(*i*) *Any totally higher rank group and lattice have* $(\mathrm{F}_{[\mathcal{H}]})$. *In particular, they have* $(\mathrm{T}_{[\mathcal{H}]})$.

(*ii*) *Any rank* 1 *algebraic group and lattices therein fail to have* $(\mathrm{T}_{[\mathcal{H}]})$. *In particular, they also fail to have* $(\mathrm{F}_{[\mathcal{H}]})$.

Item (*ii*) of this theorem shows $(\mathrm{T}_{[\mathcal{H}]})$ (and hence $(\mathrm{F}_{[\mathcal{H}]})$) are strictly stronger than (T). For instance, consider $\mathrm{Sp}_{m,1}$ $(m \geq 2)$.

To the best knowledge of the author, it is unknown whether infinite hyperbolic group can have $(\mathrm{T}_{[\mathcal{H}]})$.

Finally, we define a generalization of Kazhdan constants in this setting. It is more convenient to consider uniformly bounded representations on Hilbert spaces than isometric representations on $[\mathcal{H}]$. Therefore we define in the following situation. With noting that a class of uniformly uniformly bounded representations are stable under L^2 direct sum, we define in the following manner.

Definition 3.5.2. ([Mim1]) Let $G \trianglerighteq N$ be a pair of groups, and S be a compact subset of Γ. For $M \geq 1$, we define $\mathcal{A}_M$ as the class of all pairs $(\rho, \mathfrak{H})$ of a uniformly bounded G-representation and a representation Hilbert space that satisfy $|\rho| \leq M$. We define the *relative Kazhdan constant for uniformly bounded representations* by the following equality:

$$\overline{\mathcal{K}}(\Gamma, N; S; M) := \inf_{(\rho,\mathfrak{H})\in\mathcal{A}_M} \inf_{\xi\in S\left(\mathfrak{H}'_{\rho(N)}\right)} \sup_{s\in S} \|\xi - \rho(s)\xi\|_{\mathfrak{H}}.$$

Chapter 4

Universal lattices

In this chapter, we define universal lattices, via the definition of elementary groups and the Suslin stability theorem. Next, we observe a celebrated method of Shalom [Sha1] to prove property (T) (and (FH)) from bounded generation. Finally, we consider bounded generation of elementary groups over certain rings, and state the difficulty in proving property (T) for universal lattices, which is a theorem of Shalom [Sha5] and Vaserstein [Vas2] (Theorem 1.0.1). This theorem shall be shown in Chapter 5. A main reference of this chapter is a paper [Sha1] of Shalom. See also Chapter 4 of [BHV] for details on Section 4.2 in this thesis.

4.1 Definition

We define elementary group over a (possibly noncommutative) ring.

4.1.1 Elementary group

Recall from Chapter 0 we always assume all rings are associative and unital. For a ring R (possibly noncommutative), let $\mathrm{M}_m(R)$ denote the $m \times m$ matrix ring over R.

Definition 4.1.1. Let R be a ring and $m \geq 2$.

(*i*) An *elementary matrix* in $\mathrm{M}_m(R)$ is a matrix of the form $E_{i,j}(r) = I_m + r \cdot e_{i,j}$ ($1 \leq i \leq m$, $1 \leq j \leq m$, $i \neq j$, and $r \in R$). Here $e_{i,j}$ denotes the matrix with the (i,j)-th entry 1 and the other entries 0.

(*ii*) The *elementary group* over R, written as $\mathrm{E}_m(R)$, is the multiplicative group in $\mathrm{M}_m(R)$ generated by all elementary matrices in $\mathrm{M}_m(R)$.

(*iii*) The *general linear group* over R, written as $\mathrm{GL}_m(R)$, is the multiplicative group of all invertible elements in $\mathrm{M}_m(R)$.

(*iv*) Suppose $A = R$ is commutative. Then the *special linear group* over A, written as $\mathrm{SL}_m(A)$, is the subgroup of $\mathrm{GL}_m(A)$ consisting of all elements of determinant 1.

We note in some literature, $\mathrm{E}_m(R)$ is written as $\mathrm{EL}_m(R)$ in order to distinguish it from exceptional groups. Also observe for a commutative ring A, $\mathrm{E}_m(A)$ is a priori a subgroup of $\mathrm{SL}_m(A)$. It depends on the ring A and m whether these two coincide.

Remark 4.1.2. We note that in many cases, situations are completely different between $m \geq 3$ and $m = 2$.

(*i*) Suppose $m \geq 3$. Then it is straightforward to see the following commutator relation, which plays an important role:

$$\text{for any } i \neq j,\ j \neq k,\ k \neq i,\ \ [E_{i,j}(r_1), E_{j,k}(r_2)] = E_{i,k}(r_1 r_2).$$

Here $1 \leq i, j, k \leq m$ and $r_1, r_2 \in R$, and our commutator convention is $[g, h] = ghg^{-1}h^{-1}$. In particular, the following holds:

Lemma 4.1.3. *Let $m \geq 3$. Let R be a ring and S be a generating* (*as a ring*) *subset. Then the group $\mathrm{E}_m(R)$ is generated by the subset $\{E_{i,j}(s) : 1 \leq i, j \leq m,\ i \neq j,\ s \in S\}$.*

In particular, if R is a finitely generating ring and S is a finite generating set, then the group $\mathrm{E}_m(R)$ is finitely generated with a finite generating subset

$$\{E_{i,j}(s) : 1 \leq i, j \leq m,\ i \neq j,\ s \in S\}.$$

(*ii*) The commutator relation above shows that for any ring R and $m \geq 3$, $\mathrm{E}_m(R)$ is perfect, namely, $[\mathrm{E}_m(R), \mathrm{E}_m(R)] = \mathrm{E}_m(R)$. As we will see in Lemma 4.1.6, for any m $\mathrm{E}_m(\mathbb{Z}) = \mathrm{SL}_m(\mathbb{Z})$. Hence $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ is perfect.

(*iii*) Suppose $m = 2$. Then the commutator relation above is empty, because there are no i, j, k with $i \neq j$, $j \neq k$, $k \neq i$. For instance, it is known that the group $\mathrm{SL}_2(\mathbb{F}_q[x])$ is *not* finitely generated, where q is a positive power of a prime. As we will see in Subsection 4.1.2, $\mathrm{SL}_2(\mathbb{F}_q[x])$ coincides with $\mathrm{E}_2(\mathbb{F}_q[x])$. Thus $\mathrm{E}_2(\mathbb{F}_q[x])$ is not finitely generated, although the ring $\mathbb{F}_q[x]$ is finitely generated.

(*iv*) Suppose A and A' are commutative rings and there is a ring homomorphism $\phi\colon A \to A'$. Let $m \geq 2$. Then ϕ induces group homomorphisms $\Phi_1\colon \mathrm{E}_m(A) \to \mathrm{E}_m(A')$ and $\Phi_2\colon \mathrm{SL}_m(A) \to \mathrm{SL}_m(A')$. If ϕ is surjective, then by the very definition, Φ_1 is surjective for all m. However, it is *not* necessarily true that Φ_2 is surjective. This argument for Φ_1 also holds in noncommutative ring settings.

One corollary is that $\mathrm{E}_2(\mathbb{Z}[x])$ is *not* finitely generated. Indeed, for a prime q, $\mathbb{Z}[x]$ maps onto $\mathbb{F}_q[x]$. Therefore there is a surjection $\mathrm{E}_2(\mathbb{Z}[x]) \twoheadrightarrow \mathrm{E}_2(\mathbb{F}_q[x])$. Then item (*ii*) above ends the proof.

4.1.2 The Suslin stability theorem

It is a problem of high interest and importance in algebraic K-theory to determine whether $\mathrm{E}_m(A)$ coincides with $\mathrm{SL}_m(A)$ for $m \in \mathbb{Z}_{\geq 2}$ and a commutative ring A. Note that there is a concept of the *general elementary group* $\mathrm{GE}_m(R)$ for a noncommutative group. It is reasonable to think whether $\mathrm{GE}_m(R)$ coincides with $\mathrm{GL}_m(R)$ in noncommutative ring settings. See a paper of P. M. Cohn [Coh] for instance.

One tool to consider this problem is the *stable range* of H. Bass [Bas].

Definition 4.1.4. Let R be a (possibly noncommutative) ring.

(i) A sequence $a_1, \ldots, a_n$ in R is said to be *left unimodular* if $Ra_1 + \cdots + Ra_n = R$. If $n \geq 2$, we say this sequence is *left reducible* if there exists $r_1, \ldots, r_{n-1} \in R$ such that
$$R(a_1 + r_1 a_n) + \cdots + R(a_{n-1} + r_{n-1} a_n) = R.$$

(ii) Let $n \geq 1$. The ring R is said to have *left stable range*$\leq n$ if any sequence of length $\geq n+1$ is reducible. We define the *left stable range* of R as the infimum of such n (if there exists no such n, then we set the stable range as ∞).

There is a parallel notion of *right stable range*. However, Vaserstein [Vas1] shows these two coincide. Therefore, we call them *stable range* of R, and write as $\mathrm{sr}(R)$.

We warn that this definition differs 1 from the original definition of Bass. Our definition of stable range implies that $\mathrm{sr}(R) \leq 2$ for any euclidean ring R. For instance, $\mathrm{sr}(\mathbb{Z}) = 2$. We note that Grunewald–Mennicke–Vasestein [GMV2] shows $\mathrm{sr}(\mathbb{Z}[x]) = 3$. In general, it is known $\mathrm{sr}(\mathbb{Z}[x_1, \ldots, x_k]) \leq k+2$. It seems that it is unknown whether $\mathrm{sr}(\mathbb{Z}\langle x_1, \ldots, x_k \rangle) < \infty$ for $k \geq 2$. Here $\mathbb{Z}\langle x_1, \ldots, x_k \rangle$ denotes the noncommutative polynomial ring over $\mathbb{Z}$.

The following theorem is due to Dennis–Vaserstein [DeVa], based on a result of Bass [Bas] in the case of $m - r = 2$:

Proposition 4.1.5. ([Bas], [DeVa]) *Let A be a commutative ring with* $\mathrm{sr}(a) \geq r$. *Then for any $m \geq r$, we have*

$$\mathrm{SL}_m(A) = ULUL\mathrm{SL}_r(A).$$

Here $U = U_m A$ and $L = L_m A$ respectively means the group of upper triangular unipotent matrices and that of lower triangular unipotent matrices.

Proof. We only give a proof in the case of $r = 2$ and $m = 4$. This proof is based on a pair of U. Hadad [Had], the proof of Lemma 3.2. Let $g \in \mathrm{SL}_4(A)$. Consider the $(1,i)$-th entries a_i of g $(i = 1,2,3,4)$. Also consider $(i,1)$-th entries b_i of $g^{-1} \in \mathrm{SL}_4(A)$ $(i = 1,2,3,4)$. Then we have

$$b_1 a_1 + b_2 a_2 + b_3 a_3 + b_4 a_3 = 1$$

Therefore, the sequence $(a_1, a_2, b_3a_3+b_4a_4)$ is unimodular. By stable range condition, there exist $t_1, t_2 \in A$ such that

$$A(a_1 + t_1(b_3a_3 + b_4a_4)) + A(a_2 + t_2(b_3a_3 + b_4a_4)) = A.$$

Therefore, there exist $x_1 \in A$ and $x_2 \in A$ such that

$$x_1(a_1 + t_1(b_3a_3 + b_4a_4)) + x_2(a_2 + t_2(b_3a_3 + b_4a_4)) = 1.$$

This implies, in terms of elementary matrices, that the following matrix has $(4,4)$-th entry 1:

$$E_{4,2}((a_4 - 1)x_2)E_{4,1}((a_4 - 1)x_1)E_{2,3}(t_2b_3)E_{2,4}(t_2b_4)E_{1,3}(t_1b_3)E_{1,4}(t_1b_4)g.$$

For details, we refer to the proof of Lemma 3.2 in [Had].

Therefore by some other mulplications of U and L by left, we can eliminate all $(i,4)$-th and $(4,j)$-th entries for $i = 1, 2, 3$.

In a similar way, we can bring g to a matrix in $\mathrm{SL}_2(A)$ in a left upper corner of $\mathrm{SL}_4(A)$. By taking these procedure carefully, we have

$$\mathrm{SL}_4(A) = ULUL\mathrm{SL}_2(A),$$

as desired. □

Also, it can be done by euclidean algorithm to express any element in $\mathrm{SL}_2(A)$ as a product of elementary matrices, provided that the ring A is euclidean. Thus combining this with Proposition 4.1.5, we obtain the following:

Lemma 4.1.6. *For any euclidean domain A and for any $m \geq 2$, $\mathrm{E}_m(A) = \mathrm{SL}_m(A)$.*

Note that in the lemma above, the minimum number of expressing each element$g \in \mathrm{SL}_2(A)$ as a product of elementary matrices in $\mathrm{M}_2(A)$ is usually *unbounded* on $g \in \mathrm{SL}_2(A)$ (if A is a field, then it is bounded). However, if we consider the case of $m \geq 3$, then the situation may change. For details, see Subsection 4.2.2; and Subsection 4.3.1. Also we note that Lemma 4.1.6 itself can be deduced without Proposition 4.1.5, but that Proposition 4.1.5 gives more precise information about elimination procedures from $\mathrm{SL}_m(A)$ to $\mathrm{SL}_2(A)$.

Consider the case of $A = \mathbb{Z}[x_1, \ldots, x_k]$. As soon as $k \geq 1$, this ring becomes *noneuclidean.* The stable range of A will increase if k increases. Nevertheless, A. S. Suslin shows the following celebrated stability theorem:

Theorem 4.1.7. (*Suslin* [Sus]) *Let $k \in \mathbb{N}$. Then for any $m \geq 3$, there are equalities*

$$\mathrm{E}_m(\mathbb{Z}[x_1, \ldots, x_k]) = \mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k]);$$
$$\mathrm{E}_m(\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]) = \mathrm{SL}_m(\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]).$$

In fact, Suslin shows that these equalities hold true even if $\mathbb{Z}$ is replaced with any euclidean domain. This proof employs high techniques in algebraic K-theory and is not algorithmic, but later Park–Woodburn [PaWo] gave an algorithmic proof for the case of $\mathrm{SL}_m(F[x_1,\ldots,x_k])$, where F is any (commutative) field.

Remark 4.1.8. It is worth stating that the situation is completely different if $m = 2$. Indeed, the following is shown by Cohn.

Proposition 4.1.9. *The group* $\mathrm{E}_2(\mathbb{Z}[x])$ *is a proper subgroup of* $\mathrm{SL}_2(\mathbb{Z}[x])$*:* $\mathrm{E}_2(\mathbb{Z}[x]) \lneqq \mathrm{SL}_2(\mathbb{Z}[x])$*.*

For instance, he shown the following matrix is in $\mathrm{SL}_2(\mathbb{Z}[x]) \setminus \mathrm{E}_2(\mathbb{Z}[x])$:

$$\begin{pmatrix} 1+2x & x^2 \\ -4 & 1-2x \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[x]).$$

It is an open problem whether $\mathrm{E}_2 = \mathrm{SL}_2$ for $\mathbb{Z}[x, x^{-1}]$.

The following terminology *universal lattice* is due to Shalom [Sha1], and it is the main object in this thesis. Recall from Chapter 0 (, thanks to Theorem 4.1.7,) we let k be any natural number, and do not distinguish each other, unless otherwise stating.

Definition 4.1.10. Let $m \geq 3$.

A *universal lattice* of degree m denotes a group

$$\mathrm{SL}_m(\mathbb{Z}[x_1,\ldots,x_k])(= \mathrm{E}_m(\mathbb{Z}[x_1,\ldots,x_k])).$$

A *noncommutative universal lattices* of degree m denotes a group

$$\mathrm{E}_m(\mathbb{Z}\langle x_1,\ldots,x_k\rangle).$$

We note before Shalom, interests in universal lattices from aspects of rigidity date back to those of A. Lubotzky.

Remark 4.1.11. (i) Universal lattices are universal in the following sense: by item (ii) of Remark 4.1.2, for $m \geq 3$ and any finitely generated commutative ring A, some universal lattice of degree m maps onto $\mathrm{E}_m(A)$. Therefore, for a property P which passes to group quotients, if all universal lattices of degree m has P, then it implies for *any* commutative finitely generated ring A the group $\mathrm{E}_m(A)$ fulfills the property. Such examples of $\mathrm{E}_m(A)$ includes $\mathrm{SL}_m(\mathcal{O})$ for a ring of integers $\mathcal{O}$ of a local field, which is a lattice in a semi-simple algebraic group. However they also include groups not coming from these:

Lemma 4.1.12. *No universal lattice* (*with* $k \geq 1$) *can be realized as an arithmetic lattice in a semisimple algebraic group.*

We note that if universal lattice can be realized as a lattice in a semisimple algebraic group, then it will be automatically arithmetic. This follows from the Margulis arithmeticity theorem.

Proof. The Margulis finiteness theorem states the following:

any normal subgroup of a higher rank (*and hence arithmetic*) *lattice is either finite or of finite index.*

However, $\mathrm{SL}_m(\mathbb{Z}[x])$ has an infinite normal subgroup of infinite index. Indeed, for instance, consider the congruence subgroup H associated with the ideal $(x) \trianglelefteq \mathbb{Z}[x]$, namely, H is the kernel of a map

$$\mathrm{SL}_m(\mathbb{Z}[x]) \twoheadrightarrow \mathrm{SL}_m(\mathbb{Z}), \text{ which sends } x \text{ to } 0.$$

□

One of the main motivations of Shalom to deal with universal lattices is a relation to property (T). As we see in the paragraph above, if universal lattices above have (T), then this result will be able to regarded as the universal reseults among all groups of the form $\mathrm{E}_m(A)$, such as $\mathrm{SL}_m(\mathbb{Z})$, $\mathrm{SL}_m(\mathbb{Z}[1/2])$, $\mathrm{SL}_m(\mathbb{Z}[\sqrt{2}, \sqrt{3}])$, $\mathrm{SL}_m(\mathbb{F}_q[x])$, and $\mathrm{SL}_m(\mathbb{Z}[x, x^{-1}])$. Note that all but last one examples are arithmetic lattices, however the last example is *not*, for the same reason as in above. Also, since Kazhdan constant does not decrease under group quotients, if universal lattices above have (T), then this will imply Kazhdan constants of $\mathrm{E}_m(A)$ (with a finite generating set in Lemma 4.1.3) is bounded below from zero, and a lower bound only depends on m and the number of generators of A.

As we mentioned repeatedly, property (T) for universal lattices is proven [Sha5], [Vas2]. Also, a problem on property (T) for noncommutative universal lattices is of great significance, and Ershov–Jaikin-Zapirain have answered positively in [ErJa]. We will also see this in Subsection 9.2.2.

(*ii*) Although structures of universal lattices have not been well-studied, universal lattices are linear groups in the sense in Definition 2.7.4. This is verified by sending each x_i to algebraically independent transcendental numbers. Therefore by Theorem 2.7.10, these groups have property A of Yu, and the Novikov conjectures for these hold true.

On the other hand, it is shown by Kassabov–Sapir [KaSa] that noncommutative universal lattices (with $k \geq 1$) are *non*-linear. It is an open problem to determine these has property A, or these admit uniform embeddings into Hilbert spaces (Definition 2.7.7).

Definition 4.1.13. Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ or $R = \mathbb{Z}\langle x_1, \ldots, x_k\rangle$. For $m \geq 2$, a *unit elementary matrix* in $\mathrm{E}_m(A)$ means a matrix of the form $E_{i,j}(\pm x_l)$ with $i \neq j$ and $0 \leq l \leq m$. Here x_0 means 1.

The set of unit elementary matrices is finite. By Lemma 4.1.3, unit elementary matrices generate the whole group, provided $m \geq 3$.

4.2 Bounded generation and (T)

In this section, we see the argument of Shalom [Sha1] in proving property (T) for certain elementary group $\mathrm{E}_m(R)$ with an explicit estimate for Kazhdan constants. An attempt to make an estimation of a Kazhdan constant for $\mathrm{SL}_3(\mathbb{Z})$ dates back to a work of M. Burger [Bur], and Shalom [Sha1]. Shalom employs bounded generation for this.

4.2.1 Relative (T) for $\mathbf{SL_2}(\mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$

First, we show the following theorem due to Shalom, which is a quantitative version of Corollary 2.6.2. Throughout this subsection, we keep the following setting.

Set $G = H \ltimes N$ with $H = \mathrm{SL}_2(\mathbb{Z})$ and $N = \mathbb{Z}^2$. We identify $G \trianglerighteq N$ with

$$\left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{SL}_2(\mathbb{Z}),\, v \in \mathbb{Z}^2 \right\} \quad \trianglerighteq \quad \left\{ \left(\begin{array}{c|c} I_2 & v \\ \hline 0 & 1 \end{array} \right) : v \in \mathbb{Z}^2 \right\}.$$

Set a finite generating set S of G as the set of unit elementary matrices in $\mathrm{SL}_3(\mathbb{Z})$ which lie in G with the identification above. Also, we set $S_0 = S \cap H$ and $S_1 = S \cap N$. Then $S = S_0 \cup S_1$. Namely, with the identification above,

$$S_0 = \{h_1^{\pm}, h_2^{\pm}\},\ S_1 == \{l_1^{\pm}, l_2^{\pm}\}.$$

Here $h_1 = E_{1,2}(1)$, $h_1 = E_{2,1}(1)$; and $l_1 = E_{1,3}(1)$, $l_2 = E_{2,3}(1)$.

Theorem 4.2.1. (*Shalom* [Sha1]) *The pair* $\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \trianglerighteq \mathbb{Z}^2$ *has relative* (T). *Moreover, with the setting in this subsection, there is an inequality*

$$\mathcal{K}(G, N; S) > \frac{1}{10}.$$

Proof. Suppose $(S, 1/10)$ is not a relative Kazhdan pair. Then there exists a unitary G-representation $(\pi, \mathfrak{H})$ such that the following two conditions are satisfied:

(a) The restriction of π on N satisfies $\pi\mid_N \not\supseteq 1_N$.

(b) The representation π admist an $(S, 1/10)$-invariant vector, namely, there exists $\xi \in S(\mathfrak{H})$ such that the folloiwng holds:

$$\epsilon := \sup_{s \in S} \|\xi - \pi(s)\xi\| < 1/10.$$

We fix this π and ξ. Thanks to spectral theory on the additive group N, $\pi\mid_N\colon N \to U(\mathfrak{H})$ yields a $*$-homomorphism $\sigma\colon C(\hat{N}) \to \mathbb{B}(\mathfrak{H})$. We identify $\hat{N}$ with $\mathbb{T}^2 \cong [-1/2, 1/2)$ with the identification

$$\mathbb{T}^2 \ni t \longleftrightarrow \chi_t \in \hat{N};\quad \chi_t(n) = \exp(2\pi\sqrt{-1}n\cdot t),$$

where for $t = {}^t(t_1, t_2) \in \mathbb{T}^2$ and $n = {}^t(n_1, n_2) \in N$, $n \cdot t = n_1t_1 + n_2t_2$. With this identification, we obtain a $*$-homomorphism

$$\sigma\colon C(\mathbb{T}^2) \to \mathbb{B}(\mathfrak{H}),$$

which sends each z_i $(i = 1, 2)$ to $\pi(l_i) \in U(\mathfrak{H})$, where $z_i(t) := \exp(2\pi\sqrt{-1}t_i)$. Consider a positive linear functional on $C(\mathbb{T}^2)$, $f \mapsto \langle\sigma(f)\xi|\xi\rangle$, where ξ is taken by item (b). Then by the Riesz–Markov–Kakutani theorem, there exists a (positive) regular Borel measure ν such that

$$\text{for any } f \in C(\mathbb{T}^2),\quad \int_{\mathbb{T}^2} f d\nu = \sigma(f)\xi|\xi\rangle.$$

Note that by pointwisely approximating a characteristic function by positive elements in $C(\mathbb{T}^2)$ monotone decreasing way (here we consider the convergence in $\mathrm{B}(\mathfrak{H})$ in the strong operator topology), we obtain the spectral projection

$$E\colon \mathcal{B}(\mathbb{T}^2) \to \mathbb{B}(\mathfrak{H}),$$

where $\mathcal{B}(\mathbb{T}^2)$ denotes the Borel σ-algebra of $\mathbb{T}^2$. In particular, for any $Z \in \mathcal{B}(\mathbb{T}^2)$ the following holds:

$$\nu(Z) = \int \chi_Z d\nu = \langle E(Z)\xi, \xi\rangle = \langle\sigma(\chi_Z)\xi, \xi\rangle.$$

Here χ_Z is the characteristic function.

We take the following decomposition of $\mathbb{T}^2$:

$$\begin{aligned}
\{0\};\ D_0 &= \{|t_1| \geq 1/4 \text{ or } |t_2| \geq 1/4\};\\
D_1 &= \{|t_2| \leq |t_1| < 1/4 \text{ and } t_1t_2 > 0\};\\
D_2 &= \{|t_1| < |t_2| < 1/4 \text{ and } t_1t_2 \geq 0\};\\
D_3 &= \{|t_1| \leq |t_2| < 1/4 \text{ and } t_1t_2 < 0\};\\
D_4 &= \{|t_2| < |t_1| < 1/4 \text{ and } t_1t_2 \leq 0\}.
\end{aligned}$$

Firstly, we claim that $\nu(D_0) \leq \epsilon^2$. Indeed, for $i = 1, 2$, set $D_{0,i} = D_0 \cap \{|t_i| \geq 1/4\}$. Then $t \in D_{0,i}$ implies $|1 - z_i|^2 \geq 2$ for $i = 1, 2$. Therefore, for each $i = 1, 2$, by item (b),

$$\begin{aligned}
2\nu(D_{0,i}) &\leq \int \chi_{D_{0,i}}|1 - z_i|^2 d\nu \leq \int |1 - z_i|^2 d\nu\\
&= \int (1 - \overline{z_i})(1 - z_i) d\nu = \langle\sigma((1 - \overline{z_i})(1 - z_i))\xi|\xi\rangle\\
&= \langle(I - \pi(l_i)^*)(I - \pi(l_i))\xi|\xi\rangle\\
&= \|\xi - \pi(l_i)\xi\|^2 \leq \epsilon^2.
\end{aligned}$$

Here $\overline{z_i}$ means the complex conjugation of z_i in $C(\mathbb{T}^2)$. By observing $D_0 = D_{0,1} \cup D_{0,2}$, we confirm the claim.

Secondly, as we have seen in Subsection 2.6.1, the associated action of $H = \mathrm{SL}_2(\mathbb{Z})$ on $(\hat{N} \cong)\mathbb{T}^2$ is the multiplication of inverse transpose. In other words,

$$\text{for any } h \in H, \text{ and } Z \in \mathcal{B}(\mathbb{T}^2), \quad E({}^t h^{-1} \cdot Z) = \pi(h)E(Z)\pi(h^{-1}).$$

We now claim that for any $Z \in \mathcal{B}(\mathbb{T}^2)$, the following holds:

$$\text{for any } h \in S_0 (= \{h_1^{\pm}, h_2 \pm\}), \quad |\nu({}^t h^{-1} \cdot Z) - \nu(Z)| < 2\epsilon.$$

Indeed, we have for any Z and any $h \in S_0$,

$$\begin{aligned}
|\nu({}^t h^{-1} \cdot Z) - \nu(Z)| &= |\langle \pi(h)E(Z)\pi(h^{-1})\xi|\xi\rangle - \langle E(Z)\xi|\xi\rangle| \\
&= |(\langle \pi(h)E(Z)\pi(h^{-1})\xi|\xi\rangle - \langle E(Z)\pi(h^{-1})\xi|\xi\rangle) \\
&\quad + (\langle E(Z)\pi(h^{-1})\xi|\xi\rangle - \langle E(Z)\xi|\xi\rangle)| \\
&\leq |\langle E(Z)\pi(h^{-1})\xi|\pi(h^{-1})\xi - \xi\rangle| + |\langle E(Z)(\pi(h^{-1})\xi - \xi)|\xi\rangle| \\
&\leq 2\epsilon.
\end{aligned}$$

Here in the last line of the inequalities in above, we employ item (b), the Cauchy–Schwartz inequality, and the fact $E(Z)$ is projection (and hence $\|E(Z)\| \leq 1$).

Finally, note that ${}^t h_1^{-1} \cdot (D_1 \sqcup D_2) \subseteq D_2 \sqcup D_0$. By the two claims above, this inclusion implies $\nu(D_1) \leq 2\epsilon + \epsilon^2$. Similarly, we have the same estimate for any $\nu(D_i)$ $(i = 1, 2, 3, 4)$. Therefore, we conclude

$$\mu(\{0\}) = 1 - \sum_{i=0}^{4} \nu(D_i) \geq 1 - 8\epsilon - 5\epsilon^2.$$

At the beginning we set $\epsilon < 1/10$, and thus we have

$$\mu(\{0\}) > 0.$$

However this means $E(\{0\})$, the projection onto $\mathfrak{H}^{\pi(N)}$, is non-zero. This contradicts item (a). Therefore we have established the assertion. □

By Lemma 2.1.12, we obtain the following:

Corollary 4.2.2. *Keep the setting in this subsection. Let $(\pi, \mathfrak{H})$ be any unitary representation of $G = H \ltimes N$. For a vector ξ, set $\delta_\xi := \sup_{s \in S} \|\xi - \pi(s)\xi\|$. Then there is an inequality:*

$$\textit{for any } l \in N, \quad \|\xi - \pi(l)\xi\| \leq 20\delta_\xi.$$

Shalom shows this argument also works if $\mathbb{Z}$ is replaced with $\mathbb{Z}[x]$, or with $\mathbb{Z}[x_1, \ldots x_k]$, with some estimation of relative Kazhdan constant. Kassabov [Kas1] improved the estimate. In [Kas2], he noticed in fact it works even noncommutative ring setting, namely in the case of $\mathbb{Z}\langle x_1, \ldots x_k\rangle$, because for N part in our setting we only consider additive group structure. Thus we see the following result:

Theorem 4.2.3. (*Shalom* [Sha1]*; Kassabov* [Kas2]) *Let $R_k = \mathbb{Z}\langle x_1, \ldots x_k\rangle$. Then the pair $\mathrm{E}_2(R_k) \ltimes R_k^2 \trianglerighteq R_k^2$ has relative property* (T)*. Moreover, for the finite generating set S consisting of all unit elementary matrices in $\mathrm{E}_2(R_k) \ltimes R_k^2$ (inside $\mathrm{E}_3(R_k)$), there is a following estimate of the relative Kazhdan constant:*

$$\mathcal{K}(\mathrm{E}_2(R_k) \ltimes R_k^2, R_k^2; S) > \frac{1}{3\sqrt{2}(\sqrt{k}+3)}.$$

In particular, the statements above hold if R_k is replaced with the commuttaive ring $A_k = \mathbb{Z}[x_1, \ldots x_k]$.

For the precise estimate, we refer to the original paper [Kas2] of Kassabov.

4.2.2 Bounded generation of Carter–Keller

We define the following concepotion of high importance:

Definition 4.2.4. Let G be a group and S be a subset of G which contains e_G. Then we say G is *boundedly generated* by S if there exists $n \in \mathbb{N}$ such that $G = S^n$, namely, if there exists a universal constant n such that any $g \in G$ can be expressed as a product of n elements in S.

Let $(S_j)_{j\in J}$ are subsets of G containing e_G. Then we say G is *boundedly generated* by $(S_j)_{j\in J}$ if $S = \bigcup_{j\in J} S_j$ boundedly generates G.

Note that G always boundedly generates G itself, and therefore properties for the generating set S matter in study if bounded generation. Also, we warn in other literture, the bounded generation is defined in the following *confined* sense: J is a finite set, and each S_j is a cyclic group.

The following theorem, due to Carter–Keller [CaKe], plays a fundamental role on bounded generation and its application.

Theorem 4.2.5. (*Carter–Keller* [CaKe]) *Let $m \geq 3$. Then $\mathrm{SL}_m(\mathbb{Z})$ is boundedly generated by the set of all elementary matrices. Moreover, every $g \in \mathrm{SL}_m(\mathbb{Z})$ can be written as a product of at most*

$$v_m := \frac{1}{2}(3m^2 - m) + 36$$

elementary matrices.

Note that they also obtain a similar result for the case of $\mathrm{SL}_m(\mathcal{O})$. Here $\mathcal{O}$ is the ring of integers in a local field. For the proof of this theorem, see Section 4.1 in [BHV].

We warn that this result is deep in the following sense. Firstly, the proof employs a special property for $\mathbb{Z}$, precisely, Dirichlet's arithmetic progression theorem. Secondly, there exists an example of a euclidean domain with respect to which, SL_m is *not* boundedly generated by elementary matrices. For the latter part, see Subsection 4.3.1.

4.2.3 Property (T) for non-algebraic matrix groups

First we deduce property (T) for $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$, *without* appealing to the fact that they are lattices in $\mathrm{SL}_m(\mathbb{R})$. The keys are relative property (T) (Theorem 4.2.1; Corollary 4.2.2) and bounded generation by elementary matrices (Theorem 4.2.5).

Theorem 4.2.6. (*Shalom* [Sha1]) *Let* $m \geq 3$. *Then* $\mathrm{SL}_m(\mathbb{Z})$ *has property* (T). *Moreover, for the finite generating set* S *consisting of all unit elementary matrices there is an inequality*

$$\mathcal{K}(\mathrm{SL}_m(\mathbb{Z}); S) > \frac{1}{30m^2 - 20m + 720}.$$

Proof. We present two proofs, one gives the estimate above and the other does not. Both ways have a sprit in common to the proof(s) of Proposition 2.6.6.

(1) The first proof gives an estimate for the Kazhdan constant. Let $G_0 = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ and $N = \mathbb{Z}^2$ as abstract groups. Observe that G contains many copies of G_0 in the following sense: for any group $Z_{i,j}$ of elementary matrices with fixed (i,j) $(i \neq j)$ (, which is isomorphic to $\mathbb{Z}$,) there exists an embedding $\phi\colon G_0 \hookrightarrow G$ such that $\phi(N)$ includes $Z_{i,j}$. Set $\mathcal{S} = \bigcup_{i\neq j} Z_{i,j}$, which is the set of all elementary matrices.

    Suppose $(\pi, \mathfrak{H})$ is a unitary represenatation and take any $\xi \in S(\mathfrak{H})$. Set $\delta_\xi = \sup_{s\in S} \|\xi - \pi(s)\|$. Then by Corollary 4.2.2 and the observation above, we have

$$\text{for any } \gamma \in \mathcal{S}, \quad \|\xi - \pi(\gamma)\xi\| < 20\delta_\xi.$$

    By employing bounded generation of G by $\mathcal{S}$ (Theorem 4.2.5), we obtain:

$$\text{for any } g \in G, \quad \|\xi - \pi(g)\xi\| < 20v_m \cdot \delta_\xi.$$

    Recall the argument in item (ii) of Remark 2.1.2.

    Finally, recall $(G, 1)$ is a Kazhdan pair (see Corollary 2.1.14). Therefore, as soon as there exists $\xi \in S(\mathfrak{H})$ with $\delta_\xi \geq (20v_m)^{-1}$, π contains 1_G. This means, $(S, (20v_m)^{-1})$ is a Kazhdan pair for G, as desired.

(2) This is the other proof, which shows that G has (FH). Consider an arbitrary cocycle c with unitary coefficient. By Theorem 4.2.1 and an argument in proof (1) (with $\mathcal{S} = \bigcup_{i \neq j} Z_{i,j}$), we have the following:

$$c \text{ is bounded on } \mathcal{S}.$$

Finally through bounded generation, it follows c is bounded on G. This implies that c is a coboundary. Therefore G has (FH).

□

We stated proof (2), because we have known that in general case, (F_B) is stronger than (T_B). Proof (2) indicates, that if relative property (F_B) together with bounded generation (in appropriate sense) implies property (F_B).

We note that Kassabov [Kas1] estimated the optimal order of Kazhdan constant for $\mathrm{SL}_m(\mathbb{Z})$ with respect to m: it is $O(m^{-1/2})$.

Finally, we state the following application. Recall from Chapter 0 that the symbol S^1 denotes the unit circle.

Theorem 4.2.7. (*Shalom* [Sha1]) *Let* $m \geq 3$. *Then the loop group of* $\mathrm{SL}_m(\mathbb{C})$*, identified the* $\mathrm{SL}_m(C(\mathrm{S}^1))$ *with the topology of uniform convergence on compact subset* (*it is a non locally compact group*)*, has property* (T).

This theorem follows from the observation that $C(\mathrm{S}^1)$ contains a dense subring which is finitely generated and satisfies bounded generation by elementary matrices for E_m. For details, we refer to [Sha1].

4.3 Bounded generation for universal lattices

In this section, we state some facts on bounded generation for elementary groups over certain rings.

4.3.1 A result of van der Kallen

A question of high importance is the following: *Is a universal lattice boundedly generated by elementary matrices.* The affirmative answer will immediately imply property (T) for universal lattices, in the argument in this Section. However, even for the case of $k = 1$, this question is completely open. In fact, it is open whether $\mathrm{SL}_m(\mathbb{Q}[x])$ is boundedly generated by elementary matrices, for $m \geq 3$, although $\mathbb{Q}[x]$ is a euclidean domain and the treatment is much easier than that of $\mathbb{Z}[x]$.

The following theorem, due to W. van der Kallen, is surprising: it states even for euclidean domains, bounded generation of SL_m by elementary matrices can be *false*.

Theorem 4.3.1. (*van der Kallen* [vdKa]) *Let K a (commutative) field of infinite transcendence degree over a subfield. Then for any $m \geq 3$, $\mathrm{SL}_m(K[x])$ is* not *boundedly generated by elementary matrices.*

Here are examples of fields satisfying the condition of this theorem: firstly, any uncountable fields (such as $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ here p is a prime); secondly, it follows from a counting argument that there exists a countable field with this property. Also, by theorems on transcendental numbers, the field generated by $e^{\sqrt{p_n}}$'s , where $(p_n)_{n\geq 1}$ is a strictly increasing sequence of primes, has this property.

The proof of van der Kallen is based on the fact that for such field, $K_2(K)$ is not boundedly generated by the Steinberg symbols. We do not go further on algebraic K-theory in this thesis. We remark that I. V. Erovenko [Ero] found a constructive proof of this theorem. Namely, he found a sequence of explicit elements in $\mathrm{SL}_m(K[x])$ of which the word lengths (with respect to the set of elementary matrices) tends to infinity.

4.3.2 Vaserstein's bounded generation

As we see in the subsection above, bounded generation by elementary matrices is not known for universal lattices. However certain weak form of this bounded generation has been established by Vaserstein [Vas2]. This result, together with Shalom's machinery (Subsection 5.3) are the keys to proving property (T) for universal lattices:

Theorem 4.3.2. (*Vaserstein* [Vas2]) *Let $A_k = \mathbb{Z}[x_1, \ldots, x_k]$ for any k. Then for any $m \geq 3$, $\mathrm{SL}_m(A_k)$ is boundedly generated by the set of elementaly matrices and the subgroup $\mathrm{SL}_2(A_k)$. Here $\mathrm{SL}_2(A_k) \leqslant \mathrm{SL}_m(A_k)$ sits in the left upper corner.*

Chapter 5

Reduced cohomology and ultraproduct

By taking a closure of the coboundary space $B^1(G;\rho,B)$ (with respect to the Fréchet topology on the cocycle space $Z^1(G;\rho,B)$), we can consider the reduced cohomology $\overline{H}^1(G;\rho,B)$. We see a relation between this concept and ultraproduct of Banach spaces. As an application, we introduce Shalom's machinery, which enables us to approach property (T) for universal lattices. References on reduced cohomology with unitary coefficients are the original paper of Shalom [Sha2], and Chapter 3 of [BHV]. References on ultraproducts on metric spaces are a book of Aksoy–Khamsi [Ak], a book of Bridson–Haefliger [BrHa], and a paper of S. Heinrich [Hei]. For Shalom's machinery, we refer to the original paper [Sha5] of Shalom.

In this chapter, *we assume all groups are discrete* (*and hence countable by* σ-*compactness assumption*) in order to avoid continuity arguments on ultraproduct of actions.

5.1 Ultraproduct of metric spaces

Before proceeding into theory of reduced cohomology, we consider the concept of ultraproducts of metric spaces.

5.1.1 Definition of ultralimits and properties

We begin with the definitirion of ultrafilters on $\mathbb{N}$:

Definition 5.1.1. A *non-principal ultrafiler* ω on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ that satisfies the following properties:

(i) The collection ω satisfies $\omega \not\ni \emptyset$, and $\omega \ni \mathbb{N}$.

(ii) For any $A, B \in \omega$, $A \cap B \in \omega$.

(*iii*) For any $A \in \omega$ and $A \subseteq C$, $C \in \omega$.

(*iv*) For any $A \subseteq \mathbb{N}$, either $A \in \omega$ or $A^c \in \omega$. Here A^c means the complement.

(*v*) No finite subset of $\mathbb{N}$ is in ω

The existence of non-principal ultrafiler based on Zorn's lemma. Also, each non-principal ultrafilter can be seen as an element of Stone–Čech boundary of $\mathbb{N}$.

Throughout this chapter, we fix a non-principal ultrafilter and write it ω.

Definition 5.1.2. Let $(a_n)_n$ be a sequence in $\mathbb{C}$. We say a_n *converges to* $a \in \mathbb{C}$ with respect to ω, if the following holds:

$$\text{for any } \epsilon > 0, \quad \{n \in \mathbb{N} : |a_n - a| < \epsilon\} \in \omega.$$

In this case we call a the *ultralimit* of (a_n), and write $a = \lim_\omega a_n$.

It is easy to see if an ultralimit exists, then it is unique. Also, ultralimits preserves sums: indeed, suppose $(a_n)_n$, $(b_n)_n$ in $\mathbb{C}$ such that $\lim_\omega a_n = a$ and $\lim_\omega b_n = b$ for some $a, b \in \mathbb{C}$. Then for any $\epsilon > 0$,

$$\{n \in \mathbb{N} : |(a_n + b_n) - (a + b)| < 2\epsilon\} \supseteq \{n \in \mathbb{N} : |a_n - a| < \epsilon\} \cap \{n \in \mathbb{N} : |b_n - b| < \epsilon\}.$$

This ensures $\lim_\omega (a_n + b_n) = a + b$. Also, it is clear that the ultralimit preserves scalar multiplication.

The following is one of the most important features of ultralimits.

Proposition 5.1.3. *The ultralimit* $\lim_\omega a_n$ *exists for any* bounded *sequence* $(a_n)_n$ *in* $\mathbb{C}$.

Proof. Thanks to the observations above, we may assume $(a_n)_n$ lies in $\mathbb{R}$. Choose positive M such that $|a_n| \leq M$ for all n. Consider subsets of $\mathbb{N}$ $S_0^+ := \{n : a_n \geq 0\}$ and $S_0^- := \{n : a_n \leq 0\}$. Since $(S_0^+)^c \subseteq S_0^-$, either $S_0^+ \in \omega$ or $S_0^- \in \omega$. We may assume $S_0^+ \in \omega$. Then consider $S_1^+ := \{n : a_n \geq M/2\}$ and $S_1^- := \{n : a_n \leq M/2\}$ and either of these is in ω. By diminishing intervals, we obtain $a \in \mathbb{R}$ such that for any $m \geq 1$, $\{n : |a_n - a| \leq M/2^m\} \in \omega$. This shows $\lim_\omega a_n = a$. $\square$

Now we restrict ourselves to consider bounded sequence in $\mathbb{C}$. This corresponds to consider an element of $\ell^\infty(\mathbb{N})$. It means, we consider the *ultralimit* with respect to ω as a map $\ell^\infty(\mathbb{N}) \to \mathbb{C}$. Note that both spaces $\ell^\infty(\mathbb{N})$ and $\mathbb{C}$ are equipped with $*$-operations (, namely, complex conjugations).

Lemma 5.1.4. *The ultralimit with respect to* ω $\lim_\omega \colon \ell^\infty(\mathbb{N}) \to \mathbb{C}$ *satisfies the following properties:*

(*i*) *The map* $\lim_\omega$ *is a linear functional.*

(ii) *The map* $\lim_\omega$ *is unital: sends* $1_{\mathbb{N}}$ *to* 1.

(iii) *The map* $\lim_\omega$ *is positive: sends nonnegative functions to nonnegative reals.*

(iv) *The map* $\lim_\omega$ *preserves* $*$*-structure:* $\lim_\omega(\overline{a_n}) = \overline{\lim_\omega a_n}$.

(v) *If* $(a_n)_n$ *is a converging sequence in usual sense, then* $\lim_\omega a_n = \lim_{n\to\infty} a_n$.

(vi) *The map* $\lim_\omega$ *is multiplicative:* $\lim_\omega(a_n \cdot b_n) = (\lim_\omega a_n)(\lim_\omega b_n)$.

Proof. Items (i)-(v) are directly shown. For item (vi), let $\lim_\omega a_n = a$ and $\lim_\omega b_n = b$. Observe

$$a_n b_n - ab = (a_n - a)b_n + a(b_n - b)$$

and $(a_n)_n, (b_n)_n \in \ell^\infty(\mathbb{N})$. We have the conclusion. □

Therefore, we can regard an ultralimit as a unital positive *multiplicative* functional ($*$-preserving)

$$\lim_\omega \colon \ell^\infty(\mathbb{N}) \to \mathbb{C}$$

which coincides with the usual limit on converging sequences.

5.1.2 Ultraproduct of metric spaces, and scaling limits

Although we treat Banach spaces, for treatments of affine actions, it is convenient to consider Banach space as affine Banach spaces. Therefore, here we define an ultraproduct of metric spaces with base points.

Definition 5.1.5. (i) Let $(X_n, d_n, z_n)_n$ be a sequence of triples of metric spaces, metrics, and base points. We define the ultraproducts of pointed metric spaces $(X_n, d_n, z_n)_n$ with respect to ω as follows: set

$$\tilde{X} := \{(x_n)_n : x_n \in X_n,\ \sup_n d_n(x_n, z_n) < \infty\}.$$

We define a metric d_ω on $\tilde{X}$ by

$$d_\omega((x_n)_n, (y_n)_n) := \lim_\omega d_n(x_n, y_n)$$

(observe that by triangle inequality, $(d_n(x_n, y_n))_n \in \ell^\infty(\mathbb{C})$). Finally, set a equivalence relation

$$(x_n)_n \simeq (y_n)_n \quad \text{if} \quad d_\omega((x_n)_n, (y_n)_n) = 0.$$

The *ultraproduct* of $(X_n, d_n, z_n)_n$ with respect to ω is defined as the space $\tilde{X}/\simeq$, equipped with the canonical quotient metric d_ω and with the canonical base point. We write this as $(X_\omega, d_\omega, z_\omega)$, or $\lim_\omega(X_n, d_n, z_n)$.

(*ii*) Let (X, d) be a single metric space. Then we say a *scaling limit* of (X, d) is a metric space of the following form

$$\lim_{\omega}(X, a_n \cdot d, z_n),$$

where $(a_n)_n$ is a positive increasing sequence tending to infinity as $n \to \infty$.

(*iii*) Let $\mathcal{X}$ be a class of metric spaces. Then we say $\mathcal{X}$ is *stable under ultraproducts* if any sequence $(X_n, d_n)_n$ in $\mathcal{X}$ and base points $(z_n)_n$ ($z_n \in X_n$), the ultrapower is an element in $\mathcal{X}$.

We say $\mathcal{X}$ is *stable under scaling limits* if any element (X, d) in $\mathcal{X}$ and any base points $(z_n)_n$ ($z_n \in X$) and any sequence $(a_n)_n$ of positive reals with $a_n \nearrow \infty$, the resulting scaling limit is an element in $\mathcal{X}$.

We note that if we start from affine Banach spaces, then the resulting ultraproduct is also an affine Banach space (in particular, it is closed). Also, in that case, the choices of base points does not affect the resulting Banach space as affine spaces. In general, ultraproduct of complete metric spaces is also complete.

Definition 5.1.6. Let $(B_n, \|\cdot\|_n, 0)_n$ be a sequence of Banach spaces (with the base point as origin). Let $(T_n)_n$, $T_n \in \mathbb{B}(B_n)$ be a sequence of bounded linear operators which satisfies

$$\sup_n \|T_n\|_{\mathbb{B}(B_n)} < \infty.$$

Then the *ultraproduct of* $(T_n)_n$ with respect to ω, written as $T_\omega = \lim_\omega T_n$, is defined as the operator on $\lim_\omega(B_n, \|\cdot\|_n, 0)_n$ induced by the following operation:

$$(\xi_n)_n \mapsto (T_n\xi_n)_n.$$

In the definition above, it is a routine to check that this is well-defined, and that there is a following formula:

$$\|\lim_{\omega} T_n\|_{\mathbb{B}(B_\omega)} = \lim_{\omega} \|T_n\|_{\mathbb{B}(B_n)}.$$

Example 5.1.7. (*i*) Consider a single Banach space B with a fixed norm $\|\cdot\|$ and a base point 0. We claim that whenever B is infinite dimensional, the ultraproduct $\lim_\omega B = \lim_\omega(B, \|\cdot\|, 0)$ is *not* separable. Indeed, from infinite dimensionality, there exists a sequence $(\xi_n)_{n\geq 0}$ in $S(B)$ such that for any $n \neq m$, $\|\xi_n - \xi_m\| \geq 1/2$. For positive real number t, set $[t]$ be the maximum of integers which is at most t. For any real number $\alpha > 0$, we consider the limit $\xi_\omega^\alpha := \lim_\omega(\xi_{[n^\alpha]})_n$. Then it is easy to see the following: for any distinct $\alpha, \beta > 0$, ξ_ω^α and ξ_ω^β has difference at least 1/2. This implies that $\lim_\omega B$ is not separable. In particular, if the original B is separable, then B and $\lim_\omega B$ are never isomorphic.

(ii) The class $\mathcal{H}$, consisting of all Hilbert spaces, is stable under ultraproducts. This follows from the fact that (with setting all base points as origins), the ultralimit of inner products is again an inner product, after separation. For the class $\mathcal{L}_p$, it is unclear whether it is stable under ultraproducts. See Subsection 5.1.3.

(iii) If we consider a class $\mathcal{C}$ of Banach spaces, then scaling limits inside $\mathcal{C}$ are special case of ultrapowers inside $\mathcal{C}$. Therefore, if $\mathcal{C}$ is stable under ultrapowers, then it is stable under scaling limits. We note that the converse in *not* true. We will see it below.

Firstly, the class $[\mathcal{H}]$ is stable under scaling limits. Indeed, any single element $X \in [\mathcal{H}]$ has a compatible norm to that of a Hilbert space $\mathfrak{H}$. Thanks to the inner product on $\mathfrak{H}$, one can show $\lim_\omega X$ also has a compatible Hilbert norm (recall that $\lim_\omega \mathfrak{H}$ is a Hilbert space and Definition 5.1.6). Secondly, we see $[\mathcal{H}]$ is *not* stable under ultraproducts. Indeed, in this case we can choose a sequence $(X_n)_n$ whose norm ratios to norms of Hilbert spaces tends to infinity. Hence the resulting ultrapower no longer has a compatible norm in general.

Among metric spaces (with base points), we consider the *ultraproduct* of isometric actions of a (fixed) group. Recall that we always assume groups are discrete in this chapter. Therefore, there is no worry about continuity on resulting isometric action.

Definition 5.1.8. Let G a group. Let (X_n, d_n, z_n) be a sequence of metric spaces with base points. Let $(\alpha_n)_n$ be a sequence of isometric G-actions on (X_n, d_n, z_n) (, namely, for each n $\alpha_n \colon G \to \mathrm{Isom}(X_n)$, where $\mathrm{Isom}(X)$ denotes the group of isometries on X). Suppose $(\alpha_n)_n$ satisfies the following condition:

$$\text{for any } g \in G, \quad \sup_n d_n(z_n, \alpha_n(g)z_n) < \infty.$$

Then we define the *ultraproduct of* (α_n) with respect to ω, written as $\alpha_\omega = \lim_\omega \alpha_n$ as the isometric action induced by the following map: for each $g \in G$

$$(x_n)_n \mapsto (\alpha(g)x_n)_n.$$

Remark 5.1.9. Here are remarks on the definition above.

1. If $(X_n, d_n)_n = (B_n, \|\cdot\|_n)$ is a sequence of affine Banach spaces and α_n's are *affine* isometric action, then the ultrapower α_ω is also affine on the affine Banach space B_ω.

2. If the group G is finitely generated and S is a finite generating set, then it suffices to check the following condition on $(\alpha)_n$ for the well-definedness of α_ω:

$$\text{for any } s \in S, \quad \sup_n d_n(z_n, \alpha_n(s)z_n) < \infty.$$

In next section, we focus on the case of finitely generated groups.

5.1.3 Some facts on ultraproducts

We collect some facts needed for our theory. Firstly, we state stability results under ultraproducts for certain classes, which is due to Heinrich [Hei]. Recall we always assume p is a real in $(1, \infty)$, as in Chapter 0.

Theorem 5.1.10. (*Heinrich* [Hei]) *The following classes of Banach spaces are stable under ultraproducts:*

(*i*) *The class* $\mathcal{H}$.

(*ii*) *For any* $M \geq 1$, *the subclass* $\mathcal{H}_M$ *of* $[\mathcal{H}]$ *which is defined as follows:* $\mathcal{H}_M$ *is the class of all Banach spaces which admits compatible norms to those of Hilbert spaces, with norm ratio* $\leq M$.

(*iii*) *The class* $\mathcal{L}_p$.

In particular, these classes are stable under scaling limit. The class $[\mathcal{H}]$ *itself is stable under scaling limit.*

Proof. The cases of item (*i*) and item (*ii*) are confirmed in Example 5.1.7. The proof of the case of item (*iii*) is deep, and this is what Heinrich did. He appealed to the Theorem of Bohnenblust and Nakano that a Banach space B is in $\mathcal{L}_p$ if (and only if) B is an *abstract* L^p space. Here an abstract L^p space is a Banach lattice E which satisfies the following condition:

$$\text{for any } \xi, \eta \in E \text{ with } \xi \wedge \eta = 0, \ \ \|\xi + \eta\|^p = \|\xi\|^p + \|\eta\|^p.$$

Here $\xi \wedge \eta$ is the greatest lower bound, as in the definition of Banach lattices. Heinrich shown that the class of all Banach lattices is stable under ultraproducts, and that the condition above is also stable. Therefore, $\mathcal{L}_p$ is stable under ultraproducts. For details, we refer to [Hei] and Chapter 1 and Chapter 5 of a book of H. E. Lacey [Lac].

The second half is obvious by Example 5.1.7. □

Next, we need some stability result for modulus of convexity. Recall from Definition 3.1.1 that the modulus of convexity of a Banach space $(E, \|\cdot\|)$ is defined by the following formula: for $0 < \epsilon < 2$,

$$d_{\|\cdot\|}(\epsilon) = \inf \left\{ 1 - \frac{\|\xi + \eta\|}{2} : \xi, \eta \in B(E), \text{ and } \|\xi - \eta\| \geq \epsilon \right\}.$$

Theorem 5.1.11. *Let* $(B_n, \|\cdot\|_n)_n$ *be a sequence of Banach spaces. Suppose for any* $0 < \epsilon < 2$, *there exists* $t_\epsilon > 0$ *such that*

$$\textit{for all } n, \ \ d_{\|\cdot\|_n}(\epsilon) \geq t_\epsilon.$$

Then there is an inequality: for any $0 < \epsilon < 2$,

$$d_{\|\cdot\|_\omega}(\epsilon) \geq t_\epsilon.$$

In particular, in the setting above, the ultraproduct B_ω *is uniformly convex.*

For the proof, see (and imitate) the proof of Theorem 4.4 of a book in Aksoy–Khamsi [Ak].

5.2 Reduced cohomology

In this section, we define reduced (1-)cohomology with Banach coefficients. We prove one proposition of high importance, so-called the *Proposition of scaling limits*. This proposition gives a deep corollary on reduced cohomology of a *finitely generated* groups.

5.2.1 Definition and uniform actions

Recall groups in this chapter are assumed to be discrete (and hence countable). Consider an isometric G-representation ρ on a Banach space B. In the proof of Guchardet's direction in Theorem 2.4.13 or its generalization 3.2.9, we consider the Fréchet topology on the cocycle space $Z^1(G;\rho,B)$ induced by the separating seminorms

$$Z^1(G;\rho,B) \ni c \mapsto \sup_{s\in S} \|c(s)\|,$$

where S moves in a countable family of finite subsets of G, whose union is G. In general the subspace of coboundaries $B^1(G;\rho.B)$ is *not* closed with respect to this topology, and that was a key to proving these theorems.

Definition 5.2.1. Let G be a group. Let ρ be an isometric G-representation on a Banach space B.

(i) The space $\overline{B}^1(G;\rho,B)$ is the closure of $B^1(G;\rho.B)$ in $Z^1(G;\rho,B)$ with respect to the Fréchet topology defined as the paragraph above.

(ii) The *reduced* (1-)*cohomology* $\overline{H}^1(G;\rho,B)$ is defined as the following quotient space:

$$\overline{H}^1(G;\rho,B) = Z^1(G;\rho,B)/\overline{B}^1(G;\rho,B).$$

In [Sha2], Shalom defined the notion of *uniform* affine isometric action on a Hilbert space. We extend this notion to isometric group action on a metric space.

Definition 5.2.2. Let G be a group and (X,d) be a metric space. Let $\alpha\colon G \to \mathrm{Isom}(X)$ be an isometric G-action on X.

(*i*) We define the *displacement function* associated with S by the following formula:

$$\delta_S \colon X \to \mathbb{R}_{\geq 0}; \delta_S(x) := \sup_{s \in S} d(x, \alpha(s)x).$$

(*ii*) We say α is a *uniform action* if there exists a finite subset $S \subseteq G$ and a positive number $\epsilon > 0$ such that for any $x \in X$,

$$\delta_S(x) \geq \epsilon$$

holds.

Remark 5.2.3. In the definition above, α is not uniform if and only if for any finite subset $S \subseteq G$ and any $\epsilon > 0$ there exists $x \in B$ with

$$\delta_S(x) < \epsilon.$$

Suppose G is finitely generated. Then for above condition, it is easy to see that we only have to consider the case of that S is a generating set. Then since S is generating, we have the following: α is not uniform if and only if for some (equivalently, any) finite generating subset $S \subseteq G$, there exists (x_n) $x_n \in B$ with

$$\lim_{n \to \infty} \delta_S(x_n) = 0.$$

In the setting above, choice of finite generating sets S is not essential. Therefore if G is finitely generated, then we also write a displacement function as δ for short.

Lemma 5.2.4. *Let G be a group and ρ be an isometric G-representation on a Banach space B. Then for any ρ-cocycle c, the following are equivalent:*

(*i*) *The cocycle c belongs in $\overline{B}^1(G; \rho)$.*

(*ii*) *The affine isometric action associated with (ρ, c) is* not *uniform.*

Proof. Let α be isometric action associated with (ρ, c) and let $S \subseteq G$ be a finite subset. Then for any $\xi \in B$,

$$\delta_S(\xi) = \sup_{s \in S} \|(\xi - \rho(s)\xi) - c(s)\| = \sup_{s \in S} \|c(s) - (\xi - \rho(s)\xi)\|.$$

Recall that any coboundary $b \in B^1(G; \rho)$ has the form $b(g) = \xi - \rho(g)\xi$. Now by definition the equivalence holds true. $\square$

5.2.2 Proposition of scaling limits

The following proposition is so-called the *Proposition of scaling limits*, and plays a key role in this chapter. Here we note the assumption of *finite generation* is necessary.

Proposition 5.2.5. (*Proposition of scaling limits*) *Let G be a* finitely generated *group. Let (X, d) be a complete metric space, and $\alpha\colon G \to \mathrm{Isom}(X)$ be an isometric action.*

Suppose α is not uniform, but has no G-fixed point. Then there exist a sequence (z_n) in X and a sequence of positive reals (b_n) with $b_n \nearrow \infty$ such that the resulting isometric action from the scaling limit $\lim_\omega(X, b_n \cdot d, z_n)$ is uniform.

Note that the existence of scaling limit action is a part of the statement.

The proof which we will present here seems to be due to R. Schoen. See also, [Gro4] and [Gro6]. For the proof, we need the following lemma. From the assumption of *finite generation* for G, we take a finite generating set $S \subseteq G$ and consider the displacement function $\delta = \delta_S\colon X \to \mathbb{R}$ for the action α (recall Remark 5.2.3). By assumption, δ satisfies the following two conditions:

(1) There exists a sequence $(x_n)_n$ $(x_n \in X)$ such that

$$\lim_{n\to\infty} \delta(x_n) = 0.$$

(2) There does *not* exist x satisfying $\delta(x) = 0$.

For the proof of the proposition above, we need the following lemma:

Lemma 5.2.6. *Keep the setting as in Proposition* 5.2.5*, and let S and δ be as in above.*

Let $t > 0$. Then for any $x \in X$, there exists $x' \in B(x, t\delta(x))$ such that the following holds:

$$\textit{for any } x'' \in B(x', \frac{t}{2}\delta(x')), \quad \delta(x'') \geq \frac{1}{2}\delta(x') \ \ \textit{holds.}$$

Here $B(x, r)$ denotes the closed ball centered at x of radius r.

We make a remark that for the proof of the lemma above, condition (2) in above is only needed and condition (1) is not employed.

Proof. (Lemma 5.2.6) Suppose the contrary. Then there exists $x \in X$ such that for any $x' \in B(x, t\delta(x))$,

$$\text{for some } x'' \in B(x, \frac{t}{2}\delta(x')), \ \delta(x'') < \frac{1}{2}\delta(x') \ \text{ holds.}$$

Firstly, we apply this to the case of $x' = x$ and obtain $x_1 \in B(x, t/2 \cdot \delta(x))$ such that $\delta(x_1) < 1/2 \cdot \delta(x)$. Observe that

$$d(x_1, x) \leq \frac{t}{2}\delta(x) < t\delta(x).$$

Therefore, secondly we apply the assumption above to the case of $x' = x_1$ and get $x_2 \in B(x_1, t/4\delta(x))$ (recall the condition above) such that

$$\delta(x_2) < \frac{1}{2} \cdot \delta(x_1) < \frac{1}{4} \cdot \delta(x).$$

Also observe that

$$d(x_2, x) \leq d(x_2, x_1) + d(x_1, x) < \frac{3t}{4}\delta(x) < t\delta(x).$$

By iterating this procedure, we obtain a sequence $(x_n)_{n\geq 1}$ in X such that the following two condition are satisfied:

- For any n, $x_{n+1} \in B(x_n, (1/2)^n t\delta(x))$.
- For any n, $\delta(x_n) < (1/2)^n \delta(x)$.

This sequence is a Cauchy sequence, and from the completeness of X, there exists a convergent point x_∞. Then $\delta(x_\infty) = 0$, and this contradicts condition (2) in Proposition 5.2.5. $\square$

Proof. (Proposition 5.2.5) By condition (1), there exists a sequence (y_n) in Y such that $\delta(y_n) \searrow 0$ as $n \to \infty$. Choose a sequence (c_n) of positive reals satisfying $c_n \nearrow \infty$ and $c_n\delta(y_n) \nearrow 0$ (for instance, $c_n = (\delta(y_n))^{-1/2}$). Then by Lemma 5.2.6, for each n there exists $z_n \in B(y_n, c_n\delta(y_n))$ such that the following holds:

$$\text{for any } x_n \in B(z_n, \frac{c_n}{2}\delta(z_n)), \quad \delta(x_n) \geq \frac{1}{2}\delta(z_n) \ \text{ holds.}$$

Set a sequence (b_n) of positive reals with $c_n \nearrow \infty$ by

$$b_n = \frac{2}{\delta(z_n)}.$$

We will show in below that the scaling limit of the action α with respect to the sequence $(X, b_n \cdot d, z_n)$ exists, and this scaling limit action α_ω is uniform. Firstly,

$$\sup_n \sup_{s\in S} b_n d(z_n, \alpha(s)z_n) = \sup_n b_n \cdot \delta(z_n) = 2 < \infty.$$

Hence it follows the scaling limit isometric action α_ω exists (see also Remark 5.1.9). Secondly, we see α_ω is uniform. For any sequence (x_n) in X which satisfies $\sup d(x_n, z_n) < \infty$, exists sufficiently large n_x such that the following holds:

$$\text{for any } n \geq n_x, \quad d(x_n, z_n) \leq \frac{c_n}{2}\delta(z_n)$$

(this is because $c_n \nearrow \infty$). Therefore, if we consider the displacement function (associated with S) on $(X, b_n \cdot d)$ and write it δ_n, then we have

$$\text{for any } n \geq n_x, \quad \delta_n(x_n) \geq 1.$$

This means for $n \geq n_x$, there exists $s_n \in S$ such that $d_n(x_n, \alpha(s)x_n) \geq 1$, where we set $d_n = b_n \cdot d$. For $s \in S$, set

$$I(s) = \{n : d_n(x_n, \alpha(s)x_n) \geq 1\},$$

and consider $I(S) = \bigcup_{s\in S} I(s)$. Then $I(S)$ includes $\{n : n \geq n_x\}$ and hence is in ω. Therefore $\lim_\omega \chi_{I(S)} = 1$, where χ is characteristic unction (on $\mathbb{N}$). Here we consider the unital positive $*$-preserving linear functional $\lim_\omega \colon \ell^\infty(\mathbb{N}) \to \mathbb{C}$, as we argued in Subsection 5.1.1. Since $\chi^2 = \chi$ for any characteristic function and $\lim_\omega$ is multiplicative (see Lemma 5.1.4), $\lim_\omega \chi_{I(s)}$ is either 0 or 1. Finally, by the inequality

$$\lim_\omega \sum_{s\in S} \chi_{I(s)} \geq \lim_\omega \chi_{I(S)},$$

there exists $s \in S$ such that $\lim_\omega \chi_{I(s)} = 1$. With this s, we have

$$d_\omega((x_n), (\alpha(s)x_n)) \geq 1.$$

This shows that for any $x_\omega \in X_\omega$,

$$\delta_\omega(x_\omega) := \sup_{s\in S} d_\omega(x_\omega, \alpha_\omega(s)x_\omega) \geq 1.$$

Hence α is uniform. □

In the view of Definition 5.2.1 and Lemma 5.2.4, we establish the following theorem.

Theorem 5.2.7. *Suppose $\mathcal{C}$ be a class of Banach spaces which is* stable under scaling limit. *Then the following two conditions for a* finitely generated *group G are equivalent:*

(i) The group G has $(\mathrm{F}_{\mathcal{C}})$. Namely, for any isometric G-representation ρ on $B \in \mathcal{C}$, $H^1(G; \rho, B) = 0$.

(ii) The group G satisfies the reduced cohomology vanishing with all isometric coefficients in $\mathcal{C}$. That means, for any isometric G-representation ρ on $B \in \mathcal{C}$, $\overline{H}^1(G; \rho, B) = 0$.

Note that a priori condition (ii) is much weaker than condition (i) in general setting. Also, we note that even among finitely generated groups, $\overline{H}^1(G; \rho, B) = 0$ does *not* imply $H^1(G; \rho, B) = 0$ for a single pair (ρ, B).

We note Shalom [Sha2] shows in fact we can consider the cases of locally compact and *compactly generated* groups with $\mathcal{C} = \mathcal{H}$ (unitary coefficients), in Theorem 5.2.7. His argument is based on study of conditionally negative definite kernels, and is completely different from the argument above. See [Sha2] or Section 3.2 in [BHV].

5.2.3 Applications

There are some direct applications of Theorem 5.2.7:

Theorem 5.2.8. *The following two conditions for a* finitely generated *group G are equivalent:*

(*i*) *The group G has* (T)*.*

(*ii*) *For any irreducible unitary G-representation π, $H^1(G;\pi) = 0$.*

This follows from the fact that reduced cohomology is decomposable according to (even for infinite) L^2-direct sums.

Next application is on finite presentability of Kazhdan groups. As we have seen in Remark 2.2.4 it is not always true, Shalom, nevertheless, show the following:

Theorem 5.2.9. (*Shalom* [Sha2]) *Suppose a group G has* (T)*. Then there exists a* finitely presented *group $\tilde{G}$ with* (T) *and normal subgroup $H \trianglelefteq \tilde{G}$ such that $\tilde{G}/H$ is isomorphic to G.*

Proof. By assumption, G is finitely generated. Let S be a finite generating set of G and set $n = |S|$. Let

$$\phi\colon F_n \twoheadrightarrow G$$

be a homomorphism and set N be the kernel. Let $w_1, w_2, \ldots$ be an enumeration of the elements in N. For $k \in \mathbb{N}$, let N_k is the normal closure in F_n of the group generated by $w_1, \ldots, w_k$. Set $\Gamma_k = F_k/N_k$. Thus we have

$$F_n \twoheadrightarrow \Gamma_1 \twoheadrightarrow \Gamma_2 \twoheadrightarrow \cdots \twoheadrightarrow G.$$

We claim there exists $k \in \mathbb{N}$ such that Γ_k has (T)($\Leftrightarrow$ (FH)). Suppose the contrary. Then by Proposition 5.2.5, for any Γ_k there exists an affine isometric action α_k of Γ_k on a Hilbert space which is uniform. Consider the ultralimit of α_k (with an appropriate scaling), then this yields a uniform action on a Hilbert space of G. Indeed, consider the ultraproduct of affine isometric actions of F_n. Then the kernel of resulting action contains N so that this action factors through an action of G. This contradicts (FH) for G. $\square$

5.3 Shalom's machinery

We introduce a powerful machinery, invented by Shalom [Sha5]. Shalom employed this machinery to establish property (T) for universal lattices.

In [Mim1], the author extended the conception of the *Shalom property*, which is found in Definition 12.1.13 of book [BrOz] of Brown–Ozawa.

Definition 5.3.1. (Definition 5.4 [Mim1]) Let B be a Banach space and G be a finitely generated group. A triple of subgroups (H, N_1, N_2) of G is said to have the *Shalom property for* (F_B) if all of the following four conditions hold:

(i) The group G is generated by N_1 and N_2 together.

(ii) The subgroup H normalizes N_1 and N_2.

(iii) The group G is boundedly generated by H, N_1, and N_2 (in the sense in Definition 4.2.4).

(iv) For both $i \in \{1, 2\}$, $N_i \leqslant G$ has relative (F_B).

The following theorem is called Shalom's machinery in a paper [Mim1] of the author. Shalom [Sha5] shows in the case of $\mathcal{C} = \mathcal{H}$, and it is extended to the following general case by the author [Mim1].

Theorem 5.3.2. ([Sha5]*; Shalom's Machinery, Theorem* 5.5 [Mim1]) *Let $\mathcal{C}$ be a class of superreflexive Banach spaces which is stable under ultraproducts. Let G be a finitely generated group with finite abelianization. Suppose there exist subgroups H, N_1, and N_2 of G such that (H, N_1, N_2) has the Shalom property for* $(\mathrm{F}_{\mathcal{C}})$*. Then G has property* $(\mathrm{F}_{\mathcal{C}})$*.*

We note that in this theorem, we need the stability of the class under ultraproducts, not only under scaling limits.

For the proof below, recall the canonical decomposition of a superreflexive Banach space B as

$$B = B^{\rho(G)} \oplus B_{\rho(G)}{}' = B_0 \oplus B_1.$$

associated with an isometric representation ρ of a group G (Proposition 3.1.12). Also, for a given affine isometric G-action α with linear part ρ, there is a canonical decomposition of α associated with the decomposition of B above. More precisely, it is constructed in the following manner: consider the cocycle part c of α, and decompose $c = c_0 + c_1$ according to the decomposition $B = B_0 \oplus B_1$. Namely, for $i = 0, 1$, c_i is the image of c by the projection from B onto B_i associated with the decomposition $B = B_0 \oplus B_1$. By $\rho(G)$-invariance of these spaces, each c_i is also a ρ-cocycle on B_i. The decomposition $\alpha = \alpha_0 + \alpha_1$ is the associated affine isometric action with the decomposition $c = c_1 + c_2$ of the cocycle.

Proof. Suppose the contrary. Then by Theorem 5.2.7, there must exist an affine isometric action α_0 on some $B_0 \in \mathcal{C}$ which is uniform. For simplicity, we may assume that B_0 is uc by choosing an appropriate norm. Fix a finite generating set S of G. We define $\mathcal{A}$ as the class of all pairs (α, E) of an affine isometric G-action (on E) and a uc Banach space which satisfy the following conditions:

(a) For any $\xi \in E$, $\sup_{s \in S} \|\xi - \alpha(s)\xi\|_E \geq 1$.

(*b*) For all $0 < \epsilon < 2$, $d_{\|\cdot\|_E}(\epsilon) \geq d_{\|\cdot\|_0}(\epsilon)$. Here $d_{\|\cdot\|}$ is the modulus of convexity, and $\|\cdot\|_0$ be the (uc) norm of B_0.

Firstly, note that this class $\mathcal{A}$ is non-empty, because (α_0, B_0), with an appropriate rescaling, is an element. Secondly, we claim $\mathcal{A}$ is stable under ultraproducts. Indeed, condition (*b*) is stable by Theorem 5.1.11. To see condition (*a*) is also stable, imitate the last part of the proof of Proposition 5.2.5.

Next we define a real number D by the following formula:

$$D = \inf\{\|\xi^1 - \xi^2\| : (\alpha, E) \in \mathcal{A}\},$$

where for $i \in \{1, 2\}$, ξ^i moves through all $\alpha(N_i)$-fixed points in E. By condition (*iv*) of the Shalom property, the set is non-empty and hence D is well-defined. By taking an ultraproduct, one can show that D is in fact a minimum. Let ξ^1_∞ and ξ^2_∞ be vectors which attain the minimum D. Also let $(\alpha_\infty, E_\infty) \in \mathcal{A}$ be the associated affine isometric G-action and ρ_∞ be the linear part of α_∞.

Decompose the action α_∞ into $\alpha_{\infty,\mathrm{triv}}$ and $\alpha_\infty{}'$, where the former takes values in $E_\infty^{\rho_\infty(G)}$ and the latter tales values in $E'_{\infty,\ \rho_\infty(G)}$. Firstly, we consider orbits by $\alpha_{\infty,\mathrm{triv}}$. This part is easy: by the assumption of finite abelianization, $\alpha_{\infty,\mathrm{triv}}$ is indeed a trivial action. Hence every $\alpha_{\infty,\mathrm{triv}}(G)$ orbit is bounded. This is the first goal in this proof.

Next, we deal with orbits by the action $\alpha_\infty{}'$. For each $i = 1, 2$, we decompose the vector ξ^i_∞ according to the decomposition

$$E = E_\infty^{\rho_\infty(\Gamma)} \oplus E'_{\infty,\rho_\infty(\Gamma)}.$$

and write as $\xi^i_\infty = \xi^i_{\infty,\mathrm{triv}} + \xi^i_\infty{}'$. Firstly, we will see in below that each $\xi^i_\infty{}'$ is an $\alpha_\infty{}'(H)$-fixed point. To see this, let $h \in H$ be any element. For each $i = 1, 2$, set $\eta^i_\infty = \alpha_\infty(h)\xi^i_\infty$. Since α_∞ is an isometric action, we have

$$\|\eta^1_\infty - \eta^2_\infty\| = \|\xi^1_\infty - \xi^2_\infty\| = D.$$

We now claim that for each $i = 1, 2$, η^i_∞ is an $\alpha_\infty(N_i)$-fixed point. This follows from condition (*ii*) of the Shalom property because for any $l_i \in N_i$,

$$\alpha_\infty(l_i)\alpha_\infty(h)\xi^i_\infty = \alpha_\infty(h)\alpha_\infty(h^{-1}l_ih)\xi^i_\infty = \alpha_\infty(h)\xi^i_\infty.$$

For $i = 1, 2$, set ζ^i_∞ as the midpoint between ξ^i_∞ and η^i_∞, namely,

$$\zeta^i_\infty = \frac{\xi^i_\infty + \eta^i_\infty}{2}.$$

Since α_∞ is affine, for each $i = 1, 2$, ζ^i_∞ is also an $\alpha_\infty(N_i)$-fixed point. By the triangle inequality, we have

$$\|\zeta^1_\infty - \zeta^2_\infty\| \leq \frac{\|\xi^1_\infty - \xi^2_\infty\| + \|\eta^1_\infty - \eta^2_\infty\|}{2} = D.$$

Recall that D is a minimum and the Banach space E_∞ is uniformly convex (in particular strictly convex). Therefore, the inequality above forces that the four points ξ^1_∞, η^1_∞, η^2_∞, ξ^2_∞ form a rectangle with this order. Namely, one has

$$\xi^1_\infty - \eta^1_\infty = \xi^2_\infty - \eta^2_\infty.$$

Set this vector Ω.

Since for each $i = 1, 2$, ξ^i_∞ and η^i_∞ are $\alpha_\infty(N_i)$-fixed, the difference $\xi^i_\infty - \eta^i_\infty$ is $\rho_\infty(N_i)$-invariant. By condition (1) of the Shalom property, the equality above implies Ω is a $\rho_\infty(G)$-invariant vector. However, this forces $\Omega = 0$. Indeed, recall the decomposition

$$\xi^i_\infty = \xi^i_{\infty,\mathrm{triv}} + {\xi^i_\infty}'.$$

Then from the decomposition of α_∞ into $\alpha_{\infty,\mathrm{triv}}$ and ${\alpha_\infty}'$, we have the following: for each $i = 1, 2$

$$\eta_i = \xi^i_{\infty,\mathrm{triv}} + {\alpha_\infty}'(h){\xi^i_\infty}'.$$

Here we use the observation above that $\alpha_{\infty,\mathrm{triv}}$ is a trivial action. This implies for each $i = 1, 2,$

$$\Omega = {\xi^i_\infty}' - {\alpha_\infty}'(h){\xi^i_\infty}' \in E'_{\infty,\rho_\infty(G)}.$$

However the space $E'_{\infty,\rho_\infty(G)}$ has no non-zero $\rho_\infty(G)$-invariant vectors by construction. Hence Ω must be zero. Again by the equality above, this implies each ${\xi^i_\infty}'$ is ${\alpha_\infty}'(H)$-fixed.

Secondly, the existence of ${\alpha_\infty}'(H)$-fixed points implies that every ${\alpha_\infty}'(H)$-orbit is bounded. Condition (4) already shows for $i = 1, 2$ that every ${\alpha_\infty}'(N_i)$-orbit is bounded. With the use of condition (*iii*) (bounded generation), this imlplies that every ${\alpha_\infty}'(G)$-orbit is bounded. This is the second goal in this proof.

Finally, boundedness of $\alpha_{\infty,\mathrm{triv}}(G)$-orbits and ${\alpha_\infty}'(G)$-orbits means that indeed every $\alpha_\infty(G)$-orbit is bounded. This implies that α_∞ has a G-fixed point, and in particular α_∞ is *not* uniform. This contradicts the construction of $(\alpha_\infty, E_\infty)$. This ends our proof. □

5.4 Universal lattices have (T)

We restate Theorem 1.0.1, a deep theorem due to Shalom and Vaserstein.

Theorem 5.4.1. (*Shalom* [Sha5], *Vaserstein* [Vas2]) *Universal lattice*

$$G = \mathrm{SL}_{m\geq 3}(\mathbb{Z}[x_1, \ldots, x_k])$$

has property (T).

Therefore, for any commutative finitely generated ring A, $\mathrm{E}_{m\geq 3}(A)$ has property (T).

Proof. We have prepared all needed facts for the proof.

Let $A = \mathbb{Z}[x_1, \ldots, x_k]$. Set $G = \mathrm{SL}_m(A)$, $H \cong \mathrm{SL}_{m-1}(A)$, and $N_1, N_2 \cong A^{n-1}$. Here in G we realize H as in the left upper corner (, namely, the $((1\text{-}(m-1)) \times (1\text{-}(m-1)))$-th parts), realize N_1 as in the $((1\text{-}(m-1)) \times m)$-th unipotent parts, and realize N_2 as in the $(m \times (1\text{-}(m-1)))$-th unipotent parts. Then we claim that Shalom's machinery applies with $\mathcal{C} = \mathcal{H}$.

Indeed, (H, N_1, N_2) is has the Shalom property for (FH). Conditions (i) and (ii) are confirmed directly. Condition (iv) follows from Theorem 4.2.3 (also recall the proof of Proposition 2.6.6). Condition (iii) follows from Theorem 4.3.2, a deep result of Vasestein on bounded generation. Also G has trivial abelianization (see Remark 4.1.2).

Thus we have established property (FH), which is equivalent to (T). $\square$

We have one historical remark. Shalom's original argument in [Sha5] uses stable range condition for the ring $\mathbb{Z}[x_1, \ldots, x_k]$ (recall Definition 4.1.4 and Proposition 4.1.5), because at that moment Vaserstein's bounded generation had not been proven. Therefore, in [Sha5], there is some condition on m in terms of the number of generators k.

Remark 5.4.2. We note that at the moment, Shalom's machinery does not apply to the case of noncommutative universal lattices. The obstruction in applying this is the lack of bounded generation theorem (condition (iii)). It is not known whether an analogue of Vaserstein's bounded generation holds for noncommutative universal lattices. Nevertheless Ershov–Jaikin-Zapirain have established property (T) for noncommutative universal lattices [ErJa].

Corollary 5.4.3. *Let $m \geq 3$.*

Let A be a commutative finitely generated ring with a finite generating set S. Set $\mathcal{S}$ be the following finite generating set of $\mathrm{E}_m(A)$:

$$\mathcal{S} = \{E_{i,j}(s) : 1 \leq i \leq m, 1 \leq j \leq m, i \neq j; s \in S\}.$$

Then there exists a constant $\kappa_{m,k} > 0$, only depending on m and $k = |S|$ such that the following estimate for the Kazhdan constant holds true:

$$\mathcal{K}(\mathrm{E}_m(A); \mathcal{S}) \geq \kappa_{m,k}$$

The Kazhdan constant for universal lattices is estimated in Theorem 12.1.14 in [BrOz] (although they considered the case of $\mathrm{SL}_3(\mathbb{Z})$, their proof applies to general case). Also, Ershov–Jaikin-Zapirain [ErJa] has given estimation even for noncommutative universal lattices.

Chapter 6

Property ($\mathrm{F}_{\mathcal{L}_p}$) and ($\mathrm{F}_{[\mathcal{H}]}$) for universal lattices

This part is one of the main parts in this thesis. In this chapter, we shall establish property ($\mathrm{F}_{\mathcal{L}_p}$) (for all $p \in (1, \infty)$) and property ($\mathrm{F}_{[\mathcal{H}]}$) for universal lattices with degree at least 4 (this is Theorem A). There is one key trick to deduce them, and we will see that in Section 6.2. The main reference is the original paper [Mim1] of the author.

Here we sketch the philosophy of the proof of Theorem A. In the view of Shalom's machinery (Theorem 5.3.2), to establish property ($\mathrm{F}_{\mathcal{C}}$) for a class of superreflexive Banach spaces being stable under ultraproducts, it suffices to establish *relative property* ($\mathrm{F}_{\mathcal{C}}$) for the pair $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ (here $A = \mathbb{Z}[x_1, \ldots, x_k]$). Therefore, we consider the following two steps:

- **Step1**: establish relative property ($\mathrm{T}_{\mathcal{C}}$) for the pair $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$.
- **Step2**: deduce relative property ($\mathrm{F}_{\mathcal{C}}$) for $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$, from relative property ($\mathrm{T}_{\mathcal{C}}$) obtained in Step1.

We have succeeded in Step1 for special case, but have failed in the original Step2. Nevertheless, we have succeeded in proving that, if we consider the "larger" pair $\mathrm{SL}_3(A) \ltimes A^3 \trianglerighteq A^3$, then Step2 *always* works. Thus we need the assumption of degree≥ 4 in establishing ($\mathrm{F}_{\mathcal{C}}$) for the class $\mathcal{C} = \mathcal{L}_p$ and $\mathcal{C} = [\mathcal{H}]$.

6.1 Relative property ($\mathrm{T}_{[\mathcal{H}]}$)

First we show the following theorem, which amounts to Step1 in the philosophy explained in this chapter page.

Recall from Chapter 0 we always assume $p \in (1, \infty)$, and we use the symbol k for representing any nonnegative integer.

Theorem 6.1.1. (*Theorem* 1.4 [Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots, x_k]$. *Then the pair* $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ *has relative property* $(\mathrm{T}_{\mathcal{C}})$. *Here* $\mathcal{C}$ *stands for* $\mathcal{L}_p$ *and* $[\mathcal{H}]$.

By Theorem 3.3.10, relative (T) implies relative $(\mathrm{T}_{\mathcal{L}_p})$. Therefore in the statement above, the case of $\mathcal{C} = \mathcal{L}_p$ is trivial from Theorem 4.2.3.

Hence in this section, we deal with the case of $\mathcal{C} = [\mathcal{H}]$. In other words, we examine *uniformly bounded* representations on Hilbert spaces.

6.1.1 Dixmier's unitarization of uniformly bounded representation

The key to proving Theorem 6.1.1 is the following proposition by J. Dixmier [Dix], which states any uniformly bounded representation on a Hilbert space of an *amenable* group is unitarizable. For amenable groups, see Subsection 2.5.2.

Proposition 6.1.2. (*Dixmier* [Dix]) *Suppose* Λ *is* amenable. *Then for any uniformly bounded representation* ρ *on a Hilbert space* $\mathfrak{H}$ *of* Λ, *there exists an invertible operator* $T \in \mathbb{B}(\mathfrak{H})$ *such that*

$$\pi := \mathrm{Ad}(T) \circ \rho = T \circ \rho \circ T^{-1}$$

is a unitary representation. Moreover, one can choose T *such that*

$$\|T\| \|T^{-1}\| \leq |\rho|^2.$$

Here $\|T\|$ *means the operator norm of* T *in* $\mathbb{B}(\mathfrak{H}, \|\cdot\|_{\mathfrak{H}})$.

Later we will change norms on $\mathfrak{H}$, and then the operator norm of T in $\mathbb{B}(\mathfrak{H}, \|\cdot\|')$ will be changed according to norms $\|\cdot\|'$ on $\mathfrak{H}'$.

Proof. We use the characterization (*ii*) in Theorem 2.5.7 for amenability. Let $E\colon UCB(\Lambda) \to \mathbb{C}$ be a Λ invariant mean. We define the following map

$$\Phi\colon (\xi, \eta) \in \mathfrak{H} \times \mathfrak{H} \mapsto E(\phi_{\xi,\eta}),$$

where $\phi_{\xi,\eta} \in UCB(\Lambda)$ is defined as

$$\phi_{\xi,\eta}(g) = \langle \rho(g)\xi | \rho(g)\eta \rangle.$$

Note that the map Φ is well-defined because ρ is uniformly bounded. Then Φ gives a *new* (positive-definite) inner product on $\mathfrak{H}$, and $\rho(G)$-invariant by (right) G-invariance of E. Therefore ρ is unitary with respect to this Hilbert space structure on $\mathfrak{H}$. Again since $|\rho| < \infty$, this structure is compatible to the original Hilbert space structure in certain quantitative sense. □

We note that in general, there exists a uniformly bounded representation on a Hilbert space which is *not* unitarizable. It is known that any group which includes a subgroup isomorphic to F_2 admits such representation. It is a long-standing open problem to determine whether nonexistence of such representations implies amenability.

6.1.2 Proof of relative $(\mathbf{T}_{[\mathcal{H}]})$

The proof of Theorem 6.1.1 is based on the argument in the proof of Theorem 4.2.1. Therefore here we give a sketched proof. Also, there is a quantitative version of this theorem (Proposition I) as we mentioned in Chapter 1.

Proof. (Theorem 6.1.1, Outlined) For simplicity, we shall show the case of $k = 0$. Namely, we will prove relative property $(\mathrm{T}_{[\mathcal{H}]})$ for $N = \mathbb{Z}^2 \trianglelefteq \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 = G$. Set $H = \mathrm{SL}_2(\mathbb{Z})$. We also use other notation as in the proof of Theorem 4.2.1, such as, $h_1 = E_{1,2}(1)$, $h_2 = E_{2,1}(1)$, $l_1 = E_{1,3}(1)$, $l_2 = E_{2,3}(1)$, $S_0 = \{h_1^{\pm}, h_2^{\pm}\}$, $S_1 = \{l_1^{\pm}, l_2^{\pm}\}$, and $S = S_0 \cup S_1$.

Suppose that there exist a ucus Banach space $B \in [\mathcal{H}]$ and an isometric representation (ρ, B) of G such that ρ admits almost invariant vectors in $B'_{\rho(N)}$. We may assume that $B'_{\rho(N)} = B$ because $B'_{\rho(N)}$ is also an element in $[\mathcal{H}]$. Consider the compatible Hilbert norm on B, and we write $\mathfrak{H}$ as the space B equipped with this norm. Then the representation ρ, viewed as a representation on $\mathfrak{H}$, is uniformly bounded. Thanks to the amenability of N and Proposition 6.1.2, we may also assume $(\rho, \mathfrak{H})$ is a *unitary* representation on N (by replacing ρ with $\mathrm{Ad}(T) \circ \rho$, where T is defined as in Proposition 6.1.2). We choose any vector $\xi \in S(B)$ and fix it. We let $\delta_\xi = \sup_{s \in S} \|\xi - \rho(s)\xi\|_B$ and $\delta^*{}_\xi = \sup_{s \in S} \|\xi^* - \rho^\dagger(s)\xi^*\|_{B^*}$. Here recall that $\rho^\dagger$ denotes the contragredient representation on B^* and $\xi \mapsto \xi^*$ is the duality mapping (see Definition 3.1.6 and Definition 3.1.4).

Then from a chosen vector ξ and the duality on B, we can construct a spectral measure $\nu = \nu_\xi$ on the unitary dual $\widehat{\mathbb{Z}^2} \cong \mathbb{T}^2 \cong \left[-\frac{1}{2}, \frac{1}{2}\right)^2$. The method for constructing the measure is similar to one in the proof of Theorem 4.2.1. More precisely, we set ν on $\mathbb{T}^2$ such that the following holds: for any $f \in C(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} f d\nu = \langle \sigma(f)\xi, \xi^* \rangle.$$

Here $\sigma\colon C(\mathbb{T}^2) \to \mathbb{B}(\mathfrak{H})$ is the $*$-homomorphism induced by the *unitary* representation $\rho\mid_N$.

Note that we use the duality on B, *not* the inner product on $\mathfrak{H}$ in the construction above. Therefore unlike the original case, the measure ν is complex-valued in general. However, we obtain the positive part ν_+ by taking the Hahn–Jordan decomposition of ν. We can also verify the following three facts in an argument similar to one in the original proof of Theorem 4.2.1:

(i) The inequality $\nu_+(\mathbb{T}^2) \geq 1$ holds.

(ii) For any Borel set Z being far from the origin 0 of $\mathbb{T}^2$ (in certain quantitative sense),

$$\nu_+(Z) = O(\delta_\xi \cdot \delta^*{}_\xi),$$

as δ_ξ, $\delta^*{}_\xi \to 0$.

(*iii*) For any Borel subset $Z \subset \mathbb{T}^2$ and $h \in S_0$,

$$|\nu_+({}^t h^{-1} \cdot Z) - \nu_+(Z)| = O(\delta_\xi + \delta^*{}_\xi),$$

as δ_ξ, $\delta^*{}_\xi \to 0$.

Now let $\xi \in S(B)$ move among almost invariant vectors with $\delta_\xi \to 0$. Then by uniform continuity of the duality mapping (Lemma 3.1.7), $\delta^*{}_\xi$ also tends to 0. Hence there must exist some vector $\xi \in S(B)$ such that the associated positive measure ν_+ has a non-zero value on $\{0\}(\subset \mathbb{T}^2)$. This contradicts our assumption that $B^{\rho(N)} = 0$. □

We refer to the Appendix for details and a certain quantitative treatment.

6.2 A key trick

The following theorem amounts to Step 2 in the philosophy explained at the beginning of this chapter, and is the key in the paper [Mim1] of the author.

Theorem 6.2.1. (*Theorem* 1.3 [Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots, x_k]$. *Suppose* B *is any superreflexive Banach space. Then, if the pair* $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ *has relative property* (T_B), *then the pair* $\mathrm{SL}_3(A) \ltimes A^3 \geqslant A^3$ *has relative property* (F_B).

We note that in the statement of this theorem, we use the symbol $G \geqslant N$ for relative (F_B) because normality of N is not necessarily to define relative (F_B) (compare with the symbol $G \trianglerighteq N$ for relative (T_B)).

We state the following elementary lemma, which shall be used in the proof of Theorem 6.2.1:

Lemma 6.2.2. *Let* ρ *be an isometric* G*-representation and* c *be a* ρ*-cocycle. Then the following hold:*

(*i*) *For any* $g, h \in G$, $\|c(gh)\| \leq \|c(g)\| + \|c(h)\|$.

(*ii*) *For any* $g \in G$, $c(g^{-1}) = -\rho(g^{-1})c(g)$.

(*iii*) *For any* $g, l \in G$, $c(glg^{-1}) = \rho(g)c(l) + (I - \rho(glg^{-1}))c(g)$. *In particular, if* $\rho(glg^{-1})c(g) = c(g)$, *then*

$$c(glg^{-1}) = \rho(g)c(l).$$

Before proceeding to the main part of proof, in below we consider some setting in the proof of the theorem above.

Let $H = \mathrm{SL}_3(A)$, $N = A^3$ and $G = H \ltimes N$. We identify G with the following subgroup of $\mathrm{SL}_4(A)$:

$$\left\{ \left(\begin{array}{c|c} W & v \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{SL}_3(A),\, v \in A^3 \right\} \trianglerighteq \left\{ \left(\begin{array}{c|c} I_3 & v \\ \hline 0 & 1 \end{array} \right) : v \in A^3 \right\}.$$

We also write an element $g \in G$ as (W, v) $(W \in \mathrm{SL}_3(A), v \in A^3)$ associated with in the identification above. Also, we identify elements of the form $(I, v) \in N$ with column vectors in A^3.

We let N_1 be the subgroup of $N(\subseteq \mathrm{SL}_4(A))$ of all elements whose $(2,4)$-th and $(3,4)$-th entries are 0. Namely, N_0 is identified with the additive groups of column vectors ${}^t(v_1, 0, 0) \in A^3$ as in the identification above. Take any isometric G-representation ρ on B and any ρ-cocycle c. Fix one ucus and $\rho(G)$-invariant norm on B as in Proposition 3.1.10. Take a decomposition of B:

$$B = B^{\rho(N)} \oplus B'_{\rho(N)} =: B_0 \oplus B_1,$$

and obtain the associated decomposition of the cocycle c

$$c = c_0 + c_1 \quad (c_0(g) \in B_0,\ c_1(g) \in B_1).$$

From the $\rho(G)$-invariance of B_0 and B_1, each c_j, $j \in \{0, 1\}$ is a ρ-cocycle. For any elements $h = (W, 0) \in H$ and $r = (I, v) \in N$, $hrh^{-1} =(I, Wv) =\in N$ holds, where Wv means the multiplication of the matrix W to the column vector v. In particular, by noting that $\rho\mid_N=\mathrm{id}$ on B_0, we have the following equality: for any $h = (W, 0) \in H$ and $r = (I, v) \in N$,

$$c_0((I, Wv)) = \rho((W, 0))c_0((I, v))$$

(see item *(ii)* in Lemma 6.2.2).

Finally, we claim the following:

(a) The set $c_0(N)$ is bounded (and hence actually is equal to 0).

(b) If $c_1(N_1)$ is bounded, then $c_1(N)$ is bounded.

Indeed, for item (a), the key is that any vector $v = {}^t(v_1, v_2 v_3) \in N$ is decomposed as

$${}^t(v_1, v_2 v_3) = {}^t(1, v_2 - 1, v_3 - 1) + {}^t(v_1 - 1, 1, 0) + {}^t(0, 0, 1).$$

Then for each u of the three terms in the right hand side, there exists $h = (W, 0) \in H = \mathrm{SL}_3(A)$ such that $Wu = {}^t(1, 0, 0) =: e_1$. Therefore, in the view of the formula on $c_0((I, Wv))$ above, we have

$$\|c_0((I, v))\| \leq 3\|c_0((I, e_1))\|.$$

This gives the uniform bound of $c_0(N)$. For item (b), consider some alternating matrix $W \in \mathrm{SL}_3(A)$, which sends first column to the second (, namely, for instance, take W satisfying $W^t(v_1, v_2, v_3) = {}^t(v_2, v_1, -v_3)$). By assumption, we know that c is uniformly bounded on $N_1 = \{v = {}^t(v_1, 0, 0) : v_1 \in A\}$. Then since $(W, 0)(I, v)(W^{-1}, 0) = (I, Wv)$, we have

$$\|c_1((I, Wv))\| \leq \|c_1((I, v))\| + \|c_1((W, 0))\| + \|c_1((W^{-1}, 0))\|$$

Since W is taken independently of v, this shows c_1 is also bounded on the group $\{{}^t(0, v_2, 0) : v_2 \in A\} \leqslant N$. In a similar way, c_1 is also bounded on the group $\{{}^t(0, 0, v_3) : v_3 \in A\} \leqslant N$. Finally, observe

$$\|c_1({}^t(v_1, v_2, v_3))\| \leq \|c_1({}^t(v_1, 0, 0))\| + \|c_1({}^t(0, v_2, 0))\| + \|c_1({}^t(0, 0, v_3))\|,$$

and get item (b).

Proof. (Theorem 6.2.1) Keep the setting as in above. Thanks to item (a) and item (b) above and Lemma 3.2.5, for our proof it suffices to verify the boundedness of $c_1(N_1)$. We define a *finite* subset S_0 and two subgroups G_1, G_2 of G by the following expressions respectively:

$$\left\{\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & * & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 1 & {}^t v' & 0 \\ 0 & W' & 0 \\ 0 & 0 & 1 \end{pmatrix}\right\} \text{ and } \left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & W' & v' \\ 0 & 0 & 1 \end{pmatrix}\right\}.$$

Here in the first definition, the expression means that for each element in S_0, only one of the above $*$'s is $\pm x_l$ $(0 \leq l \leq k)$ and the others are 0. Also in the second and the third expressions, W' moves among all elements in $\mathrm{E}_2(A)$ and v' moves among all elements in A^2. We let $C = \sup_{s \in S_0} \|c_1(s)\| < \infty$. We set L $(\trianglelefteq G_1)$ as the group of all elements in G_1 with $R' = I$ and N_2 $(\trianglelefteq G_2)$ as the group of all elements in G_2 with $W' = I$. A crucial point here is that N_1 commutes with S_0: therefore for any $r_1 \in N_1$ and any $s \in S_0$, we have the following inequalities:

$$\begin{aligned} \|\rho(s)c_1(r_1) - c_1(r_1)\| &= \|c_1(sr_1) - c_1(r_1) - c_1(s)\| \\ &\leq \|c_1(r_1 s) - c_1(r_1)\| + \|c_1(s)\| \\ &= \|\rho(r_1)c_1(s)\| + \|c_1(s)\| \\ &= 2\|c_1(s)\| \leq 2C. \end{aligned}$$

We set a number $\mathcal{K}$ as the minimum of the two relative Kazhdan constants $\mathcal{K}(G_1, L; S_0 \cap G_1, \rho\mid_{G_1})$ and $\mathcal{K}(G_2, N_2; S_0 \cap G_2, \rho\mid_{G_2})$ (recall definition 3.2.7). Then by relative (T_B) for the pair $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$, $\mathcal{K}$ is strictly positive. Hence by Lemma 3.2.8, for any $\xi \in c_1(N_1)$ there are following inequalities:

$$\begin{aligned} \text{for any } l \in L, \quad &\|\xi - \rho(l)\xi\| \leq 8\mathcal{K}^{-1}C, \\ \text{and for any } r_2 \in N_2, \quad &\|\xi - \rho(r_2)\xi\| \leq 8\mathcal{K}^{-1}C. \end{aligned}$$

Next, note that the group N_1 is obtained by single commutators between L and N_2: for any $r \in N$, there exist $r_1 \in N_1$, $r_2 \in N_2$, $r' \in N_2$, and $l \in L$ such that $r = r_1 r_2$ and $r_1 = l r' l^{-1} r'^{-1}$. Hence for any $\xi \in c_1(N_1)$ and $r \in N$, the following inequality holds:

$$\begin{aligned}\|\xi - \rho(r)\xi\| &= \|\xi - \rho(l r' l^{-1} r'^{-1} r_2)\xi\| \\ &\leq 32\mathcal{K}^{-1}C.\end{aligned}$$

Note that the upper bound of the inequality above is *independent* of the choices of $\xi \in c_1(N_1)$ and of $r \in N$.

Finally, suppose that $c_1(N_1)$ is not bounded. Then one can choose $\xi \in c_1(N_1)$ such that the following holds:

$$\text{for all } r \in N,\ \ \|\xi - \rho(r)\xi\| < \|\xi\|.$$

However, in a similar argument as in the proof of Corollary 2.1.14, this inequality implies existence of a non-zero $\rho(N)$-invariant vector in $B_1 = B_{\rho(N)}{}'$. This is a contradiction. ☐

6.3 Proof of Theorem A

Here we restate Theorem A in Chapter 1 for convenience:

Theorem 6.3.1. (*Theorem* 1.2, [Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots x_k]$ $(k \in \mathbb{N})$. *Then for* $m \geq 4$, *the universal lattice* $G = \mathrm{SL}_m(A)$ *has property* $(\mathrm{F}_{\mathcal{C}})$. *Here* $\mathcal{C}$ *stands for either the class* $\mathcal{L}_p$ $(1 < p < \infty)$ *or the class* $[\mathcal{H}]$.

Proof. We will see that Shalom's machinery (Theorem 5.3.2) works for $\mathrm{SL}_m(A)$ in the following setting: set $H \cong \mathrm{SL}_{m-1}(A)$, and $N_1, N_2 \cong A^{n-1}$. Here in G we realize H as in the left upper corner (, namely, the ((1-$(m-1)$) $\times$ (1-$(m-1)$))-th parts), realize N_1 as in the ((1-$(m-1)$) $\times$ m)-th unipotent parts, and realize N_2 as in the ($m \times$ (1-$(m-1)$))-th unipotent parts. Namely, we take

$$H = \mathrm{SL}_{m-1}(A) := \left\{ \left(\begin{array}{c|c} W & 0 \\ \hline 0 & 1 \end{array} \right) : W \in \mathrm{SL}_{m-1}(A) \right\},$$

$$N_1 := \left\{ \left(\begin{array}{c|c} I_{m-1} & v \\ \hline 0 & 1 \end{array} \right) : v \in A^{m-1} \right\} \cong A^{m-1},$$

$$N_2 := \left\{ \left(\begin{array}{c|c} I_{m-1} & 0 \\ \hline {}^t v' & 1 \end{array} \right) : v' \in A^{m-1} \right\} \cong A^{m-1}.$$

We have seen that this triple satisfies conditions (i), (ii), and (iii) of the Shalom property (for any class $\mathcal{C}$) in the proof of Theorem 5.4.

Firstly we claim that Shalom's machinery applies with $\mathcal{C} = \mathcal{L}_p$. The stability of $\mathcal{L}_p$ under ultraproducts follows from item (iii) of Theorem 5.1.10. It only remains to check that (H, N_1, N_2) satisfies condition (iv) of the Shalom property for $(\mathrm{F}_{\mathcal{L}_p})$. This part is the main work of this chapter (and the paper [Mim1] of the author): Theorem 6.2.1 together with Theorem 6.1.1 insures the conclusion. Therefore, Shalom's machinery works.

In the case of $\mathcal{C} = [\mathcal{H}]$, we need an additional care, because $[\mathcal{H}]$ itself is not stable under ultraproducts. What needs here is to just replace $[\mathcal{H}]$ with $\mathcal{H}_M$, the class defined in Theorem 5.1.10, for every $M \geq 1$. Again Theorem 6.2.1 together with Theorem 6.1.1 shows (H, N_1, N_2) satisfies condition (iv) of the Shalom property for $(\mathrm{F}_{\mathcal{H}_M})$ for each M. Therefore for any $M \geq 1$, Shalom's machinery works with $\mathcal{C} = \mathcal{H}_M$, and hence G has property $(\mathrm{F}_{\mathcal{H}_M})$. Finally, observe

$$[\mathcal{H}] = \bigcup_{M \geq 1} \mathcal{H}_M.$$

Thus we have established property $(\mathrm{F}_{[\mathcal{H}]})$ for $\mathrm{SL}_{m\geq 4}(A)$. $\square$

We note that for the case of $\mathcal{C} = \mathcal{L}_p$, there is a shortcut in proving this theorem. Namely, we have a way to prove property $(\mathrm{F}_{\mathcal{L}_p})$ *without* appealing to Shalom's machinery. We will discuss it in Section 8.1. With the aid of this shortcut, we have established the fixed point property on *p-Schatten class operators* in the paper [Mim3] of the author.

Finally, we state one immediate corollary of Theorem 6.3.1, which follows from the fact that (F_B) always implies (T_B) (Theorem 3.2.9).

Corollary 6.3.2. ([Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots x_k]$ $(k \in \mathbb{N})$*. Then for* $m \geq 4$*, the universal lattice* $\mathrm{SL}_m(A)$ *has property* $(\mathrm{T}_{[\mathcal{H}]})$*.*

Chapter 7

Property (TT) and (FF_B)

In this chapter, we introduce a notion of *property* (FF_B) in the paper [Mim1] of the author, which is an extension of *property* (TT) of Monod (we refer to Chapter 12 of a book of Monod [Mon1]). Also, we explain some connection to the concept of *bounded cohomology*, whose theory has been invented by Gromov [Gro1]. Finally, we state some previously known facts on property (TT), which accounts to two papers [BuMo1], [BuMo2] of Burger–Monod; two papers of Monod–Shalom [MoSh1], [MoSh2]; and a paper [MMS] of Mineyev–Monod–Shalom. Some results of the author on (FF_B) (and (TT)) shall be stated and proven in Chapter 8.

The main references in this section are a book [Mon1] and an ICM proceedings paper [Mon2] of Monod.

7.1 Property (FF_B) and bounded cohomology

In this section, we give the definition of *property* (FF_B) and explain relation to bounded cohomology. Property (FF_B) is a *quasification* of property (F_B). That means, we consider a map which is a cocycle *up to bounded error*, and we call this map a *quasi-cocycle*. Property (FF_B) is defined as the boundedness property of such quasi-cocycles.

7.1.1 Quasi-cocycles and (FF_B)

Definition 7.1.1. Let B be a Banach space and G be a group.

(i) Let ρ be an isometric G-representation on B. A continuous map $b\colon G \to B$ is called a *quasi-ρ-cocycle* if the following holds:

$$\sup_{g,h\in G} \|b(gh) - b(g) - \rho(g)b(h)\| < \infty.$$

(*ii*) A (strongly) continuous map β from G to the set of all affine isometries on B is called a *quasi-action* if the following holds:

$$\sup_{g,h\in G}\sup_{\xi\in B}\|\beta(gh)\xi-\beta(g)\beta(h)\xi\|<\infty.$$

In the definition of quasi-actions, one can decompose the map β into the linear part ρ and the transition part b, namely, $\beta(g)\xi=\rho(g)\xi+b(g)$ for any $g\in G$ and $\xi\in B$. Then the map β is a quasi-action if and only if ρ is a group representation and b is a quasi-ρ-cocycle. Indeed, "if" part is trivial. For "only if" part, what we need to show is that the linear part ρ is indeed a representation (a group homomorphism). Suppose there exist $g,h\in G$ such that $\rho(g)\rho(h)-\rho(gh)\neq 0$. Then by taking η with $(\rho(g)\rho(h)-\rho(gh))\eta\neq 0$ and considering $t\eta$ with $t\to\infty$, we have that this β is not a quasi-action. Therefore ρ must be a group homomorphism, and we are done.

Remark 7.1.2. We warn that the terminology of *quasi-actions* is *not* a standard one, although the terminology *quasi-cocycle* is standard. Usually, the word "quasi-action" is used in relation to *quasi-isometry* among spaces (compare with Definition 11.4.3). On this topic, see [Roe], [Gro2], [Gro4], and [Gro5].

The following terminology is due to the author [Mim1]:

Definition 7.1.3. ([Mim1]) Let B be a Banach space.

(*i*) A pair $G\geqslant N$ of groups is said to have *relative property* (FF_B) if for any isometric representation ρ of G on B and any quasi-ρ-cocycle b, $b(N)$ is bounded. Equivalently, if for any quasi-action on B, some (or equivalently, any) N-orbit is bounded.

(*ii*) A group G is said to have *property* (FF_B) if $G\geqslant G$ has relative (FF_B).

This notion has its origin to *property* (TT) of Monod [Mon1]. He defined property (TT) as boundedness property of all quasi-cocycles with unitary coefficients. This property is identical to property ($\mathrm{FF}_{\mathcal{H}}$) in our definition. We use the terminology (FF_B) because this property is a quasification of (F_B), *not* of (T_B), and in general setting, (F_B) is not identical to (T_B) (we have seen that it is usually much stronger).

This is an obvious observation from Lemma 3.2.5.

Lemma 7.1.4. *If a Banach space B is superreflexive, then property* (FF_B) *implies property* (F_B)*.*

This implication may happen to be false in the case of that B is non-superreflexive.

7.1.2 Relation to bounded cohomology

Bounded cohomology is initiated around a book of B. E. Johnson [Joh]. By an inventional paper [Gro1] of Gromov, this theory has been expanded. In [Gro1], much attention has been paid on the trivial coefficient ($(1_G, \mathbb{R})$) case. Here we consider bounded cohomology with isometric Banach coefficient (we shall pay attention to the trivial coefficient case as well in Chapter 10). For details of bounded cohomology with Banach coefficients, we refer to [Mon1], [Mon2]. We give a definition of bounded cohomology, which is based on *inhomogeneous* standard cochain complex:

Definition 7.1.5. Let G be a group, and ρ be an isometric G-representation on a Banach space B. We define the (continuous) *bounded cohomology* $H^{\bullet}_{\mathrm{cb}}(G;\rho,B)$ in the following way:

(i) The *cochain* $C^n_{\mathrm{b}}(G;\rho,B)$ consists of all continuous *bounded* functions $c\colon G^{(n)} \to B$. Here we define $C^0_{\mathrm{b}}(G;\rho,B) := B$.

(ii) The *coboundary* map $\delta = \delta^n\colon C^n_{\mathrm{b}} \to C^{n+1}_{\mathrm{b}}$ is the restriction of the coboundary map in the ordinary cohomology theory on C^n_{b}: namely, we define δ above by the following formula:

$$\begin{aligned}\delta c(g_0,\ldots,g_n) := \rho(g_0)c(g_1,\ldots,g_n) + \sum_{i=0}^{n-1}(-1)^{i+1}c(g_0,\ldots,g_ig_{i+1},\ldots,g_n) \\ + (-1)^{n+1}c(g_0,\ldots,g_{n-1}).\end{aligned}$$

(iii) The *bounded cohomology* is defined as

$$H^n_{\mathrm{cb}}(G;\rho,B) := Z^n_{\mathrm{cb}}(G;\rho,B)/B^n_{\mathrm{cb}}(G;\rho,B).$$

Here $Z^n_{\mathrm{cb}}(G;\rho,B)$ denotes $\mathrm{Ker}\delta^n$ and $B^n_{\mathrm{cb}}(G;\rho,B)$ denotes $\mathrm{Im}\delta^{n-1}$.

If ρ is obvious in context, then we write $H^{\bullet}_{\mathrm{cb}}(G;\rho,B)$ shortly as $H^{\bullet}_{\mathrm{cb}}(G;B)$. Also if (ρ,B) is the trivial representation on reals (, namely, $(1_G,\mathbb{R})$,), then we write it as $H^{\bullet}_{\mathrm{cb}}(G,\mathbb{R})$, or just $H^{\bullet}_{\mathrm{cb}}(G)$.

Similar to the definition of group (1-)cohomology, we use the symbol $H^{\bullet}_{\mathrm{b}}(G;\rho,B)$ if G is discrete.

In below, we will explain relation between bounded cohomology and quasi-cocycles. Let (ρ,B) is an isometric representation of a group G, and consider the space of all quasi-ρ-cocycles. This space itself is ridiculously large because it contains *all* continuous map $G \to B$ with *bounded* range, and it contains all ρ-cocycles. In order to get a reasonable space, we need to *mod out* these maps.

Definition 7.1.6. Let G be a group and (ρ,B) be an isometric G-representation.

(*i*) Define $QH_{\mathrm{c}}(G;\rho,B)$ as the space of all quasi-ρ-cocycles.

(*ii*) Define $\widetilde{QH}_{\mathrm{c}}(G;\rho,B)$ as the following quotient space:

$$\begin{aligned}&\widetilde{QH}_{\mathrm{c}}(G;\rho,B)\\ :=&QH_{\mathrm{c}}(G;\rho,B)/(\{\rho\text{-cocycles}\}+\{\text{continuous maps } G\to B \text{ with bounded range}\}).\end{aligned}$$

We call $\widetilde{QH}_{\mathrm{c}}(G;\rho,B)$ the *actual space of quasi-ρ-cocycles.*

If ρ is obvious in context, then we write $QH_{\mathrm{c}}(G;\rho,B)$ shortly as $QH^{\bullet}_{\mathrm{c}}(G;B)$ and $\widetilde{QH}_{\mathrm{c}}(G;\rho,B)$ shortly as $\widetilde{QH}^{\bullet}_{\mathrm{c}}(G;B)$. Also if (ρ,B) is $(1_G,\mathbb{R})$, then we write them respectively as $QH_{\mathrm{c}}(G)$ and $\widetilde{QH}_{\mathrm{c}}(G)$.

We also omit the symbol c if G is discrete. We warn that the terminology *actual space of quasi-cocycles* and the symbols $QH_{\mathrm{c}}(G;\rho,B)$, $\widetilde{QH}_{\mathrm{c}}(G;\rho,B)$ are not standard. This QH is named after *Quasi-Homomorphisms* (see Chapter 10), and hence the symbols $QH_{\mathrm{c}}(G)$ and $\widetilde{QH}_{\mathrm{c}}(G)$ are standard.

Next, we define the *comparison map*.

Definition 7.1.7. Let G be a group and (ρ,B) be an isometric G-representation. Let $n\geq 1$. Then a natural homomorphism (between vector spaces)

$$\Psi^n_{\mathrm{cb}}\colon H^n_{\mathrm{cb}}(G;\rho,B)\to H^n_{\mathrm{c}}(G;\rho,B)$$

is induced by the natural inclusion of complexes. Here $H^n(G;\rho,B)$ denotes the ordinary group cohomology. This map is called the *comparison map* in degree n.

Again, we omit the symbol c if G is discrete.

We note that comparison map is *neither* injective *nor* surjective in general. The following lemma connects bounded cohomology in degree 2 and the actual space of quasi-cocycles. Note that if B is superreflexive, then $H^1_{\mathrm{cb}}(G;\rho,B)=0$ for any group G and isometric representation ρ on B (this amounts to Lemma 7.1.4, namely, any bounded cocycle on B is a coboundary). Therefore, the case of our main concern and high importance is dimension 2 case.

Lemma 7.1.8. *Let G be a group and (ρ,B) be an isometric G-representation. Then there is a natural isomorphism among vector spaces,*

$$\widetilde{QH}_{\mathrm{c}}(G;\rho,B)\cong \mathrm{Ker}\Psi^2_{\mathrm{cb}}.$$

Here $\mathrm{Ker}\Psi^2_{\mathrm{cb}}$ *denotes the comparison map in degree* 2

$$\Psi^2_{\mathrm{cb}}\colon H^2_{\mathrm{cb}}(G;\rho,B)\to H^2_{\mathrm{c}}(G;\rho,B).$$

Proof. Consider the following homomorphism among vector spaces:

$$QH_{\mathrm{c}}(G;\rho,B) \to \mathrm{Ker}\Psi^2_{\mathrm{cb}};\ b \mapsto [\delta b]_{\mathrm{cb}},$$

where $[\cdot]_{\mathrm{cb}}$ is the bounded cohomology class. Then it is a routine to check that it is surjective, and that the kernel of this map is the space of all maps of the form of "ρ-cocycles + maps with bounded range." Therefore, this homomorphism induces the isomorphism in the lemma above. □

Corollary 7.1.9. *Let G be a group and B be a Banach space. If G has* (FF_B), *then for any isometric G-representation ρ on B, $H^2_{\mathrm{cb}}(G;\rho,B)$ naturally injects into $H^2_{\mathrm{c}}(G;\rho,B)$.*

Some examples on property (TT)$(=(\mathrm{FF}_{\mathcal{H}}))$ will be examined in Section 7.2. Also, we will see some examples of quasi-cocycle for the case of $(\rho,B)=(1_G,\mathbb{R})$ in Subsection 10.2.2.

7.2 Property (TT) and some facts

Recall that *property* (TT) of Monod is identical to property $(\mathrm{FF}_{\mathcal{H}})$ in our definition, and it means that every quasi-cocycle with any unitary coefficient is bounded. The theorem of Monod–Shalom [MoSh1], [MoSh2] and Mineyev–Monod–Shalom [MMS] in below implies that property (TT) is strictly stronger than property (T).

Definition 7.2.1. (i) A discrete group is said to be *virtually abelian* if it contains an abelian subgroup of finite index (we regard $\{e\}$ as an abelian group as well).

(ii) A hyperbolic group (recall Definition 2.6.19) is said to be *non-elementary* if it is not virtually abelian.

It is known that hyperbolic groups cannot contain $\mathbb{Z}^2$ (see [Gro2]). Therefore a hyperbolic group is elementary if and only if it is finite or it is virtually $\mathbb{Z}$ (, namely, it contains $\mathbb{Z}$ as a finite index subgroup). All virtually abelian groups are amenable. On the other hand, it is known that any non-elementary hyperbolic group contains F_2, and hence is not amenable. We note that it is a theorem of Gromov–Cheeger that for amenble groups, bounded cohomology with reasonable coefficients always vanishes. In contrast, I. Mineyev [Min] shows that for non-elementary hyperbolic groups, comparison maps in degree≥ 2 with any reasonable coefficient are all surjective.

Theorem 7.2.2. ([MoSh1], [MoSh2], [MMS]) *Let G be a non-elementary hyperbolic group. Then $\widetilde{QH}(G;\ell^2(G))$ is infinite dimensional. Here the isometric G-representation on $\ell^2(G)$ is the left regular representation.*

Note that F_2 is hyperbolic, and also that $H^2(F_2;\ell^2(F_2)) = 0$ because it is a free group. Therefore, $H^2_{\mathrm{b}}(F_2;\ell^2(F_2))$ has infinite dimension, and the comparison map in degree 2 sends it to the zero-space. This example shows that sometimes behavior of bounded cohomology is pathological.

On the contrary, Burger–Monod [BuMo1], [BuMo2] show that for higher rank lattices, this sort of pathology does not occur, at least in degree 2. Recall our terminology *totally higher rank lattices* in Chapter 0.

Theorem 7.2.3. (*Burger–Monod* [BuMo1], [BuMo2]) *Any totally higher rank algebraic group and totally higher rank lattice have* (TT)*.*

Remark 7.2.4. They show this theorem first for totally higher rank algebraic group. In [BuMo1], they show (TT) passes to cocompact lattices. In [BuMo2], they deal with noncocompact cases. Note that (p-, and)2-induction of quasi-cocycles is available for (p-, and)2-integrable lattices in a similar manner to that in Subsection 3.4.2. However that even for cocompact case, there is a gap in just imitating the argument of heredity of (($\mathrm{F}_{\mathcal{L}_p}$), and) (T) to lattices. The problem lies in how to deduce boundedness of the original quasi-cocycle from the boundedness of the induced quasi-cocycle. For details, see [BuMo1], [BuMo2], and also a paper [Oza] of N. Ozawa.

However, as long as considering discrete case (, namely, the case of finite index subgroups), induction has no difficulties: the boundedness of the induced quasi-cocycle obviously implies the boundedness of the original quasi-cocycle. Therefore, it is easy to see that property (TT) passes to finite index subgroups.

Finally, we briefly state the strong negation of property (TT), which is weakening of the Haagerup property:

Definition 7.2.5. A group G is said to have the a-TT-*manablity* (or the *ha-haagerup property*) if there exists a proper quasi-cocycle on G with some unitary coefficient.

The following result is implicitly stated in [Min]. See also [Oza].

Theorem 7.2.6. (*Mineyev* [Min]) *Every hyperbolic group is* a-TT-*menable.*

Chapter 8

Property (FF_B)/T

In this chapter, we introduce a notion of *property* (FF_B)/T, which is a priori a weaker version of property (FF_B). This property means the following: any quasi-cocycle with any isometric coefficient on B is bounded *modulo trivial part*. This notion is needed, at the present, in deducing the property for universal lattices. More precisely, we deduce property $(\mathrm{FF}_{\mathcal{L}_p})$/T for all p and property $(\mathrm{FF}_{[\mathcal{H}]})$/T for universal lattices of degree at least 4. These are one of the main results in the paper [Mim1] of the author, and stated as Theorem B in this thesis. However at the moment, it seems to be open to determine whether $(\mathrm{FF}_{\mathcal{L}_p})$ (or $(\mathrm{FF}_{[\mathcal{H}]})$) is satisfied for universal lattices. It is worth noting that even $(\mathrm{FF}_{\mathcal{H}})$/T for universal lattices of degree at least 4 is a *new* result.

Also, we introduce a shortcut to deduce (FF_B)/T for universal lattices (with degree≥ 4) which does *not* appeal to Shalom's machinery. This implies in particular that there is a possibility to apply this shortcut to the case of that the class B of Banach spaces (or, even a single Banach space) is *not* stable under ultraproducts. We note that, however, this shortcut is deeply based on the fact that universal lattices have (T), and hence philosophically this is not free from Shalom's machinery. Nevertheless, this shortcut applies to the case of p-Schatten class spaces C_p. This shows property (FF_{C_p})/T for universal lattices (with degree≥ 4), and this result can be seen as a generalization of Theorem B in (some) noncommutative setting. This is one of the main results in the paper [Mim3] of the author, and is stated as Theorem C in this thesis.

Finally, we study more on property $(\mathrm{FF}_{\mathcal{H}})$/T, which is also called *property* (TT)/T. By employing a theorem of Ozawa, we prove universal lattice of degree 3 in fact enjoys this property. As we will see in Chapter 11, (TT)/T has intriguing application to homomorphism rigidity into mapping class groups (of surfaces) and into automorphism groups of free groups.

Works in this chaper are based on the papers [Mim1], [Mim2]. On the fixed point property (F_{C_p}) on the p-Schatten classes C_p, it isalso inspired by a previous work of Puschnigg [Pus]. Also, we refer to a book [PiXu] of Pisier–Xu for comprehensive treatment for noncommutative spaces. For property (TT**T**) of **T**aka Ozawa, see the original paper [Oza].

8.1 A shortcut without ultraproducts

In this section, we see a shortcut of the proof of prperty $(\mathrm{F}_{\mathcal{L}_p})$ for universal lattices of degree≥ 4, which is based on the Shalom–Vaserstein's theorem of property (T) for this group. Also, by employing this shortcut, we show property $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ for this group, which is stronger than property $(\mathrm{F}_{\mathcal{L}_p})$. We introduce this notion of property $(\mathrm{FF}_B)/\mathrm{T}$ in this section.

8.1.1 Definition of property $(\mathrm{FF}_B)/\mathrm{T}$

Let G be a group, and ρ is an isometric G-representation on a Banach space B. Recall if B is superreflexive, then there is a canonical decomposition as $\rho(G)$-spaces:

$$B = B^{\rho(G)} \oplus B_{\rho(G)}{}' =: B_0 \oplus B_1$$

(see Proposition 3.1.12). Let $b\colon G \to B$ be a quasi-ρ-cocycle. Then for the decomposition

$$b = b_0 + b_1,$$

associated with the decomposition of B, b_0 and b_1 are also quasi-ρ-cocycles. This follows from $\rho(G)$-invariance of spaces and a norm estimate in Proposition 3.1.12.

In [Mim1], the author has come up with the following technical notion:

Definition 8.1.1. ([Mim1]) Let B be a Banach space. A group G is said to have *property* $(\mathrm{FF}_B)/\mathrm{T}$ (, which is called "*property* (FF_B) *modulo trivial part*",) if for any isometric G-representation ρ on B and any quasi-ρ-cocycle b, $b'(G)$ is bounded. Here $b'\colon \Gamma \to B/B^{\rho(G)}$ is the natural quasi-cocycle constructed from the projection of b associated the canonical quotient map $B \twoheadrightarrow B/B^{\rho(G)}$. If B is superreflexive, then this definition is equivalent to the following condition: for any isometric representation ρ of Γ on B and any quasi-ρ-cocycle b, $b_1(G)$ is bounded. Here we decompose b as $b_0 + b_1$ such that b_0 takes values in $B_0 = B^{\rho(\Gamma)}$ and b_1 takes values in $B_1 = B'_{\rho(\Gamma)}$.

In particular, the following holds: if G has $(\mathrm{FF}_B)/\mathrm{T}$, then for any isometric G-representation ρ on B *satisfying* $\rho \not\supseteq 1_G$, every quasi-ρ-cocycle is bounded. Property $(\mathrm{FF}_B)/\mathrm{T}$ is a priori a weaker notion than that of property (FF_B), but at the moment we have no example which distinguishes these. As we will see in below, universal lattices of degree≥ 4 have $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ but it is not known whether they enjoy $(\mathrm{FF}_{\mathcal{L}_p})$. Therefore these groups are possible candidates. The following is easy to see:

Lemma 8.1.2. *Let $\mathcal{C}$ be $\mathcal{L}_p$ or $[\mathcal{H}]$. Then property* $(\mathrm{FF}_{\mathcal{C}})/\mathrm{T}$ *implies property* $(\mathrm{F}_{\mathcal{C}})$.

Proof. Note that spaces in such $\mathcal{C}$ are superreflexive, and hence that all bounded cocycle with isometric coefficients on these are coboundary. Therefore for the proof, it suffices to show the abelianization $H = G/\overline{[G,G]}$ is compact. Suppose not. Then since H is abelian, H has unbounded λ_H-cocycle on $L^p(H)$ (it can be constructed from the fact that $\lambda_H \succeq 1_H$, where λ_H denotes the left-regular representation). Here if $\mathcal{C} = [\mathcal{H}]$, then set $p = 2$. Pull-back this cocycle to a cocycle of G. Also, since H is non-compact, the pull-back of the representation does not contain 1_G. This contradicts $(\mathrm{FF}_{\mathcal{C}})$ for G. $\square$

This proof shows Lemma 8.1.2 can be true in much more general situations. Also, the proposition below follows from Lemma 3.2.5, and the proof of Theorem 3.2.9:

Proposition 8.1.3. *Let B be a superreflexive Banach space. Then property $(\mathrm{FF}_B)/\mathrm{T}$ implies property (T_B).*

Next, we see some permanence properties.

Proposition 8.1.4. *Let G be a group, and B be a Banach space. Suppose G has $(\mathrm{FF}_B)/\mathrm{T}$.*

(i) Suppose there is a continuous homomorphism $G \to H$ to a group H with dense range. Then H has $(\mathrm{FF}_B)/\mathrm{T}$.

(ii) Suppose G is discrete. Let $\Gamma \leqslant G$ be a finite index subgroup. Let $p \in (1,\infty)$, and consider p-induction $E = \ell^p(G/\Gamma, B)$ (recall Subsection 3.4.2). Then Γ has $(\mathrm{FF}_E)/\mathrm{T}$.

In particular, $(\mathrm{FF}_B)/\mathrm{T}$ passes to group quotients; and property $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ and property $(\mathrm{FF}_{[\mathcal{H}]})/\mathrm{T}$ pass to finite index subgroups.

Proof. Item (i) is obvious. For item (ii), recall Remark 7.2.4 and also check on the invariant spaces. Induction shows the assertions. $\square$

With more effort, it might be possible to obtain a similar result to cocompact lattices (and even more to p-integrable lattices under some condition). However we do not proceed in this direction, because our groups of main concern are discrete groups.

We warn that there is no reason to expect that $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$, for instance, is stable under extensions. In fact, there is no reason to expect this even for direct product (the reason is that $(G_1 \times G_2)$-invariant space is in general much smaller than each G_i-invariant spaces).

8.1.2 Property $(\mathrm{FF}_B)/\mathrm{T}$ for universal lattices

Here we introduce a shortcut in proving (F_B) or strongly, $(\mathrm{FF}_B)/\mathrm{T}$ *without* taking ultraproducts. Firstly we see the following generalization of Theorem 6.2.1, which can be directly obtained by imitating the original proof:

Theorem 8.1.5. (*Theorem* 6.4 [Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots, x_k]$. *Suppose B is any superreflexive Banach space. Then, if the pair* $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ *has relative property* (T_B), *then the pair* $\mathrm{SL}_3(A) \ltimes A^3 \geqslant A^3$ *has relative property* (FF_B).

The following is the shortcut theorem. It is a generalization of Proposition 6.6 in [Mim1], and plays a key role to establishing (TT)/T for symplectic universal lattices (Section 9.3). For the proof, we extend our definition of relative (FF_B) for (closed) *subsets*.

Definition 8.1.6. Let G be a group and B be a Banach space. Let $Q \subseteq G$ be a closed *subset*.

(*i*) A pair $G \supseteq Q$ is said to have *relative* (FF_B) if for any isometric G-representation ρ on B, every ρ-cocycle is bounded on Q.

(*ii*) A pair $G \supseteq Q$ is said to have *relative* $(\mathrm{FF}_B)/\mathrm{T}$ if for any isometric G-representation ρ on B and any quasi-ρ-cocycle b, $b'(S)$ is bounded, where $b' \colon \Gamma \to B/B^{\rho(G)}$ is the natural quasi-cocycle constructed from the projection of b associated the canonical quotient map $B \twoheadrightarrow B/B^{\rho(G)}$. If B is superreflexive, then this definition is equivalent to the following condition: for any isometric representation ρ of Γ on B and any quasi-ρ-cocycle b, $b_1(S)$ is bounded. Here we decompose b as $b_0 + b_1$ such that b_0 takes values in $B_0 = B^{\rho(\Gamma)}$ and b_1 takes values in $B_1 = B'_{\rho(\Gamma)}$.

In this definition, if Q is a subgroup of G, then we prefer to use the symbol "$G \geqslant Q$ has relative $(\mathrm{FF}_B)/\mathrm{T}$."

We note in the definition of relative $(\mathrm{FF}_B)/\mathrm{T}$, "T" always means the $\rho(G)$-trivial part, even if we consider subsets or subgroups.

Theorem 8.1.7. ([Mim4]) *Let G be a group and B be a Banach space. Suppose the group G and a pair (H, Q), where $H \leqslant G$ is a subgroup and $Q \subseteq G$ is a closed* subset, *satisfies the following five conditions:*

(*i*) *The group G has* (T_B).

(*ii*) *The set Q generates G.*

(*iii*) *The set Q is invariant under the conjugation of elements in H. Namely, for any $h \in H$, $hQh^{-1} \subseteq Q$.*

(*iv*) *The pair* $G \supseteq Q$ *has relative* $(\mathrm{FF}_B)/\mathrm{T}$.

(*v*) *The group* G *is boundedly generated by* Q *and* H.

Then G *has property* $(\mathrm{FF}_B)/\mathrm{T}$.

Even without condition (*v*)*, we have that* $G \geqslant H$ *has relative* $(\mathrm{FF}_B)/\mathrm{T}$. *Namely, if* (G, H, Q) $(H \leqslant G,\ Q \subseteq G)$ *satisfies conditions* (*i*), (*ii*), (*iii*) *and* (*iv*)*, then* $G \geqslant H$ *has relative* $(\mathrm{FF}_B)/\mathrm{T}$.

Proof. We will show the latter assertion. Then the former assertion is confirmed by the bounded generation (condition (*v*)). Let ρ be an isometric G-representation, and b be a quasi-ρ-cocycle. Consider the representation $\rho' \colon G \to O(B/B^{\rho(G)})$ and the ρ'-cocycle $b' \colon G \to B/B^{\rho(G)}$, which are naturally determined by the quotient map $B \twoheadrightarrow B/B^{\rho(G)}$. Since b' is a quasi-cocycle, we set

$$C_1 := \sup_{g_1, g_2 \in G} \|b'(g_1 g_2) - b'(g_1) - \rho(g_1) b'(g_2)\| < \infty.$$

Also by condition (*iv*), b' is bounded on Q. Set

$$C_2 := \sup_{q \in Q} \|b'(q)\| < \infty.$$

Set $C = \max\{C_1, C_2\} < \infty$.

Take any $h \in H$ and any $q \in Q$. Then we have the following inequalities:

$$\begin{aligned}
\|\rho'(q)b'(h) - b'(h)\| &\leq \|b'(qh) - b'(q) - b'(h)\| + C \\
&\leq \|b'(qh) - b'(h)\| + 2C \\
&= \|b'(h(h^{-1}qh)) - b'(h)\| + 2C \\
&\leq \|b'(h) + \rho'(h)b'(h^{-1}qh) - b'(h)\| + 3C \\
&\leq \|b'(h^{-1}qh)\| + 3C \\
&\leq 4C.
\end{aligned}$$

Here in the last line we use condition (*iii*) (and condition (*iv*)). Note that the last dominating term $4C$ is *independent* of the choices of $h \in H$ and $q \in Q$.

Now suppose $b'(H)$ is not bounded. Then by the inequalities above, this means for any ϵ, ρ' admits a (Q, ϵ)-invariant vector. By condition (*ii*), this means $\rho' \succeq 1_G$ (see the proof of Lemma 2.1.3). However this contradicts condition (*i*). $\square$

Also, the proof shows that there is an (F_B)-version of this theorem. Note that unless B is superreflexive, boundedness of cocycle does not necessarily imply the cocycle being a coboundary (there are some examples of Banach spaces *beyond* superreflexive ones with respect to which the implication above is true: for instance, any separable reflexive Banach space is known to satisfy the implication above).

Theorem 8.1.8. ([Mim4]) *Let G be a group and B be a Banach space. Suppose the group G and a pair (H,Q) where $H \leqslant G$ be a subgroup and $Q \subseteq G$ be a closed* subset *satisfies the following five conditions:*

(i) *The group G has* (T_B).

(ii) *The set Q generates G.*

(iii) *The set Q is invariant under the conjugation of elements in H.*

(iv) *The pair $G \supseteq Q$ has relative* (F_B).

(v) *The group G is boundedly generated by Q and H.*

Suppose in addition B is superreflexive and G has compact abelianization. Then G has property (F_B).

Even without condition (v), we have that $G \geqslant H$ has relative (F_B). *Namely, if (G,H,Q) $(H \leqslant G,\ Q \subseteq G)$ satisfies conditions $(i),(ii),(iii)$ and (iv); B is superreflexive; and G has compact abelianization, then $G \geqslant H$ has relative* (F_B).

Remark 8.1.9. Here are remarks on Theorem 8.1.7 and Theorem 8.1.8.

(i) The proof of Theorem 8.1.7 is very simple, as we have seen in above. We note that this can*not* apply to establishing property (T) because a property having form (T_B) *itself* is contained in the conditions of the theorem (condition (i)). Nevertheless, Theorem 8.1.7 is very powerful: once property (T_B) is confirmed (by some other means), then existence of such pair (H,Q) implies $(\mathrm{FF}_B)/\mathrm{T}$; and thus in particular (F_B) if B is superreflexive and G has finite abelianization.

In particular, recall that property (T) implies $(\mathrm{T}_{\mathcal{L}_p})$ but that $(\mathrm{F}_{\mathcal{L}_p})$ $(p \gg 2)$ is strictly stronger than (T). However, Theorem 8.1.7 implies that if a group G has a pair (H,Q) in the theorem, then (T) for G is *sufficient* to deduce $(\mathrm{F}_{\mathcal{L}_p})$ and even $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$. We note that condition (iv) contains information on a(relative) $(\mathrm{FF}_{\mathcal{L}_p})$ and hence Theorem 8.1.7 is not free from conditions concerning (FF_B)-side. However, in general, it is much easier to find a subset (or a subgroup) with relative $(\mathrm{FF}_B)/\mathrm{T}$ (or relative (F_B)), than to verify the whole group has $(\mathrm{FF}_B)/\mathrm{T}$ (or (F_B)).

(ii) One example of such (G,H,Q) is the following: let G is a finitely generated group and a triple of subgroups in G (H,N_1,N_2) has the Shalom property for (F_B) (see Definition 5.3.1). Then if G has (T_B), then $(H,Q) = (H, N_1 \cup N_2)$ satisfies the conditions (i)-(v) in Theorem 8.1.8. If moreover for each $i = 1,2$ $G \geqslant N_i$ has relative $(\mathrm{FF}_B)/\mathrm{T}$, then $(H,Q) = (H, N_1 \cup N_2)$ satisfies the conditions (i)-(v) in Theorem 8.1.7. Note that in Shalom's machinery, it is essential that Q in above is the union of *two subgroups*, and H normalizes *each* of them.

Therefore, merits of introducing Theorem 8.1.7 are the following, in comparison to considering Shalom's machinery:

(A) One can consider *subsets* $Q \subseteq G$, free from subgroup structures.

(B) The stability of a Banach space (or a class) B under ultraproducts is *not* needed.

(C) One can apply to deduce not only (F_B), but also $(\mathrm{FF}_B)/\mathrm{T}$ (with approriate assumption).

On the other hand, the following point is the obstruction in applying Theorem 8.1.7 in general cases:

(T) One needs to have property (T_B) for G in advance.

Note that Shalom's machinery is a deep theorem, but the proof of Theorem 8.1.7 is quite easy: because the main point of this theorem is to reduce $(\mathrm{FF}_B)/\mathrm{T}$ to (T_B), and (T_B) is in advance assumed. From this point of view, we can consider property (T_B) is the *origin* of these rigidities on B, although in general (T_B) is much weaker than (F_B) or $(\mathrm{FF}_B)/\mathrm{T}$ (if B is superreflexive). Therefore, Shalom's machinery is significant because this machinery provides us with a method to deduce ((F_B) and hence) (T_B) from relative (F_B).

(*iii*) Item (B) in the remark above is essential in proving Theorem C. Item (A) is essential in the proof of Theorem D. These theorems are based on Theorem 8.1.7.

(*iv*) The proof of Theorem 8.1.7 indicates that in fact even for H, we only need the assumption $h^{-1}Qh \subseteq Q$ for all $h \in H$, and then H can be taken as a *subset* of G. We have stated in these theorems that H is assumed to be a subgroup simply because in practically use that case is a main case.

(*v*) As we have mentioned, the proof of Theorem 8.1.7 is based on property (T_B), namely, the non-existence of almost invariant vectors for induced representations on the quotient Banach space *modulo the space of invariant vectors*. Therefore, the situation is completely different for quasi-cocycles with *trivial* coefficients (this is the reason why we introduce $(\mathrm{FF}_B)/\mathrm{T}$). The study of cocycles with trivial coefficient is not difficult: it only depends on the abelianization of the group. However, the study of *quasi*-cocycles with trivial coefficient is much involved. A quasi-cocycles with trivial real coefficient $(1_G, \mathbb{R})$ is called a *quasi-homomorphism*, and this object has intensively studied by many mathematicians. We will see some part of theories on quasi-homomorphisms in Chapter 10.

We note that it seems open to determine whether all quasi-homomorphisms on universal lattices are bounded. See Subsection 10.5.3 for details.

8.1.3 Proof of Theorem B (*i*)

Here we show item (*i*) of Theorem B. We restate it of the extracted form:

Theorem 8.1.10. (*Theorem* 1.5 [Mim1]) *Let* $A = \mathbb{Z}[x_1, \ldots, x_k]$. *Then if* $m \geq 4$, *then for any p, universal lattice* $G = \mathrm{SL}_m(A)$ *has property* $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ *and property* $(\mathrm{FF}_{[\mathcal{H}]})/\mathrm{T}$.

In particular for any ρ, *an isometric* G*-representation on an* L^p *space or a uniformly bounded* G*-representation on a Hilbert space which satisfies* $\rho \not\supseteq 1_G$, *then the comparison map in degree* 2 $H_b^2(G;\rho) \to H^2(G;\rho)$ *is injective.*

Proof. Let (H, N_1, N_2) be the triples of subgroups of G which is defined as in the proof of Theorem 6.3.1 (see Section 6.3). Set $(H, Q) = (H, N_1 \cup N_2)$. Then as we see in item (*ii*) Remark 8.1.9, this pair of a subgroup and a subset of G satisfies conditions (*ii*), (*iii*) and (*v*) in Theorem 8.1.7. We set $B = \mathcal{L}_p$ or $B = [\mathcal{H}]$ in Theorem 8.1.7.

By Theorem 5.4.1 and Theorem 3.3.7, G has $(\mathrm{T}_{\mathcal{L}_p})$. By Corollary 6.3.2, G also has $(\mathrm{T}_{[\mathcal{H}]})$. Hence G fulfills condition (*i*). Finally, by combining Theorem 6.1.1 and Theorem 8.1.5, we have that (G, Q) enjoys condition (*iv*). Therefore Theorem 8.1.7 applies, and we have accomplished the proof of Theorem 8.1.10. $\square$

By examining the proof above carefully, we obtain the following criterion for a Banach space (or a class of Banach spaces) that $G = \mathrm{SL}_{m\geq 4}(\mathbb{Z}[x_1, \ldots, x_k])$ has $(\mathrm{FF}_B)/\mathrm{T}$. This is one of the main results in the paper [Mim3] of the author. Here we need one definition.

Definition 8.1.11. ([Mim3]) Let E_1 be a Banach space (or a class of Banach spaces). Let E_2 be a a Banach space (or a class of Banach spaces). Then we say *relative property* (T_{E_1}) *implies relative property* (T_{E_2}) if the following holds true: for a pair $G \trianglerighteq N$ of a group and a (possibly non-proper) *normal* subgroup, whenever $G \trianglerighteq N$ has relative (T_{E_1}), it has (T_{E_2}).

We say *relative property* (T) *implies relative property* (T_{E_2}) if for a pair $G \trianglerighteq N$ of a group and a (possibly non proper) *normal* subgroup, whenever $G \trianglerighteq N$ has relative (T), it has (T_{E_2}).

Note that in above definition, if relative (T_{E_1}) implies relative (T_{E_2}) and if a group G has (T_{E_1}), then G also has (T_{E_2}) (consider $G \trianglerighteq G$). Theorem 3.3.10 exactly means that relative (T) implies relative $(\mathrm{T}_{\mathcal{L}_p})$ in the sense above.

Theorem 8.1.12. ([Mim3]) *Let k is a non-negative integer and $A = \mathbb{Z}[x_1, \ldots, x_k]$. Let B be a class of superreflexive Banach spaces (or a superreflexive Banach space). Suppose B satisfies either of the following conditions:*

(*i*) *the class B is stable under ultraproducts, and the pair $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ has relative (T_B);*

(*ii*) *relative $(\mathrm{T}_{[\mathcal{H}]})$ implies relative (T_B), in the sense in Definition 8.1.11.*

Then for any $m \geq 4$, universal lattices $\mathrm{SL}_m(A)$ has $(\mathrm{FF}_B)/\mathrm{T}$. In particular, $\mathrm{SL}_m(A)$ has (F_B).

In particular, if relative (T) implies relative (T_B), then for any $m \geq 4$, $\mathrm{SL}_m(A)$ has $(\mathrm{FF}_B)/\mathrm{T}$ and (F_B).

This observation is the key to proving Theorem C.

Remark 8.1.13. Here is a remark for condition (*i*) (Shalom's machinery) of the theorem above.

It is a problem of high interest to determine whether totally higher rank lattices and universal lattices have property $(\mathrm{F}_{\mathcal{B}_{\mathrm{uc}}})$ for $\mathcal{B}_{\mathrm{uc}}$ being the class of all uniformly convex Banach spaces. One of the main motivations for studying this problem is this relates to uniform (non-)embeddability of expander graphs constructed by the group into a uniformly convex Banach space (see Section 2.7. We also refer to the original argument of Gromov [Gro6] for existence of groups which are not uniformly embeddabile into Hilbert spaces. For details of this argument, also see a paper [ArDe] of Arzhantseva–Delzant). That relates the coarse geometric Novikov conjecture [KaYu] and (possible counter example of the surjectivity-side of) the Baum–Connes conjecture with coefficient (we refer to [HLS] in the connection of this). There is a breakthrough by V. Lafforgue [Laf1], [Laf2], and his results imply $\mathrm{SL}_{n\geq 3}(\mathbb{K})$ ($\mathbb{K}$ is a non-archimedean local field) and cocompact lattices therein have this property (in fact, he shown property (F_B) for these groups where B is the class of all Banach spaces with type> 1. For details, see his original papers [Laf1], [Laf2]). For archimedean local field cases or noncocompact lattice cases, however, it seems no result is known for this problem.

In [BFGM], Bader–Furman–Gelander–Monod observed that for higher rank groups and lattices, in order to verify property $(\mathrm{F}_{\mathcal{B}_{\mathrm{uc}}})$ it suffices to show that pairs $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 \trianglerighteq \mathbb{R}^2$ and $\mathrm{SL}_2(\mathbb{R}) \ltimes S^{2*}(\mathbb{R}^2) \trianglerighteq S^{2*}(\mathbb{R}^2)$ have relative property $(\mathrm{T}_{\mathcal{B}_{\mathrm{uc}}})$ (here the latter pair relates to symplectic groups, see Subsection 2.6.3). Thanks to Theorem 8.1.12, we have the following analogue of this observation for universal lattices:"*if $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$ has relative property $(\mathrm{T}_{\mathcal{B}_{\mathrm{uc}}})$, then $\mathrm{SL}_{m\geq 4}(A)$ has property $(\mathrm{F}_{\mathcal{B}_{\mathrm{uc}}})$*." This is because the family of uniformly convex Banach spaces *with uniform lower bounds for modulus of convexity* is stable under ultraproducts (see Theorem 5.1.11), and hence a similar argument to one in the proof of Theorem 6.3.1 for the case of $\mathcal{C} = [\mathcal{H}]$ applies.

8.2 The case of p-Schatten spaces C_p

In this section, we prove Theorem C. This is an extension of a work of Puschnigg [Pus].

8.2.1 Basics on noncommutative L^p spaces

Firstly, we state some basic facts on noncommutative L^p spaces. We refer to [PiXu] for details. Since the p-Schatten class space is of our interest, we state in this case.

Let $\mathfrak{H}$ be a separable Hilbert space. Then there exists a countable orthonormal basis $(\xi_n)_{n\in\mathbb{N}}$, and we fix it. For a *positive* operator $A \in \mathbb{B}(\mathcal{H})$, we define the *trace* of A by the following formula:

$$\mathrm{Tr}(A) := \sum_{n\in\mathbb{N}} \langle A\xi_i | \xi_i\rangle \in [0,\infty].$$

Note that this value is independent of the choices of orthonormal basis $(\xi_n)_n$.

Set the following subspace of $\mathbb{B}(\mathfrak{H})$:

$$C_1 := \{T \in \mathbb{B}(\mathfrak{H}) : \mathrm{Tr}(|T|) < \infty\}.$$

Here $|T| := (T^*T)^{1/2}$ denotes the absolute value (operator) of T. For $T \in C_1$, $\mathrm{Tr}(T) := \sum_{n\in\mathbb{N}}\langle T\xi_i|\xi_i\rangle \in \mathbb{C}$ is well-defined, and is independent of the choices of $(\xi_n)_n$. The space C_1 is closed with respect to the norm $\|T\|_1 := \mathrm{Tr}(|T|)$. For $p \in (1,\infty)$, similarly we define the space

$$C_p := \{T \in \mathbb{B}(\mathfrak{H}) : \mathrm{Tr}(|T|^p) < \infty\},$$

which is closed with respect to the norm

$$\|T\|_p := (\mathrm{Tr}(|T|^p))^{1/p}.$$

We call $(C_p, \|\cdot\|_p)$ the *space of p-Schatten class operators.*

Note that C_2 is in fact a Hilbert space, equipped with the following inner product:

$$\langle T|S\rangle := \mathrm{Tr}(S^*T).$$

This space C_2 is also called the space of *Hilbert–Schmidt operators* (we note that noncommutative L^p spaces are constructed even from type III von-Neumann algebras with normal a semi-finite weight (the construction is due to U. Haagerup and H. Kosaki). See Section 3 in [PiXu]). For duality, the following holds:

Theorem 8.2.1. *Let $p, p' \in (1,\infty)$ with $p^{-1} + p'^{-1} = 1$. Then*

$$(C_p)^* \cong C_{p'}$$

The following inequality is called the *Clarkson type inequality.* See Section 5 in [PiXu].

Theorem 8.2.2. (*Clarkson type inequality*) *Let* p, p' *in* $(1, \infty)$ *with* $p^{-1} + p'^{-1} = 1$. *Then the following hold:*

(*i*) *If* $p \leq 2$, *then for any* $S, T \in C_p$,

$$\left[\frac{1}{2}(\|S+T\|_p^{p'} + \|S-T\|_p^{p'})\right]^{1/p'} \leq (\|S\|_p^p + \|T\|_p^p)^{1/p}.$$

(*ii*) *If* $p \geq 2$, *then for any* $S, T \in C_p$,

$$\left[\frac{1}{2}(\|S+T\|_p^{p} + \|S-T\|_p^{p})\right]^{1/p} \leq (\|S\|_p^{p'} + \|T\|_p^{p'})^{1/p'}.$$

.

By this theorem and the duality in above, it is straightforward that C_p is uniformly convex and uniformly smooth:

Corollary 8.2.3. *For* $p \in (1, \infty)$, C_p *is ucus.*

Finally, we state the following inequality of T. Ando:

Theorem 8.2.4. (*Ando* [Ando]) *Let* $0 < \alpha < 1$ *and* $p \in (1, \infty)$. *Then for any positive operators* A, B *in* C_p, *there is an inequality:*

$$\|A^\alpha - B^\alpha\|_r^r \leq \|A - B\|_p^p.$$

Here $r = p/\alpha$.

We note that all of these three theorems are extended to general noncommutative L^p setting (the extension of last one is due to Kosaki).

Remark 8.2.5. (*i*) Although there are many similarities between noncommutative L^p spaces and commutative ones, it is known that any (infinite dimensional) noncommutative L^p spaces can*not* be isometrically embedded into commutative one, whenever $p \neq 2$ (for instance, see a book [HRS] of Haagerup–Rosenthal–Sukochev). In the view of Remark 3.3.3, this implies the kernel

$$C_p \times C_p \to \mathbb{R}\colon (S, T) \mapsto \|S - T\|^p$$

is *not* conditionally positive definite if $p \neq 2$, even when $p \in (1, 2)$.

(*ii*) The space C_p is *not* stable under ultraproducts because C_p itself is separable (see Example 5.1.7). In fact, even if we consider the class of all noncommutative L^p spaces (for fixed p) associated with normal semifinite trace on type I or II von Neumann algebras, it is *not* stable. If we consider ultraproducts, then we have to deal with the case of type III von Neumann algebras. See [Ray] for details.

8.2.2 Proof of Theorem C

We restate Theorem C in below, which is a main theorem in the paper [Mim3] of the author. Recall from Chapter 0 that we always assume p is in $(1, \infty)$.

Theorem 8.2.6. ([Mim3]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ ($k \in \mathbb{N}$) and $m \geq 4$. Take any p. Then the following hold true:*

(i) The universal lattice $G = \mathrm{SL}_m(A)$ has $(\mathrm{FF}_{C_p})/\mathrm{T}$. In particular, G has (F_{C_p}).

(ii) Any finite index subgroup Γ of $\mathrm{SL}_m(A)$ has $(\mathrm{FF}_{C_p})/\mathrm{T}$. In particular, Γ has (F_{C_p}).

(iii) Any totally higher algebraic rank group and totally higher rank lattice have (F_{C_p}).

Firstly, we will prove item (*i*) in below. Recall that the criterion in Theorem 8.1.12 show there are two ways in establishing $(\mathrm{FF}_B)/\mathrm{T}$: either to use Shalom's machinery (consider ultraproducts); or verify (for instance) relative (T) implies relative (T_B) in the sense in Definition 8.1.11. Consider the case of C_p. By item (*ii*) of Remark 8.2.5, the former way does *not* work. Hence we aim to show that relative (T) implies relative (T_{C_p}), as in the case of commutative L^p spaces. To sum up, our main goal is the following theorem:

Theorem 8.2.7. ([Mim3]) *Relative* (T) *implies relative* (T_{C_p})*, in the sense in Definition* 8.1.11.

We note that item (*iii*) of Theorem 8.2.7 is essentially obtained by Puschnigg [Pus]. More precisely, he has proven the following theorem.

Theorem 8.2.8. (*Theorem* 5.8, *Corollary* 5.10 [Pus]) *Let $p \in (1, \infty)$ and $\mathfrak{H}$ be a separable Hilbert space. Denote by C_p the space of p-Schatten class operators.*

(i) Let G be a totally higher rank algebraic group. Let π be a unitary G-representation on $\mathfrak{H}$, and let ρ be the isometric G-representation on C_p induced by π, namely, for $g \in G$ and $T \in C_p$ define

$$\rho(g)T = \pi(g)T\pi(g)^{-1}.$$

Then for any π the following holds:

$$H^1_{\mathrm{c}}(G; \rho, C_p) = 0.$$

(ii) Let Γ be a totally higher rank lattice. Let π be a unitary Γ-representation on $\mathfrak{H}$, and let ρ be the isometric Γ-representation on C_p induced by π. Then for any π the following holds:

$$H^1(\Gamma; \rho, C_p) = 0.$$

Our proof has its origin in his proof of Theorem 8.2.8, and we will explain his strategy. Recall that in the case of commutative L^p spaces on σ-finite measure, there are two ways in proving Theorem 3.3.10:

(1) utilize conditionally negative definite kernel $\|\xi - \eta\|^p$;

(2) combine the Mazur map and the Banach–Lamperti theorem.

By item (i) of Remark 8.2.5, strategy (1) does *not* work for the case of C_p. The strategy of Puschnigg [Pus] is to construct an analogue of the Mazur map between (unit spheres of) C_p's. We note that in [Pus], he considered the case of that isometric representation is induced by unitary representations on $\mathfrak{H}$. This means, in his case, an analogue of the Banach–Lamperti theorem is not needed.

We explain his construction of the noncommutative analogue of the Mazur map. Recall that for any $T \in \mathbb{B}(\mathfrak{H})$, there is a *polar decomposition*:

$$T = U|T|.$$

Here $|T| = (T^*T)^{1/2}$ is the canonical positive operator, and U is a unitary operator (in general, we have many choices for U).

Definition 8.2.9. (Puschnigg [Pus]) Let $p, q \in (1, \infty)$. Define *a noncommutative Mazur map* by the following formula:

$$M_{p.q} \colon S(C_p) \to S(C_q);\ T \mapsto U|T|^{p/q}.$$

Here $T = U|T|$ is a polar decomposition of T.

Note that although there are many choices for unitary U, $U|T|^{p/q}$ defines the same operator. Therefore the definition above is well-defined.

Puschinigg deduced uniform continuity of the noncommutative Mazur map as in Definition 8.2.9 from Theorem 8.2.4:

Theorem 8.2.10. (*Corollary* 5.6 [Pus]) *For any* $p, q \in (1, \infty)$*, the noncommutative Mazur map* $M_{p,q} \colon S(C_p) \to S(C_q)$ *is uniformly continuous.*

For the proof, we refer to the original paper [Pus].

Proof. (Theorem 8.2.7) We need an analogue of the Banach–Lamperti theorem, namely, a classification of linear isometries on C_p $(p \neq 2)$. This is obtained by J. Arazy [Ara]. Since our Hilbert space $\mathfrak{H}$ is separable, by choosing a (countable) orthogonal normal basis $(\xi_n)_n$, we can identify $\mathfrak{H}$ with a square integrable sequence space $\ell^2 = \ell^2(\mathbb{N})$. Through this identification, we can consider the *transpose map* (associated with $(\xi_n)_n$); $S \mapsto {}^tS$ on $\mathbb{B}(\mathfrak{H}) \cong \mathbb{B}(\ell^2)$. Note that transpose maps are *not* canonical: it depends on the choice of $(\xi_n)_n$. However, it is easy to see the following:

Lemma 8.2.11. *Stick to the setting in above. Let $S \mapsto {}^tS$ be the transpose map on $\mathbb{B}(\ell^2)$.*

(*i*) *The transpose map is linear.*

(*ii*) *The transpose map is compatible with the adjoint operation. Namely, for any $T \in \mathbb{B}(\ell^2)$,*

$$({}^tT)^* = {}^t(T^*).$$

(*iii*) *For any $S, T \in \mathbb{B}(\ell^2)$, ${}^t(ST) = {}^tT{}^tS$.*

(*iv*) *If U is a unitary, then so is ${}^tU$.*

(*v*) *If T is positive, then so is ${}^tT$.*

(*vi*) *If T is positive and $\alpha > 0$, then ${}^t(T^\alpha) = ({}^tT)^\alpha$.*

(*vii*) *The transpose map is a* (*linear*) *isometry on each C_p.*

The following is the classification theorem of Arazy:

Theorem 8.2.12. (*Arazy* [Ara]) *Let $1 < p < \infty$ with $p \neq 2$ and C_p be the space of p-Schatten class operators on ℓ^2. Then every linear isometry Z on C_p is either of the following two forms:*

(1) *there exist unitaries $W, V \in \mathcal{U}(\ell^2)$ such that*

$$Z\colon C_p \to C_p;\ T \mapsto WTV;$$

(2) *there exist unitaries $W, V \in \mathcal{U}(\ell^2)$ such that*

$$Z\colon C_p \to C_p;\ T \mapsto W{}^tTV.$$

Thanks to Theorem 8.2.10 and Theorem 8.2.12, by following footsteps of Bader–Furman–Gelander–Monod (option (1) of the proofs of Theorem 3.3.7), we accomplish the conclusion of Theorem 8.2.7. More precisely, we follow the following argument:

Suppose a group pair $G \trianglerighteq N$ does not have prelative (T_{C_p}). We identify the Hilbert space with ℓ^2 in the way explained in above. Then there exists an isometric G-representation ρ on C_p such that $\rho' \mid_N \succ 1_N$ Here $\rho' \mid N$ is the restriction of ρ on the subspace $C'_p := C'_{p,\rho(N)}$ (here we use Corollary 8.2.3 that C_p is ucus). By employing the noncommutative Mazur map in the sense in Definition 8.2.9, define a (possibly nonlinear) map $\pi\colon C_2 \to C_2$ by

$$\pi(g) = M_{p,2} \circ \rho(g) \circ M_{2,p} \quad (g \in G).$$

We claim that this $\pi(g)$ is in fact a linear map for $g \in G$. Indeed, take any linear isometry Z on C_p, and show $\widetilde{Z} := M_{p,2} \circ Z \circ M_{2,p}$ is a linear map. By Theorem 8.2.12,

Z is either of form (1) or of form (2) in the statement of the theorem. With use of Lemma 8.2.11, we treat each case as follows (we take unitaries W, V as in the statement):

(i) in the case of (1), for any $T \in C_2$ with a polar decomposition $T = U|T|$, a polar decomposition of $WU|T|^{2/p}V$ is $(WUV)(V^*|T|^{2/p}V)$. Therefore, we have

$$\begin{aligned}\widetilde{Z} \cdot T &= WUV(V^*|T|^{2/p}V)^{p/2} \\ &= WUVV^*|T|V \\ &= WU|T|V = WTV.\end{aligned}$$

Hence $\widetilde{Z}$ is linear;

(ii) in the case of (2), for any $T \in C_2$ with a polar decomposition $T = U|T|$, a polar decomposition of $W^t(U|T|^{2/p})V = W({}^t|T|)^{2/p t}UV$ is

$$(W^tUV)\{V^*({}^tU)^*({}^t|T|)^{2/p}({}^tU)V\}.$$

Therefore, we have

$$\begin{aligned}\widetilde{Z} \cdot T &= W^tUV\{V^*({}^tU)^*({}^t|T|)^{2/p}({}^tU)V\}^{p/2} \\ &= W^tUVV^*({}^tU)^*({}^t|T|)^tUV \\ &= W^t|T|^tUV \\ &= W^t(U|T|)V = W^tTV.\end{aligned}$$

Hence $\widetilde{Z}$ is linear.

Therefore the π constructed in above is indeed a unitary representation (recall that C_2 is equipped with a natural inner product). Finally, by uniform continuity of the noncommutative Mazur maps (Theorem 8.2.10), it is not difficult to see that $\rho' \succ 1_\Lambda$ implies $\pi' \succ 1_\Lambda$ (, where π' is the restriction of π on the orthogonal complement of $(\ell^2)^{\pi(N)}$). This means $G \trianglerighteq N$ does not have relative property (T). This ends our proof. □

We note that linear isometries on a general noncommutative L^p space (with $p \neq 2$) have been classified in terms of *Jordan homomorphisms*. For details on these results, see [She] and [Yea]. Also we refer to [Sto] for a study of Jordan homomorphisms.

By Theorem 8.2.7 and Theorem 8.1.12, item (i) of Theorem 8.2.6 is proven.

Proof. (Item (ii) of Theorem 8.2.6) We employ p-induction in this setting. This is also done by Puschnigg as follows: let (X, μ) is a Borel space with finite measure (in mind, we consider $X = \mathcal{D} = G/\Gamma$, for a general G group and p-integrable lattice

$\Gamma \leqslant G$). He constructed $L^p(X, C_p)$ as the space of all measurable $\mathbb{B}(\mathfrak{H})$-valued functions with finite norms, where we define the norm on $L^p(X, C_p)$ by

$$\|\xi\|_p^p := \int_X \mathrm{Tr}((\xi(x)^*\xi(x))^{p/2}) d\mu(x).$$

Observe that again $L^2(X, C_2)$ becomes a Hilbert space.

Puschnigg shown the following:

Theorem 8.2.13. (*Corollary* 5.7, *Puschnigg* [Pus])

(*i*) *For any* p, $L^p(X, C_p)$ *is ucus.*

(*ii*) *Let* $p, q \in (1, \infty)$. *In the setting above, a noncommutative Mazur map, defined by*

$$M_{p.q} \colon S(L^p(X, C_p)) \to S(L^q(X, C_q)); \xi(x) \mapsto U_x|\xi(x)|^{p/q}$$

is uniformly continuous. Here for each x, $\xi(x) = U_x|\xi(x)|$ *is a polar decomposition.*

Now suppose a finite index subgroup Γ of $G = \mathrm{SL}_m(A)$ does not have $(\mathrm{FF}_{C_p})/\mathrm{T}$. Then there exists an isometric Γ-representation σ on C_p and a quasi-σ-cocycle c such that $c' \colon \Gamma \to C'_{p,\sigma(\Gamma)}$ is *unbounded*. Take a p-induction of c (see Subsection 3.4.2 and Proposition 8.1.4). Then we can construct the induced representation $\rho = \mathrm{Ind}_\Gamma^G \sigma$ and the induced quasi-ρ-cocycle b. Then since Γ is of finite index in G, $b' \colon G \to L^p(G/\Gamma, C_p)'_{\rho(G)}$ is *unbounded* (here we really use the assumption of finite index. For general setting of $\Gamma \leqslant G$ being a cocompact lattice, there is a gap to fix). However in the view of Theorem 8.1.7 and Theorem 6.2.1, this implies that there exists a copy of $G_0 = \mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2 = N$ such that the relative Kazhdan constant for $\rho\mid_{G_0}$,

$$\mathcal{K}(G_0, N; S, \rho\mid_{G_0})$$

is zero (see Definition 3.2.7). Here S is some finite generating set of G_0 (to see the assertion above, take negations in the arguments in the proofs of these theorems).

However, in a similar argument in the proof of Theorem 8.2.7, by interpolating by the noncommutative Mazur map (Theorem 8.2.13), we deduce that then $G_0 \trianglerighteq N$ does *not* have relative (T). This contradicts Theorem 4.2.3. Therefore, Γ has $(\mathrm{FF}_{C_p})/\mathrm{T}$, as desired. □

Item (*iii*) of Theorem 8.2.6 can be shown in the similar manner to one in the proof above, together with the arguments in Subsection 3.4.1 (recall that induction of a (*genuine*) cocycle has no problem in deducing that the original cocycle is a coboundary from the induced cocycle being a coboundary).

Thus we complete our proof of Theorem 8.2.6, which is Theorem C.

8.3 Property (TT)/T

In this section, we show item (*ii*) of Theorem B. The key to proving this is a theorem of Ozawa [Oza], in a connection to *property* (TTT), which is invented by him.

8.3.1 Ozawa's property (TTT)

Definition 8.3.1. (Ozawa, [Oza]) Let G be a group.

(*i*) A measurable map $\tilde{b}\colon G \to \mathfrak{H}$ from G to a Hilbert space $\mathfrak{H}$ is called a *wq-cocycle* if the following two conditions are satisfied:

(*a*) There exists a (measurable) *map* $\pi\colon G \to U(\mathfrak{H})$, which is *not* assumed to be a group homomorphism, such that

$$\sup_{g,h\in G} \|\tilde{b}(gh) - \tilde{b}(g) - \pi(g)\tilde{b}(h)\| < \infty.$$

(*b*) The map $\tilde{b}$ is *locally bounded*. This means, for any compact set $S \subseteq G$, the set $\{\|\tilde{b}(s)\| : s \in S\} \subseteq \mathbb{R}_{\geq 0}$ is relatively compact.

(*ii*) The group G is said to have *property* (TTT) if any wq-cocycle on G is bounded.

(*iii*) Let $N \leqslant G$ is a subgroup. We say $G \geqslant N$ has *relative property* (TTT) if any wq-cocycle on G is bounded on N.

Recall that property (TT), which is identical to ($\mathrm{FF}_{\mathcal{H}}$), is the boundedness property of all quasi-cocycles with unitary coefficients. A notion of wq-cocycles are weakening of that of quasi-cocycles. Therefore, (relative) property (TTT) is a strengthening of (relative) property (TT). At the present, there is no known example of groups with (TT) but without (TTT). For his motivations of inroducing (TTT) and application of that property, we refer to the original paper [Oza] and a paper [BOT] of Burger–Ozawa–Thom.

The following result is one of the main results in [Oza]:

Theorem 8.3.2. (*Ozawa* [Oza]) *Let* $\mathbb{K}$ *be any local field and* $m \geq 3$. *Then the group* $\mathrm{SL}_m(\mathbb{K})$ *and lattices therein have property* (TTT).

In particular, $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ *has* (TTT).

In the proof of item (*ii*) of Theorem B, we employ the following theorem. This is used by Ozawa to deduce the theorem above.

Theorem 8.3.3. (*Ozawa* [Oza]) *Let* $G = G_0 \ltimes A$ *be the semidirect product of a* abelian *group by a continuous action of* G_0. *Then the following are equivalent:*

(*i*) *The pair* $G \geqslant A$ *has relative* (T).

(*ii*) *The pair* $G \geqslant A$ *has relative* (TTT)*.*

In particular, if $G \geqslant A$ *has relative* (T)*, then it has relative* (TT)*.*

For the proof, see Proposition 3 in [Oza].

8.3.2 Universal lattices have (TT)/T

We restate the definition of property (TT)/T, because it has an interesting application (see Section 11.5).

Definition 8.3.4. ([Mim1], [Mim4]) A group G is said to have *property* (TT)/T if for any unitary G-representation π *with* $\pi \not\supseteq 1_G$, every π-cocycle is bounded.

By definition, property (TT)/T is identical to property $(\mathrm{FF}_{\mathcal{H}})$/T.

By utilizing Theorem 8.3.3, we show item (*ii*) of Theorem B. We restate the assertion:

Theorem 8.3.5. (*Remark* 6.7*,* [Mim1]) *Let* $k \in \mathbb{N}$ *and* $A = \mathbb{Z}[x_1, \ldots, x_k]$*. Then for* $m \geq 3$*,* $G = \mathrm{SL}_{m\geq 3}(A)$ *possesses property* (TT)/T*.*

Proof. Take the following subgroups (H, N_1, N_2): $H \cong \mathrm{SL}_{m-1}(A)$, and $N_1, N_2 \cong A^{n-1}$. Here in G we realize H as in the left upper corner (, namely, the ((1-$(m-1)) \times (1$-$(m-1)$))-th parts), realize N_1 as in the ((1-$(m-1)) \times m$)-th unipotent parts, and realize N_2 as in the ($m \times (1$-$(m-1)$))-th unipotent parts. And set $Q = N_1 \cup N_2$.

Then thanks to Theorem 8.3.3, this pair (H, Q) fulfills all conditions of Theorem 8.1.7 for $\mathcal{C} = \mathcal{H}$. Here the crucial point is on condition (*iv*). Therefore Theorem 8.1.7 applies, and we obtain property $(\mathrm{FF}_{\mathcal{H}})$/T(= (TT)/T). □

We have accomplished the proof(s) of Theorem B. Note that by item (*i*) of Theorem B (Theorem 8.1.10), we are already done for the case of $m \geq 4$, and the essential part of Theorem 8.3.5 is the case of $m = 3$.

Chapter 9

Symplectic universal lattices

In this chapter, we consider an analogue of the elementary group over a ring (Chapter 4) in the symplectic setting. In this case, it is called the *elementary symplectic group*. Note that this concept is only defined over a *commutative* ring.

We see an analogue of the Suslin stability theorem, which is due to Grunewald–Mennicke–Vaserstein. We define the notion of *symplectic universal lattices* in a similar manner to one in the definition of universal lattices. We see a symplectic version of Vaserstein's bounded generation for symplectic universal lattices, which is also due to Vaserstein. Note that although it looks similar to that for universal lattices, their properties are different from the viewpoint of Shalom's machinery. Namely, Shalom's machinery does *not* work for symplectic universal lattices.

Next, we see a celebrated result of Ershov–Jaikin-Zapirain that noncommutative universal lattices have (T), and its extension, which is a recent work of Ershov–Jaikin-Zapirain–Kassabov. Specially, we see their result that symplectic universal lattices as well have (T).

Finally, we appeal to this result, and deduce Theorem D. In particular, we see that symplectic universal lattices possess (TT)/T.

9.1 Definition

In this section, we introduce the notion of *symplectic universal lattices*, which is parallel to that of universal lattices.

9.1.1 Elementary symplectic groups– one realization

Firstly, for $m \geq 1$, we take the $2m$ by $2m$ alternating matrix as

$$L_m = \begin{pmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & L \end{pmatrix} \in \mathrm{M}_{2m}.$$

Here

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{M}_2.$$

This choice of alternating matrix is standard, but in connection to property (T), this choice is not good. Hence this alternating matrix L_m later shall be replaced with another one J_m in this thesis. See the next subsection for details.

Let A be a *commutative* ring. Then the *symplectic group* over A of degree $2m$ is the following group:

$$\mathrm{Sp}_{2m}(A) = \{g \in \mathrm{M}_2m(A) : {}^t g L_m g = L_m\}.$$

Here ${}^t g$ is the transpose matrix of g. We use the following permutation symbol on $\mathbb{Z}$:

$$(2i)' := 2i - 1;\ (2i-1)' = 2i \quad (i \in \mathbb{N}).$$

For any (i,j) with $1 \leq i \leq 2m$, $1 \leq i \leq 2m$ and $i \neq j$; and any $a \in A$, we define the *elementary symplectic matrix* as follows:

$$\mathrm{SE}_{i,j}(a) := \begin{cases} I_2 m + a e_{i,j} & \text{if } i = j'; \\ I_2 m + a e_{i,j} - (-1)^{i+j} a e_{j',i'} & \text{if } i \neq j'. \end{cases}$$

Here $e_{i,j}$ is the matrix with (i,j)-th entry 1 and the other entries 0. The *elementary symplectic group* over A of degree $2m$ is the group generated by all elementary symplectic matrices, and it is written as $\mathrm{Ep}_{2m}(A)$. A priori, $\mathrm{Ep}_{2m}(A)$ is a (possibly non-proper) subgroup of $\mathrm{Sp}_{2m}(A)$.

Then there are some basic commutator relations between elementary symplectic matrices. They are so many, and here we only state some important relations. For the all precise relations, we refer to Lemma 4.1 of the paper [GMV1] of Grunewald–Menniche–Vaserstein.

Lemma 9.1.1. *Let $a, b \in A$ are any element. Then there are the following formulae:*

(i) (1) $[\mathrm{SE}_{i,i'}(a), \mathrm{SE}_{i',l}(b)] = \mathrm{SE}_{i,l}(ab)\mathrm{SE}_{l',l}((-1)^{i+l}ab^2)$,
if $i \neq l, i' \neq l$.

(2) $[\mathrm{SE}_{i,i'}(a), \mathrm{SE}_{k,i'}(b)] = \mathrm{SE}_{i,i'}(2ab + (-1)^{i+k}ab^2)$,
if $i \neq k, i' \neq k$.

(3) $[\mathrm{SE}_{i,i'}(a), \mathrm{SE}_{k,i}(b)] = \mathrm{SE}_{k,i'}(-ab)\mathrm{SE}_{k,k'}(-(-1)^{i+k}ab^2)$, *if* $i \neq k, i' \neq k$.

(4) $[\mathrm{SE}_{i,i'}(a), \mathrm{SE}_{i,l}(b)] = \mathrm{SE}_{l',l}(-2ab - (-1)^{i+l}ab^2)$, *if* $i \neq l, i' \neq l$.

(*ii*) (1) $[\mathrm{SE}_{i,j}(a), \mathrm{SE}_{j,l}(b)] = \mathrm{SE}_{i,l}(ab)$, *if* $i' \neq j, j' \neq l, i \neq l, i' \neq l$.

(2) $[\mathrm{SE}_{i,j}(a), \mathrm{SE}_{j,l}(b)] = \mathrm{SE}_{i,l}(2ab)$, *if* $i' \neq j, j' \neq l, i \neq l, i' = l$.

(3) $[\mathrm{SE}_{i,j}(a), \mathrm{SE}_{k,j'}(b)] = \mathrm{SE}_{k,i'}((-1)^{i+j}ab)$, *if* $i' \neq j, j' \neq k, i' \neq k, i \neq k$.

(4) $[\mathrm{SE}_{i,j}(a), \mathrm{SE}_{k,j'}(b)] = \mathrm{SE}_{i,i'}(2(-1)^{i+j}ab)$, *if* $i' \neq j, j' \neq k, i' \neq k, i = k$.

These relation implies that if $m \geq 2$, then whenever A is finitely generated (as a ring), $\mathrm{Ep}_{2m}(A)$ is a finitely generated group. However, we warn that *we need some care on a finite generating set* of $\mathrm{Ep}_{2m}(A)$. If one looks Lemma 9.1.1 carefully, then one will notice that some rules are *different* between elementary symplectic matrices of the form $\mathrm{SE}_{i,j}(a)$ $(i' \neq j)$; and those of the form $\mathrm{SE}_{i,i'}(a)$. Structures of the latter in terms of commutators is *much more complicated* than that of the former. Indeed, let $m \geq 2$ and suppose A is finitely generated and $S = \{s_1, \ldots, s_k\}$ is a finite generating set for A (as a ring). Then the following finite set is a generating set for $\mathrm{Ep}_{2m}(A)$:

$$\begin{aligned}\mathcal{S} := &\{\mathrm{SE}_{i,j}(\pm s_l) : 1 \leq i, j \leq 2m, i \neq j, i' \neq j; 0 \leq l \leq k\} \\ &\cup \{\mathrm{SE}_{i,i'}(\pm s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k}) : 1 \leq i \leq 2m; \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\}.\end{aligned}$$

Here we regard $s_0 = 1$ and $s_l^0 = 1$.

Note that if $m \geq m_0$, then there is a natural inclusion

$$\mathrm{Ep}_{2m_0}(A) \hookrightarrow \mathrm{Ep}_{2m}(A),$$

sending to the left upper corner.

Remark 9.1.2. Some reasons why we need the *commutativity* of a ring are the following: firstly, if a ring R is not commutative, then the transpose map $\mathrm{M}_{2m}(R) \to \mathrm{M}_{2m}(R)$ is not well-defined (more precisely, if we transpose just entries, then this map is *not* an anti-homomorphism). Therefore the group $\mathrm{Sp}_{2m}(R)$ is *not* well-defined. Secondly, if R is noncommutative, then the commutator relation stated above (Lemma 9.1.1) does *not* hold.

If R itself admits the transpose map (for instance let $R = \mathbb{Z}G$ for a group G and define ${}^t\delta_g := \delta_{g^{-1}}$), then it is possible to define $\mathrm{Sp}_{2m}(R)$. However, definition of elementary symplectic matrices needs an appropriate modification. It is still possible to define $\mathrm{Ep}_{2m}(R)$ over such a ring after some modification, but there is no reason to expect finite generation for $\mathrm{Ep}_{2m}(R)$ even if R is finitely generated.

As we have mentioned in above, we use *another* realization for $\mathrm{Sp}(A)$ and $\mathrm{Ep}(A)$ in this thesis. That means, we choose *different* $2m \times 2m$ alternating matrix, as in the next subsection.

9.1.2 Another realization

Although the realization in the previous subsection is natural, it has some difficulty to examine (T)-type properties. Therefore, we use the following *another* realization throughout this thesis, even though it may look somewhat awkward.

In the rest part of this thesis, we basically use the following alternating matrix:

$$J_m := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \in \mathrm{M}_{2m}$$

Note that J_m is always conjugate to L_m, and that hence the following definitions are equivalent to ones in the previous subsection.

Definition 9.1.3. Let $m \geq 1$ and A be a commutative ring.

(i) The *symplectic group* over A of degree $2m$, written as $\mathrm{Sp}_{2m}(A)$, is defined as the multiplicative group of symplectic matrices in matrix ring $\mathrm{M}_{2m}(A)$ associated with the alternating matrix J_m, namely,

$$\mathrm{Sp}_{2m}(A) := \{g \in \mathrm{M}_{2m}(A) : {}^t g J_m g = J_m\}.$$

(ii) Matrices of the following form are called *elementary symplectic matrices*:

(1) For $1 \leq i, j \leq m$ with $i \neq j$, define

$$\begin{aligned} B_{i,j}(a) &:= I_{2m} + a(e_{i,m+j} + e_{j,m+i}), \\ C_{i,j}(a) &:= I_{2m} + a(e_{m+j,i} + e_{m+i,j})(= {}^t B_{i,j}(a)), \\ D_{i,j}(a) &:= I_{2m} + a e_{i,j} - a e_{m+j,m+i}. \end{aligned}$$

(2) For $1 \leq i \leq m$, define

$$\begin{aligned} B_{i,i}(a) &:= I_{2m} + a e_{i,m+i}, \\ C_{i,i}(a) &:= I_{2m} + a e_{m+i,i} (= {}^t B_{i,i}(a)). \end{aligned}$$

Here $a \in A$ is any element.

Namely, for $i \neq j$,

$$B_{i,j}(a) := \begin{pmatrix} I_m & a(e_{i,j}+e_{j,i}) \\ 0 & I_m \end{pmatrix}, C_{i,j}(a) := \begin{pmatrix} I_m & 0 \\ a(e_{i,j}+e_{j,i}) & I_m \end{pmatrix},$$
$$D_{i,j}(a) := \begin{pmatrix} I_m + ae_{i,j} & 0 \\ 0 & I_m - ae_{j,i} \end{pmatrix};$$

and for $i = j$,

$$B_{i,i}(a) := \begin{pmatrix} I_m & ae_{i,i} \\ 0 & I_m \end{pmatrix}, C_{i,i}(a) := \begin{pmatrix} I_m & 0 \\ ae_{i,i} & I_m \end{pmatrix}.$$

(*iii*) The *elementary symplectic group* $\mathrm{Ep}_{2m}(A)$ over A of degree $2m$ is the subgroup of $\mathrm{Sp}_{2m}(A)$ generated by all elementary symplectic matrices in the sense above

Note that in item (*ii*) above, the case of $i \neq j$ corresponds to that of $\mathrm{SE}_{l,k}(a)$ ($l' \neq k$); and the case of $i = j$ corresponds to that of $\mathrm{SE}_{l,l'}(a)$ in the previous chapter. However, the indices are permuted.

The following is mere interpretation of an observation in the previous subsection to this realization:

Lemma 9.1.4. *Let $m \geq 2$ and suppose A is finitely generated and $S = \{s_1, \ldots, s_k\}$ is a finite generating set for A (as a ring). Then the following finite set is a generating set for* $\mathrm{Ep}_{2m}(A)$*:*

$$\begin{aligned} \mathcal{S} := &\{B_{i,j}(\pm s_l), C_{i,j}(\pm s_l), D_{i,j}(\pm s_l) : 1 \leq i,j \leq m, i \neq j; 0 \leq l \leq k\} \\ &\cup \{B_{i,i}(\pm s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k}), C_{i,i}(\pm s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_k^{\epsilon_k}) : 1 \leq i \leq m; \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\}. \end{aligned}$$

Here we regard $s_0 = 1$ and $s_l^0 = 1$.

Finally, we define the following identification for certain subgroups of $\mathrm{Sp}_{2m}(A)$ (or $\mathrm{Ep}_{2m}(A)$):

Definition 9.1.5. Let A be a commutative ring.

(*i*) For $m \geq m_0 \geq 2$, by $\mathrm{SL}_{m_0}(A) \leqslant \mathrm{Sp}_{2m}(A)$ (or, $\mathrm{SL}_{m_0}(A) \hookrightarrow \mathrm{Sp}_{2m}(A)$) we mean the inclusion is realized in the following way:

$$\left\{ \begin{pmatrix} W & 0 & 0 & 0 \\ 0 & I_{m-m_0} & 0 & 0 \\ 0 & 0 & {}^tW^{-1} & 0 \\ 0 & 0 & 0 & I_{m-m_0} \end{pmatrix} : W \in \mathrm{SL}_{m_0}(A) \right\} \leqslant \mathrm{Sp}_{2m}(A).$$

(*ii*) Let $m \geq 2$. We denote by $S^{m*}(A^m)$ the additive group of all symmetric matrices in $\mathrm{M}_m(A)$. By $\mathrm{E}_m(A) \ltimes S^{m*}(A^m) \trianglerighteq S^{m*}(A^m)$, we identify these groups respectively with

$$\left\{ (W, v) := \left(\begin{array}{c|c} W & v \\ \hline 0 & {}^t(W^{-1}) \end{array} \right) : W \in \mathrm{E}_m(A),\ v \in S^{m*}(A^m) \right\}$$
$$\trianglerighteq \left\{ \left(\begin{array}{c|c} I_m & v \\ \hline 0 & I_m \end{array} \right) : v \in S^{m*}(A^m) \right\}.$$

Thus the action of $\mathrm{E}_m(A)$ on $S^{m*}(A^m)$ is:

$$(W, 0)(I_m, v)(W^{-1}, 0) = (I_m, Wv^tW) \quad (W \in \mathrm{E}_m(A), v \in S^{m*}(A^m)).$$

Here $\mathrm{E}_m(A)$ denotes the elementary group over A (see Section 4.1).

(*iii*) Let $m \geq m_0 \geq 2$. Then by $\mathrm{Sp}_{2m_0}(A) \leqslant \mathrm{Sp}_{2m}(A)$ (or, $\mathrm{Sp}_{2m_0}(A) \hookrightarrow \mathrm{Sp}_{2m}(A)$), we mean the inclusion is realized as

$$\left\{ \begin{pmatrix} P & 0 & Q & 0 \\ 0 & I_{m-m_0} & 0 & 0 \\ R & 0 & S & 0 \\ 0 & 0 & 0 & I_{m-m_0} \end{pmatrix} : \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}_{2m_0}(A) \right\} \leqslant \mathrm{Sp}_{2m}(A).$$

The advantage of this realization of $\mathrm{Sp}_{2m}(A)$ (, namely the choice of J_m) is that then it is easy to express the pair in item (*ii*). This pair is the base point for study on property (T) for symplectic groups, as we have seen in Subsection 2.6.3.

Remark 9.1.6. It is easy to see that if $m = 1$, then for any commutative ring A, $\mathrm{Sp}_2(A) = \mathrm{SL}_2(A)$, and $\mathrm{Ep}_2(A) = \mathrm{E}_2(A)$. Nevertheless, in this thesis, (for $m \geq 2$) we *distinguish* $\mathrm{Sp}_2(A) \leqslant \mathrm{Sp}_{2m}(A)$ and $\mathrm{SL}_2(A) \leqslant \mathrm{Sp}_{2m}(A)$. Namely, in this thesis *the former is realized as in item* (*iii*) *in the definition above, and the latter is realized as in item* (*i*) *in the definition above.* Similarly, we distinguish $\mathrm{Ep}_2(A) \leqslant \mathrm{Sp}_{2m}(A)$ and $\mathrm{E}_2(A) \leqslant \mathrm{Sp}_{2m}(A)$. These distinctions have important meaning on study of property (T) on symplectic universal lattices in connection to Vaserstein's bounded generation for Sp. These shall be examined in Subsection 9.2.1.

9.1.3 The Grunewald–Mennicke–Vaserstein stability theorem

As is parallel to elementary group cases, it is natural to ask in which case $\mathrm{Ep}_{2m}(A)$ coincides with $\mathrm{Sp}_{2m}(A)$. Situations are extremely different between the case of $m = 1$ and the case of $m \geq 2$. In the paper [GMV1], Grunewald–Mennicke–Vaserstein proven the following stability theorem:

Theorem 9.1.7. ([GMV1]) *Let $k \in \mathbb{N}$. Then for any $m \geq 2$, there are equalities*

$$\mathrm{Ep}_{2m}(\mathbb{Z}[x_1, \ldots, x_k]) = \mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k]);$$
$$\mathrm{Ep}_{2m}(\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]) = \mathrm{Sp}_{2m}(\mathbb{Z}[x_1^{\pm}, \ldots, x_k^{\pm}]).$$

In fact, they shown that these equalities hold true even if $\mathbb{Z}$ is replaced with any euclidean domain.

In parallel to the definition of universal lattices of Shalom, in forthcoming paper [Mim4], the author defined a notion of *symplectic universal lattices* as follows.

Definition 9.1.8. ([Mim4]) Let $m \geq 2$. Take any $k \in \mathbb{N}$. A *symplectic universal lattice* of degree $2m$ denotes a group

$$\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])(= \mathrm{Ep}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])).$$

9.2 Symplectic universal lattices have (T)

In this section, we see the fact that symplectic universal lattices have (T). First, we see some analogue of Vaserstein's bounded generation in this case (this is also due to Vaserstein [Vas2]). However, we then see this is *not* sufficient to deduce (T) (in other words, Shalom's machinery does *not* work). Finally, we state a theorem of Ershov–Jaikin-Zapirain–Kassabov on property (T) for (twisted) Steinberg groups, from which (T) for symplectic universal lattices follows.

9.2.1 Vaserstein's bounded generation for Sp

Before stating Vaserstein's generation, we state a fundamental relative property (T) concerning $\mathrm{Sp}_{2m}(A)$. This is due to M. Neuhauser [Neu], and is obtained by idea inspired by Shalom's theorem [Sha1] for elementary group case (Theorem 4.2.1, Theorem 4.2.3):

Theorem 9.2.1. (*Neuhauser, Theorem* 3.3, [Neu]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ for any integer $k \in \mathbb{Z}$. Then for $m \geq 2$, the pair $\mathrm{E}_m(A) \ltimes S^{m*}(A^m) \trianglerighteq S^{m*}(A^m)$, which is defined as in item (ii) of Definition* 9.1.5, *has relative* (T).

Here we state Vaserstein's bounded generation for symplectic universal lattices:

Theorem 9.2.2. (*Vaserstein* [Vas2]) *Let $A_k = \mathbb{Z}[x_1, \ldots l_k]$ for any k. Then for any $m \geq 2$, $\mathrm{Sp}_{2m}(A_k)$ is boundedly generated by the set of elementaly symplectic matrices and the subgroup $\mathrm{Sp}_2(A_k)$.*

Here $\mathrm{Sp}_2(A_k)$ is realized in $\mathrm{Sp}_{2m}(A_k)$ as in item (iii) of Definition 9.1.5, *namely,*

$$\mathrm{Sp}_2(A_k) := \left\{ \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & I_{m-1} & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & I_{m-1} \end{array} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_2(A_k) \right\} \leqslant \mathrm{Sp}_{2m}(A_k).$$

Now we explain why Shalom's machinery does *not* work for symplectic universal lattices. Consider for instance the case of $G = \mathrm{Sp}_4(A)$ ($A = \mathbb{Z}[x_1, \ldots x_k]$). Then in the view of Theorem 9.2.1, the following triple (H, N_1, N_2) is a standard candidate for pairs with the Shalom property (for (T)):

$$\begin{aligned}
H = \mathrm{SL}_2(A) &:= \left\{ \left(\begin{array}{c|c} W & 0 \\ \hline 0 & {}^tW^{-1} \end{array} \right) : W \in \mathrm{SL}_2(A) \right\} \\
&= \left\{ \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{array} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A) \right\} \\
N_1 &:= \left\{ \left(\begin{array}{c|c} I_2 & v \\ \hline 0 & I_2 \end{array} \right) : v \in S^{2*}(A^2) \right\} \cong S^{2*}(A^2), \\
N_2 &:= \left\{ \left(\begin{array}{c|c} I_2 & 0 \\ \hline v' & I_2 \end{array} \right) : v' \in S^{2*}(A^2) \right\} \cong S^{2*}(A^2).
\end{aligned}$$

The point here is the following, as in Remark 9.1.6: *even though the group* $\mathrm{Sp}_2(A)$ *and* $\mathrm{SL}_2(A)$ *are isomorphic as abstract groups, their realization inside* $\mathrm{Sp}_{2m}(A)$ *are completely different. Moreover, a group relating to Vaserstein's bounded generation is* $\mathrm{Sp}_2(A)$*; but a group relating to relative* (T) *is* $\mathrm{SL}_2(A)$. Therefore at the present, it seems impossible to apply Shalom's machinery.

9.2.2 A theorem of Ershov–Jaikin-Zapirain

As we have mentioned in Remark 5.4.2, property (T) for noncommutative lattices is shown by Ershov–Jaikin-Zapirain [ErJa] (with some estimation for Kazhdan constant).

Theorem 9.2.3. (*Ershov–Jaikin-Zapirain* [ErJa]) *Noncommutative universal lattice*

$$G = \mathrm{E}_{m\geq 3}(\mathbb{Z}\langle x_1, \ldots, x_k \rangle)$$

has property (T).

In particular, for any finitely generated ring and any $m \geq 3$, $\mathrm{E}_m(R)$ *has* (T).

Their argument is inspired by a work of Dymara–Januszkiewicz [DyJa]. Main idea is for given unitary representation $(\pi, \mathfrak{H})$ of a group G and finite family of subgroups $N_1, \ldots N_n$ of G that generate G, estimates angles of spaces $(\mathfrak{H}^{\rho(N_i)})_{1\leq i\leq n}$. If each $G \geqslant N_i$ has relative (T) and the angles above are sufficiently big (close to orthogonal, far from coinciding), then G itself has (T). The actual proof is much more involved, and we refer to the original paper [ErJa]. One of the point of this result is that they does *not* appeal to bounded generation. They moreover have proven property (T) for *Steinberg groups.*

Definition 9.2.4. Let $m \geq 3$ and R be a ring. The *Steinberg group* over R of degree m is defined as follows: the group generated by $\{x_{i,j}(r) : 1 \leq i, j \leq m, i \neq j\}$ (the set of formal generators) which subjects to the following commutator relations:

$$\begin{aligned} &x_{i,j}(r)x_{i,j}(s) = x_{i,j}(r+s), \\ &[x_{i,j}(r), x_{k,l}(s)] = e && \text{if } i \neq l, j \neq k, \\ &[x_{i,j}(r), x_{j,k}(s)] = x_{i,k}(rs) && \text{if } i \neq k. \end{aligned}$$

The Steinberg group is written as $\mathrm{St}_m(R)$.

Steinberg groups are generalization of elemetary groups. More precisely, there is a natural surjective homomorphism:

$$\mathrm{St}_m(R) \twoheadrightarrow \mathrm{E}_m(R);\ x_{i,j}(r) \mapsto E_{i,j}(r).$$

Therefore, the following result of Ershov–Jaikin-Zapirain is more universal than Theorem 9.2.3:

Theorem 9.2.5. (*Ershov–Jaikin-Zapirain* [ErJa]) *For any finitely generated ring and any* $m \geq 3$*, the Steinberg group* $\mathrm{St}_m(R)$ *has* (T).

Also, there is a concept of the *twisted Steinberg group* (over a commutative ring) associated with a reduced irreducible classical root system. For instance, commutator relations as in Lemma 9.1.1 (and more) yields one example of twisted Steinberg group, and in this case it maps onto elementary symplectic groups (in this case, associated root system is so-called of type C_n). In a recent work of Ershov–Jaikin-Zapirain–Kassabov, they have established property (T) for such twisted Steinberg groups, whenever the rank of the associated root system is at least 2:

Theorem 9.2.6. ([EJK]) *Let* Φ *be a reduced irreducible classical root system of rank at least* 2 *and* A *be a finitely generated commutative ring. Then* $\mathrm{St}_\Phi(A)$*, the twisted Steinberg group over* A*, has property* (T).

In particular, every symplectic universal lattice ($m \geq 2$),

$$\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])(= \mathrm{Ep}_{2m}(\mathbb{Z}[x_1, \ldots, x_k]))$$

has property (T).

In the next section, we show property (TT)/T for symplectic universal lattices, by appealing to this theorem. Observe that Theorem 9.2.6 amounts to condition (i) in Theorem 8.1.7.

Remark 9.2.7. The proof of Theorem 9.2.3 does not directly apply to the case of property $(\mathrm{F}_{\mathcal{L}_p})$ because angle estimate is special to Hilbert spaces. Therefore at the present, it seems open to determine whether noncommutative lattices have $(\mathrm{F}_{\mathcal{L}_p})$. However, Theorem 8.1.7 provides us with some relative property:

Proposition 9.2.8. ([Mim4]) *Let* $R = \mathbb{Z}\langle x_1, \ldots, x_k\rangle$ $(k \in \mathbb{N})$. *Then for any* $m \geq 4$, $\mathrm{E}_m(R) \geqslant \mathrm{E}_{m-1}(R)$ *has relative* $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$. *In particular, this pair has relative* $(\mathrm{F}_{\mathcal{L}_p})$. *Here the inclusion above is realized as the subgroup sitting in the left upper corner.*

Also, for any $m \geq 3$, $\mathrm{E}_m(R) \geqslant \mathrm{E}_{m-1}(R)$ *has relative* $(\mathrm{TT})/\mathrm{T}$.

Proof. This is a mere application of Theorem 8.1.7, and can be done in a similar argument to ones in the proof(s) of Theorem B. Note that all conditions but condition (v) (bounded generation) in Theorem 8.1.7 are valid. For condition (iv), observe that Theorem 8.1.5 can be extended to noncommutative universal lattice cases without any change in the proof. □

9.3 Symplectic universal lattices have (TT)/T

The goal of this section is to prove the following theorem, which is item (ii) of Theorem D (see Definition 8.3.4 for the definition of (TT)/T):

Theorem 9.3.1. ([Mim4]) *Let* $A = \mathbb{Z}[x_1, \ldots, x_k]$ $(k \in \mathbb{N})$. *Then for any* $m \geq 2$, *symplectic universal lattice* $\mathrm{Sp}_{2m}(A)$ *has* $(\mathrm{TT})/\mathrm{T}$.

We will see that Vaserstein's bounded generation (Theorem 9.2.2) works in connection to Theorem 8.1.7, although it is not enough for Shalom's machinery.

Proof. We only prove in the case of $m = 2$ (other case also follows from Theorem 9.2.2 and the proof below). Set $G = \mathrm{Sp}_4(A)$. We take the subgroup $\mathrm{Sp}_2(A) \leqslant G$, as in item (iii) in Definition 9.1.5, and the following (infinite) subset $P \subseteq G$. Namely, we take the following subgroup H and subset P of G:

$$H = \left\{ \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_2(A) \right\} \leqslant \mathrm{Sp}_{2m}(A),$$

$$\begin{aligned} P &= \{B_{1,2}(r), C_{1,2}(r), D_{1,2}(r), D_{2,1}(r), B_{2,2}(r), C_{2,2}(r) : r \in A\} \\ &= \left\{ \left(\begin{array}{cc|cc} 1 & 0 & 0 & r \\ 0 & 1 & r & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & r & 1 & 0 \\ r & 0 & 0 & 1 \end{array} \right), \right. \\ &\quad \left(\begin{array}{cc|cc} 1 & r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -r & 1 \end{array} \right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & -r \\ 0 & 0 & 0 & 1 \end{array} \right), \\ &\quad \left. \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & r \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right), \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & r & 0 & 1 \end{array} \right) : r \in A \right\} \\ &\subseteq \mathrm{Sp}_4(A). \end{aligned}$$

Define a new subset $Q \subseteq G$ as

$$Q := \cup_{h \in H} hPh^{-1}.$$

We will show that this pair (H, Q) in G fulfills all conditions of Theorem 8.1.7 for $\mathcal{C} = \mathcal{H}$. Firstly, condition (i), namely, property (T) for G follows from Theorem 9.2.6 (here we are appealing to this deep theorem)! Condition (iii) (invariance of Q by conjugation of H) is trivial by construction. Next, we will see condition (ii) (generation of G by Q). For each element in P, if we rewrite them in the realization by another alternating matrix L_m as in Subsection 9.1.1, then we have the following correspondences:

alternating matrix:	J_2	$\longleftrightarrow$	L_2
	$B_{1,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,4}(r)$,
	$C_{1,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{4,1}(r)$,
	$D_{1,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,3}(r)$,
	$D_{2,1}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{3,1}(r)$,
	$B_{2,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{3,4}(r)$,
	$C_{2,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{4,3}(r)$,
	$B_{1,1}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,2}(r)$,
	$C_{1,1}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{2,1}(r)$.

Therefore by Lemma 9.1.1, we have following equalities:

$$B_{1,1}(r) = D_{2,1}(-r)[B_{2,2}(r), C_{1,2}(1)],$$
$$C_{1,1}(r) = D_{1,2}(r)[C_{2,2}(r), B_{1,2}(1)].$$

These equalities show every elementary symplectic matrix is a product of at most 5 elements in P (and hence Q). This shows condition (ii). Note that Theorem 9.2.2 states G is boundedly generated by H and elementary symplectic matrices. Therefore, condition (v) (bounded generation by H and Q) holds true.

It remains to check condition (iv) (, namely, that $G \supseteq Q$ has relative (TT)/T). We will prove in fact $G \subseteq Q$ has relative (TT). Theorem 9.2.1 (relative (T)) together with Theorem 8.3.3 implies relative (TTT) (hence in particular relative (TT)) for the pair $\mathrm{E}_2(A) \ltimes S^{2*}(A^2) \trianglerighteq S^{2*}(A^2)$. Thus for any copy of $\mathrm{E}_2(A) \ltimes S^{2*}(A^2)$ in $G = \mathrm{Sp}_4(A)$, $S^{2*}(A^2)$-part has relative (TT) (with respect to G). This implies the following: "let $Y \subseteq G$ be the set of all elementary symplectic matrices. Then $G \supseteq Y$ has relative (TT)". Finally, for each

$$H \ni h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(the identification is as in above) and for each element l in P, we compute hlh^{-1}. Then we have the following:

$$hB_{1,2}(r)h^{-1} = B_{1,2}(ar)D_{2,1}(-cr),$$
$$hC_{1,2}(r)h^{-1} = C_{1,2}(dr)D_{1,2}(br),$$
$$hD_{1,2}(r)h^{-1} = C_{1,2}(cr)D_{1,2}(ar),$$
$$hD_{1,2}(r)h^{-1} = B_{1,2}(-br)D_{2,1}(dr),$$
$$hB_{2,2}(r)h^{-1} = B_{2,2}(r),$$
$$hC_{2,2}(r)h^{-1} = C_{2,2}(r).$$

Therefore, every element in Q is a product of at most 2 elements in Y. This shows that $G \supseteq Q$ has relative (TT), and hence condition (iv) is fulfilled.

We have checked all conditions (i)-(v), and thus Theorem 8.1.7 ends our proof. □

9.4 Property $(\mathrm{FF}_{\mathcal{B}})/\mathrm{T}$ for symplectic universal lattices

Here we show item (i) of Theorem D. We restate it here:

Theorem 9.4.1. ([Mim4]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ $(k \in \mathbb{N})$. Then for any $m \geq 3$, a symplectic universal lattice $\mathrm{Sp}_{2m}(A)$ has $(\mathrm{FF}_{\mathcal{L}_p})/\mathrm{T}$ and $(\mathrm{FF}_{C_p})/\mathrm{T}$. In particular, it possesses $(\mathrm{F}_{\mathcal{L}_p})$ and (F_{C_p}).*

For the proof, we employ the following result, which is parallel to Theorem 8.1.5.

Theorem 9.4.2. ([Mim4]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$. Suppose B is any superreflexive Banach space. If the pairs $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$; and $\mathrm{E}_2(A) \ltimes S^{2*}(A^2) \trianglerighteq S^{2*}(A^2)$ have relative property (T_B), then the pair $\mathrm{SL}_3(A) \ltimes S^{3*}(A^3) \geqslant S^{3*}(A^3)$ has relative property (FF_B).*

Proof. The proof of Theorem 9.4.2 goes along the line of the base case: that of Theorem 8.1.5, but it requires more care. Set $G = \mathrm{SL}_3(A) \ltimes S^{3*}(A^3)$ and $N = S^{3*}(A^3) \trianglelefteq G$, and consider G (and N) as subgroup(s) in $\mathrm{Sp}_6(A)$, as in item (ii) of Definition 9.1.5. Let ρ be an isometric G-representation on B and $b\colon G \to B$ be an (arbitrary) quasi-ρ-cocycle. Take a ucus ρ-invariant norm on B and fix it. Take a decomposition $B = B_0 \oplus B_1 := B^{\rho(N)} \oplus B'_{\rho(N)}$, and decompose b as

$$b = b_0 + b_1,$$

where $b_0\colon G \to B_0$ and $b_1\colon G \to B_1$. Then for $i = 1, 2$, by $\rho(G)$-invariance of B_i, b_i becomes a quasi-ρ-cocycle. Set the following two subgroups N_1, N_2 of N:

$$N_1 := \left\{ B_{1,1}(r) = \left(\begin{array}{c|ccc} & r & 0 & 0 \\ I_3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \hline 0 & & I_3 & \end{array} \right) : r \in A \right\},$$

$$N_2 := \left\{ B_{1,2}(r) = \left(\begin{array}{c|ccc} & 0 & r & 0 \\ I_3 & r & 0 & 0 \\ & 0 & 0 & 0 \\ \hline 0 & & I_3 & \end{array} \right) : r \in A \right\}.$$

Then in a similar way to one in the proof of Theorem 8.1.5, the following two can be verified:

(1) The quasi-cocycle b_0 is bounded on N.

(2) If b_1 is bounded both on N_1 and on N_2, then it is bounded on N.

Therefore for the proof of the theorem, it suffices to show the following two assertions:

(A1) *The set $b_1(N_1)$ is bounded.*

(A2) *The set $b_1(N_2)$ is bounded.*

In below, we shall prove assertions (A1) and (A2). Recall from Definition 9.1.3 the definitions of $B_{i,j}(r)$ and $D_{i,j}(r)$. Also recall from Lemma 9.1.4 a finite generating set of $\mathrm{Sp}_6(A)$. Set

$$\begin{aligned} \mathcal{S} = &\{B_{i,j}(\pm x_l), D_{i,j}(\pm x_l) : 1 \leq i, j \leq 3, i \neq j, 0 \leq l \leq k\} \\ &\cup \{B_{i,i}(\pm x_1^{\epsilon_1} \cdots x_k^{\epsilon_k}) : 1 \leq i \leq 3, \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\}, \end{aligned}$$

where $x_0 = 1$ and $x_l^0 = 1$. Then $\mathcal{S}$ is a finite generating set of $G(\leqslant \mathrm{Sp}_6(A))$.

Firstly, we verify assertion (A1). Define a *finite* subset $\mathcal{S}_0$ of G as follows:

$$
\begin{aligned}
\mathcal{S}_1 := \{& B_{1,2}(\pm x_l), B_{1,3}(\pm x_l), B_{2,3}(\pm x_l), D_{1,2}(\pm x_l), D_{1,3}(\pm x_l), D_{2,3}(\pm x_l), D_{3,2}(\pm x_l) : 0 \le l \le k\} \\
&\cup \{B_{i,i}(\pm x_1^{\epsilon_1} \cdots x_k^{\epsilon_k}) : 1 \le i \le 3, \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\} \\
&= \left\{ \left(\begin{array}{ccc|ccc} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & * & 1 & * & * & * \\ \hline & & & 1 & 0 & 0 \\ & 0 & & * & 1 & * \\ & & & * & * & 1 \end{array} \right) \right\} \cap \mathcal{S}.
\end{aligned}
$$

Set

$$C := \max\{\sup_{s \in \mathcal{S}_1} \|b_1(s)\|, \sup_{g,h \in G} \|b_1(gh) - b_1(g) - \rho(g) b_1(h)\|\} < \infty.$$

It is a key observation to this proof that N_1 commutes with $\mathcal{S}_1$. Therefore for any $l \in N_1$ and $s \in \mathcal{S}_1$, the following inequalities hold:

$$
\begin{aligned}
\|b_1(l) - \rho(s) b_1(l)\| &\le \|b_1(l) - b_1(sl)\| + \|b_1(s)\| + C \\
&\le \|b_1(l) - b_1(sl)\| + 2C \\
&= \|b_1(l) - b_1(ls)\| + 2C \\
&\le \|b_1(l) - b_1(l) - \rho(l) b_1(s)\| + 3C \\
&\le 4C.
\end{aligned}
$$

Note that the term in the very below is independent of the choice of $l \in N_1$. Now we define the following pairs of subgroups (G', N'); (H, H') of G:

$$
\begin{aligned}
G' :&= \left\{ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & W' & 0 & v' \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & {}^t(W')^{-1} \end{array} \right) : W' \in \mathrm{E}_2(A), v' \in S^{2*}(A^2) \right\} \\
&\trianglerighteq \{g \in G' : W' = I_2\} := N'; \\
H :&= \left\{ \left(\begin{array}{cc|cc} 1 & {}^t u & 0 & 0 \\ 0 & W' & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & {}^t(W')^{-1} u & {}^t(W')^{-1} \end{array} \right) : W' \in \mathrm{E}_2(A), u \in A^2 \right\} \\
&\trianglerighteq \{h \in H : W' = I_2\} := H'.
\end{aligned}
$$

Set a real number $\mathcal{K}$ by the following formula:

$$\mathcal{K} := \min\{\mathcal{K}(G', N'; \mathcal{S} \cap G', \rho\,|_{G'}), \mathcal{K}(H, H'; \mathcal{S} \cap H, \rho\,|_{H})\}$$

(recall the definition of the relative Kazhdan constant for property (T_B) from Definition 3.2.7). Then by the assumptions of relative property (T_B) for $\mathrm{E}_2(A) \ltimes S^{2*}(A^2) \trianglerighteq S^{2*}(A^2)$; and $\mathrm{E}_2(A) \ltimes A^2 \trianglerighteq A^2$, $\mathcal{K}$ is strictly positive. Therefore, we have the following inequalities by the inequalities above and Lemma 3.2.8:

$$\text{for any } \xi \in b_1(N_1), \qquad \|\xi - \rho(n')\| \leq 16\mathcal{K}^{-1}C,$$
$$\text{and } \|\xi - \rho(h')\| \leq 16\mathcal{K}^{-1}C.$$

Here $n' \in N'$ and $h' \in H'$ are arbitrary elements. Note the following:

$$N' = \{B_{2,3}(r), B_{2,2}(r), B_{3,3}(r) : r \in A\},$$
$$H' = \{D_{1,2}(r), D_{1,3}(r) : r \in A\}.$$

Now we observe the following correspondences among elementary symplectic matrices according to choices of alternating matrices ($J_3 \leftrightarrow L_3$):

alternating matrix:	J_3	$\longleftrightarrow$	L_3
	$B_{2,3}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{3,6}(r)$,
	$D_{1,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,3}(r)$,
	$D_{1,3}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,5}(r)$,
	$B_{2,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{3,4}(r)$,
	$B_{3,3}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{5,6}(r)$,
	$B_{1,2}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,4}(r)$,
	$B_{1,3}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,6}(r)$,
	$B_{1,1}(r)$	$\longleftrightarrow$	$\mathrm{SE}_{1,2}(r)$.

By Lemma 9.1.1 (item *(ii)* (1) and item *(i)* (3)), we have the following equalities: for any $r \in A$,

$$B_{1,3}(r) = [D_{1,2}(1), B_{2,3}(r)], \quad B_{1,1}(r) = B_{1,3}(-r)[B_{3,3}(-r), D_{1,3}(1)].$$

These equalities together with the inequalities in the paragraph above imply that for any $\xi \in b_1(N_1)$ and any $r \in A$,

$$\|\xi - \rho(B_{1,3}(r))\xi\| \leq 64\mathcal{K}^{-1}C, \text{ and } \|\xi - \rho(B_{1,1}(r))\xi\| \leq 128\mathcal{K}^{-1}C.$$

In a similar way, we also have $\|\xi - \rho(B_{1,2}(r))\xi\| \leq 64\mathcal{K}^{-1}C$ in the setting above. Therefore we conclude the following:

$$\text{for any } \xi \in b_1(N_1) \text{ and any } n \in N, \ \|\xi - \rho(n)\xi\| \leq 304\mathcal{K}^{-1}C.$$

Note that the right hand side is independent of the choices of $\xi \in b_1(N_1)$ and $n \in N$. By (a generalization to uc Banach space cases of) Corollary 2.1.14 and the trivial fact that $(B'_{\rho(N)})^{\rho(N)} = 0$, this inequality forces that

$$\sup_{\xi \in b_1(N_1)} \|\xi\| \leq 304\mathcal{K}^{-1}C.$$

This means that $b_1(N_1)$ is bounded. Thus we have shown assertion (A1).

Finally, we will confirm assertion (A2). Define a *finite* subset $\mathcal{S}_2$ of G as follows:

$$\begin{aligned}
\mathcal{S}_2 :=& \{B_{1,2}(\pm x_l), B_{1,3}(\pm x_l), B_{2,3}(\pm x_l), D_{1,2}(\pm x_l), D_{2,1}(\pm x_l), D_{1,3}(\pm x_l), D_{2,3}(\pm x_l) : 0 \le l \le k\} \\
&\cup \{B_{i,i}(\pm x_1^{\epsilon_1} \cdots x_k^{\epsilon_k}) : 1 \le i \le 3, \epsilon_1, \ldots, \epsilon_k \in \{0,1\}\} \\
=& \left\{ \left(\begin{array}{ccc|ccc} 1 & * & * & * & * & * \\ * & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ \hline & & & 1 & * & 0 \\ & 0 & & * & 1 & 0 \\ & & & * & * & 1 \end{array} \right) \right\} \cap \mathcal{S}.
\end{aligned}$$

Then the following holds true:

$$\text{for any } s \in \mathcal{S}_2 \text{ and } \gamma \in N_2, \text{ there exists } l \in N_1 \text{ such that}$$
$$s\gamma = \gamma s l.$$

More precisely, $l = e$ unless s is of the form of $D_{1,2}$ or $D_{2,1}$. Since we have already verified assertion (A1), we know that $b_1(N_1)$ is bounded. Therefore

$$C' := \max\{\sup_{s \in \mathcal{S}_2} \|b_1(s)\|, \sup_{g,h \in G} \|b_1(gh) - b_1(g) - \rho(g)b_1(h)\|, \sup_{l \in N_1} \|b_1(l)\|\}$$

is a finite real number. We have the following inequalities for any $\gamma \in N_2$ and $s \in \mathcal{S}_2$:

$$\begin{aligned}
\|b_1(\gamma) - \rho(s)b_1(\gamma)\| &\le \|b_1(\gamma) - b_1(s\gamma)\| + \|b_1(s)\| + C' \\
&\le \|b_1(\gamma) - b_1(s\gamma)\| + 2C' \\
&= \|b_1(\gamma) - b_1(\gamma s l)\| + 2C' \\
&\le \|b_1(\gamma) - b_1(\gamma) - \rho(\gamma)b_1(s) - \rho(\gamma s)b_1(l)\| + 4C' \\
&\le 6C'.
\end{aligned}$$

Here $l \in N_1$ is chosen such that $s\gamma = \gamma s l$ as in above. Note that the very below term in these inequalities is independent of the choice of $\gamma \in N_2$. Therefore we can verify assertion (A2) in a similar way to one in the proof of assertion (A1).

We have verified assertions (A1) and (A2), and thus have completed the proof of the theorem.

□

Proof. (Theorem 9.4.1) For simplicity, we give a proof for the case of $m = 3$. Let

$G = \mathrm{Sp}_6(A)$. we take the following subgroup H and subset P of G:

$$H = \mathrm{Sp}_2(A) := \left\{ \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & I_2 & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & I_2 \end{array} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_2(A) \right\} \leqslant \mathrm{Sp}_6(A),$$

$$\begin{aligned} P = \{ & B_{1,2}(r), C_{1,2}(r), D_{1,2}(r), D_{2,1}(r), B_{2,2}(r), C_{2,2}(r), \\ & B_{i,3}(r), C_{i,3}(r), D_{j,3}(r), D_{3,j}(r) : r \in A, 1 \leq i \leq 3, 1 \leq j \leq 2 \}. \end{aligned}$$

Define $Q \subseteq G$ as

$$Q = \bigcup_{h \in H} hPh^{-1}.$$

It is the same as in the proof of Theorem 9.3.1 that the pair (H, Q) in G satisfies conditions (ii), (iii), and (v). For condition (i), we use the facts that relative (T) implies relative $(\mathrm{T}_{\mathcal{L}_p})$ and relative (T_{C_p}) (Theorem 3.3.10 and Theorem 8.2.7).

For condition (iv), we utilize Theorem 9.4.2 together with Theorem 9.2.1 and Theorem 4.2.3. Theorem 8.1.7 ends our proof. □

Thus we have accomplished the proofs of Theorem D.

Remark 9.4.3. We warn that we have *no* result on property $(\mathrm{FF}_{[\mathcal{H}]})/\mathrm{T}$. This is because in the proof of the theorem above, it is crucial that relative (T) implies relative $(\mathrm{T}_{\mathcal{L}_p})$ and relative (T_{C_p}). Since Shalom's machinery does not work in symplectic setting, we do not have the statement that implies $(\mathrm{T}_{[\mathcal{H}]})$ for universal symplectic lattices. This means, if we try to apply Theorem 8.1.7, then condition (1) becomes an obstruction.

Chapter 10

Quasi-homomorphisms and stable commutator length

In this chapter, *all groups are assumes to be discrete, and we allow uncountable discrete groups.*

Until this point, we have examined quasi-cocycles *modulo trivial linear part.* In this chapter, we turn into studies on quasi-cocycles with trivial (real) coefficient, which are called *quasi-homomorphisms.* Firstly, we start from defining apparently different object, called the *scl* (Stable Commutator Length) on the commutator subgroup of a group.

Secondly, we define this celebrated concept, and see some nontrivial examples on F_2. In the views of theorems of Epstein–Fujiwara [EpFu] and Burger–Monod [BuMo1], [BuMo2], this concept has "higher rank versus rank 1" phenomena. Thirdly, we see the homogenization procedure for quasi-homomorphism, which is special for trivial coefficient cases. By utilizing this, we state the celebrated *Bavard duality theorem*, which connects (homogeneous) quasi-homomorphisms and scl.

Finally, we state a question of M. Abért and N. Monod on (un)boundedness of commutator length and scl. Then we see some answers to this question, one by A. Muranov [Mur] and another by the author [Mim2]. Specially, we prove Theorem E.

A comprehensive reference in this chapter is a book [Cal2] of D. Calegari, which has the name "*scl*."

10.1 Scl

Recall the assumption that in this chapter all groups are discrete, and that we allow uncountable discrete groups. Also recall that our commutator convention is: $[g, h] = ghg^{-1}h^{-1}$.

Definition 10.1.1. Let G be a group.

(*i*) The set of *single commutators* means

$$\{[g,h] : g,h \in G\}.$$

The *commutator subgroup*, written as $[G,G]$, is the subgroup of G generated by (single) commutators.

(*ii*) The *commutator length* $\mathrm{cl}\colon [G,G] \to \mathbb{Z}_{\geq 0}$ is the word length on $[G,G]$ with respect to the set of single commutators. Namely, for $g \in [G,G]$, $\mathrm{cl}(g)$ is the minimum of numbers which is needed to express g as a product of single commutators. We set $\mathrm{cl}(e) = 0$.

(*iii*) The *stable commutator length*, also written as *scl*, is

$$\mathrm{scl}\colon [G,G] \to \mathbb{R}_{\geq 0}; \quad \mathrm{scl}(g) := \lim_{n\to\infty} \frac{\mathrm{cl}(g^n)}{n}.$$

(*iv*) The group G is said to be *perfect* if $[G,G] = G$.

(*v*) Suppose G is perfect. Then the *commutator width* is defined as

$$\sup_{g\in G} \mathrm{cl}(g) \in \mathbb{N} \cup \{\infty\}.$$

.

Note that cl is subadditive, and hence that the limit in the definition of scl always exists.

Example 10.1.2. (*i*) Let $F_2 = \langle a, b\rangle$ be a free group of rank 2. Then

$$\mathrm{cl}([a,b]) = 1,$$

but there is a fact that

$$\mathrm{scl}([a,b]) = \frac{1}{2}.$$

We will sketch the proof of this fact in below (see Subsection 10.3.2). From the viewpoint of rigidity, what is important here is this value of scl is *strictly positive*. Namely, there exists $g \in [F_2, F_2]$ such that $\mathrm{scl}(g) > 0$ (in fact, in the case of F_2, any nontrivial element $g \in [F_2, F_2]$ has $\mathrm{scl}(g) \geq 1/2$).

(*ii*) Consider $G = \mathrm{SL}_3(\mathbb{Z})$. Then commutator relation

$$[E_{i,j}(r), E_{j,k}(s)] = E_{i,k}(rs) \quad (i \neq j, j \neq k, k \neq i; r, s \in \mathbb{Z})$$

shows G is perfect. For this G, it is known that

$$\mathrm{scl} \equiv 0 \quad \text{on } [G,G] = G.$$

One reason to extend the definition of cl to that of scl is that in many cases it is much more computable than cl (the meaning of this will be more clear after stating the Bavard duality theorem. see Subsection 10.3.2). Also, by considering scl (not cl itself), we become able to consider the comparison as in the example above: namely, "vanishing identically" versus "non-vanishing" in terms of scl. It has been revealed by works of several mathematicians that this concept scl has deep consequences and background. In this thesis, we are unable to see these beautiful theories due to the limit of contents. We refer to [Cal2] for this direction. For instance, Calegari [Cal1] shown that scl takes only rational values if G is a free group.

10.2 Quasi-homomorphisms

Here we define apparently quite distinct concept from scl, *quasi-homomorphisms.* As we have seen in Chapter 8 and Chapter 9, quasi-cocycles on *trivial* coefficient are in some sense the most difficult cases, because property (T_B)-type property gives *no* information on trivial linear part. In theories of quasi-homomorphisms, we usually consider the trivial *real* coefficient.

10.2.1 Definition

We state the definition of *quasi-homomorphisms* on a group. We note that it is (more) common to call them *quasimorphisms.* Compare with Definition 7.1.6 for the general definitions of quasi-cocycles and actual space of them. Also recall in this chapter we assume all groups are discrete, and we allow uncountable discrete groups.

Definition 10.2.1. Let G be a group.

(i) A map $\psi\colon G \to \mathbb{R}$ is called a *quasi-homomorphism* (or *quasimorphism*) if

$$\sup_{g,h\in G} |\psi(gh) - \psi(g) - \psi(h)| < \infty.$$

We write the left hand side of the formula above as $D(\psi)$, and call it the *defect* of ψ.

(ii) Define $QH(G)$ as the space of all quasi-homomorphisms on G.

(iii) Define $\widetilde{QH}(G)$ as the following quotient space:

$$\widetilde{QH}(G)$$
$$:=QH(G)/(\{\text{homomorphisms on } G\} + \{\text{maps } G \to \mathbb{R} \text{ with bounded range}\}).$$

We call $\widetilde{QH}(G)$ the *actual space of quasi-homomorphisms.*

(*iv*) We say a quasi-homomorphism ψ is *trivial* (or *cohomologically trivial*) if the image of ψ by the projection $QH(G) \twoheadrightarrow \widetilde{QH}(G)$ is zero. Namely, if ψ can be written as the sum of a homomorphism and a bounded map $G \to \mathbb{R}$.

Note that since bounded homomorphism (to $\mathbb{R}$) is a zero map, in the definition of $\widetilde{QH}(G)$ the sum of vector spaces is in fact a direct sum.

The following lemma is the special case of Lemma 7.1.8. We note that second bounded cohomology with trivial (real) coefficient is one of the most important object in the theory of bounded cohomology (for the definitions of bounded cohomology and of comparison map, see respectively Definition 7.1.5 and Definition 7.1.7).

Lemma 10.2.2. *Let G be a group. Then there is a natural isomorphism among vector spaces,*

$$\widetilde{QH}(G) \cong \mathrm{Ker}\Psi^2_{\mathrm{b}}.$$

Here $\mathrm{Ker}\Psi^2_{\mathrm{b}}$ *denotes the comparison map in degree* 2

$$\Psi^2 \colon H^2_{\mathrm{b}}(G) \to H^2(G),$$

where $H^2_{\mathrm{b}}(G) = H^2_{\mathrm{b}}(G; 1_G, \mathbb{R})$ *denotes the second bounded cohomology of G with trivial coefficient* $(1_G, \mathbb{R})$*.*

10.2.2 Examples on free groups

Let F_2 be a free group of rank 2, and a, b be free generators. In below, we will see that the actual space of quasi-homomorphisms $\widetilde{QH}(F_2)$ is enormous. In the view of Lemma 10.2.2 (recall that $H^2(F_2) = 0$), this implies behavior of bounded cohomology is quite different from ordinary cohomology (as we have seen with left regular representation coefficients in Subsection 7.1.2). First example of nontrivial quasi-homomorphism on F_2 is due to B. E. Johnson [Joh] (the author acknowledges P. de la Harpe for pointing this out to him). Here we see different examples, which are respectively due to R. Brooks [Bro] and P. Rolli [Rol].

Example 10.2.3. (*i*) (Brooks [Bro]) Consider a reduced word w in F_2 and fix it. Since F_2 is free, any element $g \in F_2$ has a canonical form as a reduced word. Set

$|g|_w :=$ maximum of numbers of disjoint w's appearing in the reduced word g.

Here we considering the order. For instance, $|ababab^{-1}|_{aba} = 1$, $|ab^2aba^{-3}|_{ab} = |abbaba^{-1}a^{-1}a^{-1}|_{ab} = 2$, $|b^{-1}a|_{ab^{-1}} = 0$, and $|(ab)^{-1}|_{ab} = |b^{-1}a^{-1}|_{ab} = 0$.

Then this $|\cdot|$ itself is *not* a quasi-homomorphism because in the multiplication process

$$(g, h) \mapsto gh,$$

there may be a plenty of cancellations. However, if we set

$$\psi_w(g) := |g|_w - |g|_{w^{-1}}$$

Then it can be shown that this ψ_w is a quasi-homomorphism. Indeed, $D(\psi_w) \leq 3$.

Next we see that there exists a reduced word w for which $\psi_w \in QH(F_2)$ is nontrivial. For example, $w = [a, b]$ does the job. Indeed, suppose ψ_w is trivial, namely, there exists a homomorphism f and a bounded map c such that

$$\psi_w = f + c.$$

Then by considering $[a, b]^n$ $(n \in \mathbb{N})$, we have

$$n = f([a, b]^n) + c([a, b]^n) = c([a, b]^n).$$

Here the last equality follows from the assumption that f is a homomorphism. Since c is bounded, this contradicts as we let $n \to \infty$.

We note in below that there is a geometric interpretation for this construction: consider the Cayley graph $\mathrm{Cay}(F_2; S)$ of F_2 with respect to the generating set $S = \{a^{\pm}, b^{\pm}\}$ (recall Definition 2.5.9). This is a 4-regular tree. Then a reduced word w can be seen as a directed path from e to w (as a group element). Recall F_2 acts on $\mathrm{Cay}(F_2; S)$ isometrically by left multiplication (, namely, $h \in F_2$ acts by $F_2 \ni l \mapsto hl$), and this induces *copies* of a directed path $w = \{h \cdot w : h \in F_2\}$.

Now observe that since F_2 is free, for any $g \in G$ there exists a *unique* geodesic path from e to g. With this setting, we have the following natural interpretation of $|\cdot|_w$:

$$|g|_w = \text{maximum of numbers of disjoint copies of } w \\ \text{which appears in the directed path from } e \text{ to } g.$$

(ii) (Rolli [Rol]) Take any function $\sigma\colon \mathbb{Z} \to \mathbb{R}$ with following two conditions:

(1) Anti-symmetry: for any $n \in \mathbb{Z}$, $\sigma(-n) = -\sigma(n)$.

(2) Boundedness: $\|\sigma\|_\infty := \sup\{|\sigma(n)| : n \in \mathbb{Z}\} < \infty$.

Note that condition (1) forces $\sigma(0) = 0$. Now note that any element in G can be expressed as the following "almost reduced word":

$$g = a^{e_1} b^{f_1} \cdots a^{e_k} b^{f_k},$$

where e_j, f_j $(1 \leq j \leq k)$ except e_1 and f_k must be non-zero. Then Rolli's construction $\sigma \mapsto \psi_\sigma$ is defined by the following formula:

$$\psi_\sigma(g) = \sum_{j=1}^{k} (\sigma(e_j) + \sigma(f_j)).$$

It is straightforward to check that $\psi_\sigma \in QH(F_2)$, which is left to the readers (indeed, $D(\psi_\sigma) \leq 3\|\sigma\|_\infty$).

We next show that there exists a nontrivial $\psi_\sigma \in QH(F_2)$. In fact, it is always the case, unless $\sigma \equiv 0$. To see this observe that if $\sigma \not\equiv 0$, then there exists $n_1, n_2 > 0$ such that $\sigma(n_2) \neq 0$ and

$$\frac{n_1}{n_2} \neq \frac{\sigma(n_1)}{\sigma(n_2)}.$$

We then claim ψ_σ is non-trivial. Indeed, suppose ψ_σ is trivial. Then there exists a homomorphism f and a bounded map c such that $\psi_\sigma = f + c$. For any $k \in \mathbb{N}$, consider $g^k = (a^{n_1}b^{n_1})^k$ and $h^k = (a^{n_1}b^{n_1})^k$. Then we have

$$\frac{\sigma(n_1)}{\sigma(n_2)} = \frac{2k\sigma(n_1)}{2k\sigma(n_2)} = \frac{\psi_\sigma(g^k)}{\psi_\sigma(h^k)} = \frac{kn_1 f(ab) + c_1}{kn_2 f(ab) + c_2}.$$

If we let $k \to \infty$, then we see that $f(ab)$ cannot be zero and that the very right hand side in the equality tends to n_1/n_2. Contradiction occurs.

In fact, both of the examples above implies the following:

Proposition 10.2.4. *The space $\widetilde{QH}(F_2)$ is infinite dimensional.*

It is not difficult to see this space in fact has continuum dimension. In the view of Lemma 10.2.2, this means the space $H_{\mathrm{b}}^2(F_2)$ has continuum dimension.

10.2.3 Higher rank versus rank one

Recall that Brooks' quasi-homomorphisms (item (i) in Example 10.2.3) has a geometric interpretation. For the construction, the following point is essential: for any two distinct points s, t in the Cayley graph $\mathrm{Cay}(F_2; S)$ (, which is tree) there exists a unique geodesic path from s to t. Epstein–Fujiwara [EpFu] extended Brooks' construction in the case of that G is hyperbolic (, namely, a Cayley graph associated with a finite generating set is a hyperbolic space in the sense of Gromov, see Definition 2.6.19) as follows: a hyperbolic group G acts on its Cayley graph $X = \mathrm{Cay}(G; S)$ (Definition 2.5.9) by isometries from the left, where S is an (arbitrarily) fixed finite symmetric generating set of G. Fix a base point $x_0 \in X$ (for instance, $x_0 = e$). Fix an (oriented) path w on X, and consider the *copies* of w:

$$\{\gamma \cdot w : \gamma \in G\}.$$

For $g \in G$, set $c_w(g) \in \mathbb{Z}_{\geq 0}$ by the following formula:

$$c_w(g) := d(x_0, g \cdot x_0) - \inf_{p:\text{path}} \{|p| - \max\{\text{number of disjoint copies of } w \text{ on } p\}\}.$$

Here p moves among (oriented) paths from x_0 to $g \cdot x_0$, d denotes the distance on X, and $|p|$ means the length of the path p. Finally, define $\phi_w \colon G \to \mathbb{R}$ as

$$\phi_w(g) := c_w(g) - c_{w^{-1}}(g).$$

Epstein–Fujiwara shown that this ϕ_w is in $QH(G)$, whenever $|w| \geq 2$. It follows from the hyperbolicity of (X, d), and this construction has its origin in (the geometric interpretation) of Brooks' construction of quasi-homomorphisms on F_2. They also examined the necessary and sufficient condition of a hyperbolic group G such that there exists a path w with $[\phi_w] \neq 0$ in $\widetilde{QH}(G)$. Thus the following theorem has been proven:

Theorem 10.2.5. (*Epstein–Fujiwara* [EpFu]) *Let G be a non-elementary hyperbolic group. Then the space $\widetilde{QH}(G)$ is infinite dimensional.*

In particular, $H^2_{\mathrm{b}}(G)$ is infinite dimensional.

Recall from Definition 7.2.1 a hyperbolic group is *non-elementary* if G is neither finite nor virtually $\mathbb{Z}$ (a group is said to be virtually $\mathbb{Z}$ if it contains $\mathbb{Z}$ with finite index). This condition that hyperbolic group G is non-elementary is equivalent to one that G is not virtually abelian; and to one that G is not amenable. This condition assures existence of such ψ_w (we need some modification to the construction from the original Brooks argument) which is *non*trivial. For example, $\mathbb{Z}$ is a hyperbolic group, but all quasi-homomorphisms on $\mathbb{Z}$ are trivial.

In fact, the same assertion holds true for any amenable group G. To see this, let $E \colon \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$ be a invariant mean. For any $\psi \in G$, define $\delta_g(h) := \psi(gh) - \psi(h)$. Then define

$$\psi' \colon G \to \mathbb{R};\ g \mapsto E(\delta_g).$$

Then by G-invariance of E, this ψ' is a homomorphism. Also by construction, the difference $\psi - \psi'$ is a bounded map. This means $\widetilde{QH}(G) = 0$ for amenable groups.

Remark 10.2.6. In Theorem 10.2.5, the following properties for a group G is essential to deduce that $\widetilde{QH}(G)$ is infinite dimensional (in particular, that $\widetilde{QH}(G) \neq 0$.)

(a) The group G acts on a hyperbolic space X isometrically $\alpha \colon G \curvearrowright X$.

(b) The action α is *properly discontinuous*, namely, for any $x \in X$ and any $r > 0$, the set

$$\{g \in G : d(x, g \cdot x) < r\}$$

is finite.

(c) There exists $g \in G$ of infinite order such that g acts on X by a *hyperbolic* isometry, and for any $n > 0$ and any $h \in G$, $hg^n h^{-1} \neq g^{-n}$. In addition, G is not virtually $\mathbb{Z}$.

For details of condition (c), we refer to [BeFu] and [CalFu]. These conditions are explicitly written in the paper [CalFu], and there they show existence of some bound form below for scl.

We note that hyperbolic groups can be seen as discrete analogue of rank 1 groups.

In contrary, Burger–Monod [BuMo1], [BuMo2] (Theorem 7.2.3) shown that totally higher rank lattices have (TT). This in particular implies that for any totally higher rank lattice Γ, $\widetilde{QH}(\Gamma) = 0$ holds. In fact, they shown the following:

Theorem 10.2.7. (*Burger–Monod* [BuMo1], [BuMo2]) *For any higher rank lattice Γ in the sense in Chapter* 0*, we have*

$$\widetilde{QH}(\Gamma) = 0.$$

Recall that for the definition of a *higher rank lattice* in a semisimple algebraic group G, we task the irreducibility condition on each rank 1 factor, namely we assume for each rank 1 factor G_i, the image of Γ by the projection $G \twoheadrightarrow G_i$ is dense in G_i. A basic example is $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$. This group is isomorphic to

$$\Gamma = \{(g, \tilde{g}) : g \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])\} \leqslant \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) = G,$$

where $g \mapsto \tilde{g}$ is the conjugation $\sqrt{2} \mapsto -\sqrt{2}$. This $\Gamma \leqslant G$ is an irreducible lattice in a rank 2 (but not totally higher rank) group, and hence $\mathrm{SL}_2(\mathbb{Z}[\sqrt{2}])$ is a higher rank lattice in the sense in Chapter 0.

Therefore in terms of QH, there is a “higher rank versus rank 1” phenomena.

10.3 Homogenization of quasi-homomorphism

In this section, we introduce a powerful tool in the study of quasi-homomorphism, which is called *homogenization*.

10.3.1 Homogeneous quasi-homomorphisms

Definition 10.3.1. Let G be a group.

(i) A quasi-homomorphism ϕ on G is said to be *homogeneous* if ϕ is a homomorphism on each cyclic subgroup of G. Namely, if for any $g \in \Gamma$ and $m \in \mathbb{Z}$, $\phi(g^m) = m \cdot \phi(g)$ holds.

(ii) We define $HQH(G)$ as the space of all homogeneous homomorphisms on G.

(iii) We define $\widetilde{HQH}(G)$ as the following vector space:

$$\widetilde{HQH}(G) := HQH(G)/\{\text{homomorphisms } G \to \mathbb{R}\}.$$

Note that any bounded homogeneous quasi-homomorphism is identical to the zero map.

Apparently it seems that homogeneity condition on quasi-homomorphism is strong, but the next proposition shows there is no loss of generality to concentrate on homogeneous cases:

Proposition 10.3.2. *Let G be a group. Then for any $\psi \in QH(G)$, there exists a unique element $\phi \in HQH(G)$ such that*

$$\|\psi - \phi\|_\infty \leq D(\psi).$$

Here $D(\psi)$ denotes the defect of ψ. More precisely, ϕ is given by the following formula:

$$\phi(g) = \lim_{n\to\infty} \frac{\psi(g^n)}{n}.$$

Proof. Take any element $g \in G$. Let $m \in \mathbb{N}$. Consider $g^{2^m} = g^{2^{m-1}} \cdot g^{2^{m-1}}$ and obtain

$$|\psi(g^{2^m}) - 2\psi(g^{2^{m-1}})| \leq D.$$

Here $D = D(\psi)$ is the defect. This inequality is rewritten as

$$\left| \frac{\psi(g^{2^m})}{2^m} - \frac{\psi(g^{2^{m-1}})}{2^{m-1}} \right| \leq \frac{D}{2^m}.$$

This implies that $(\psi(g^{2^m})/2^m)_m$ is a Cauchy sequence in $\mathbb{R}$, and thus there exists a limit. We define $\phi(g)$ as this limit. It is obvious that

$$\|\psi - \phi\|_\infty \leq D,$$

and hence $\phi \in QH(G)$. To see $\phi \in HQH(G)$, for any $n \in \mathbb{N}$ we have

$$\begin{aligned} |\phi(g^n) - n\phi(g)| &= \lim_{m\to\infty} \frac{|\psi(g^{n2^m}) - n\psi(g^{2^m})|}{2^m} \\ &\leq \lim_{m\to\infty} \frac{(n-1)D}{2^m} = 0. \end{aligned}$$

Therefore $\phi(g^n) = n\phi(g)$ for any $n \geq 0$. Also observe for any $n > 0$

$$D(\phi) \geq |\phi(g^n) + \phi(g^{-n})| = n|\phi(g) + \phi(g^{-1})|.$$

Hence for any $g \in G$,

$$\phi(g^{-1}) = -\phi(g).$$

This means $\phi \in HQH(G)$.

The uniqueness of such ϕ in the statement is trivial by homogeneity. □

Corollary 10.3.3. *For any group G, there is a natural isomorphism*

$$\widetilde{QH}(G) \cong \widetilde{HQH}(G).$$

Therefore, from this point, we focus on homogeneous quasi-homomorphisms.

10.3.2 The Bavard duality theorem

Note that any perturbation of a quasi-homomorphism by a bounded map is also a quasi-homomorphism. This implies for quasi-homomorphisms, an exact value on each point has no meaning. One good point for considering *homogeneous* quasi-homomorphism is then each exact value enjoys nice properties.

We need one fact about commutators. For the proof, see Subsection 2.2.3, Lemma 2.24 in [Cal2].

Lemma 10.3.4. *For any g, h in a group and $n \in \mathbb{N}$, $g^{2n}h^{2n}(gh)^{-2n}$ is a product of n commutators.*

Here we state basic properties of homogeneous quasi-homomorphisms.

Lemma 10.3.5. *Let G be a group. Then for any $\phi \in HQH(G)$, the following hold:*

- (i) *Let $H \leqslant G$ is a subgroup. If ϕ is bounded on H, then $\phi\mid_H \equiv 0$.*
- (ii) *For any commuting pair $g, h \in G$, $\phi(gh) = \phi(g) + \phi(h)$. In particular, if G is abelian, then $\widetilde{HQH}(G) = 0$.*
- (iii) *The map ϕ is constant on each conjugacy class. Namely, for any $g \in G$ and any $t \in G$,*
$$\phi(tgt^{-1}) = \phi(g).$$
- (iv) *There is an equality*
$$D(\phi) = \sup_{g,h \in G} |\phi([g,h])|.$$

Note that item (ii) is a special case of the fact that $\widetilde{HQH} = 0$ for amenable groups, which we have seen in the previous subsection.

Proof. Item (i) is clear.

Note that by homogeneity of ϕ for any g, h,

$$\phi(gh) - \phi(g) - \phi(h) = \lim_{n\to\infty} \frac{\phi((gh)^n) - \phi(g^n) - \phi(h^n)}{n}.$$

Item (ii) then follows because if $gh = hg$,

$$|\phi((gh)^n) - \phi(g^n) - \phi(h^n)| \leq 2D(\phi) + |\phi((gh)^n g^{-n} h^{-n})| = 2D(\phi).$$

For item (iii), since $(tgt^{-1})^n = tg^n t^{-1}$,

$$|\phi((tgt^{-1})^n) - \phi(g^n)| \leq 2D(\phi) + 2|\phi(t)|$$

for any $n \in \mathbb{N}$. Again by letting $n \to \infty$, we get the conclusion.

For item (iv), by item (iii) we have

$$|\phi([g,h])| \leq D(\phi) + |\phi(ghg^{-1}) + \phi(h^{-1})| = D(\phi) + |\phi(h) + \phi(h^{-1})| = D(\phi),$$

and hence $D(\phi) \geq \sup_{g,h\in G} |\phi([g,h])|$. To get the inverse inequality, fix $n \geq 1$. By the definition of the defect, there exist $g, h \in G$ such that

$$|\phi(gh) - \phi(g) - \phi(h)| \geq D(\phi) - \frac{1}{n^2}.$$

By Lemma 10.3.4, $g^{2n}h^{2n}(gh)^{-2n}$ is a product of n commutators $c_1, \ldots, c_n$. Therefore, by homogeneity, we have

$$\begin{aligned} 2nD(\phi) - \frac{2}{n} &\leq |\phi(c_1 \cdots c_n)| \\ &\leq (n-1)D(\phi) + \sum_{i=1}^{n} |\phi(c_i)|. \end{aligned}$$

Since each c_i is commutator, $|\phi(c_i)| \leq D(\phi)$ holds. Therefore, there exists some commutator c_i such that

$$|\phi(c_i)| \geq D(\phi) - \frac{2}{n}.$$

This confirms the inequality $D(\phi) \leq \sup_{g,h\in G} |\phi([g,h])|$. □

As we mentioned in item (i) in Example 10.1.2, we now proceed in showing the following:

Lemma 10.3.6. *Let F_2 be a free group and a, b be free generators. Then*

$$\mathrm{scl}([a,b]) = \frac{1}{2}$$

on F_2.

Proof. First we show that $\mathrm{scl}([a,b]) \leq 1/2$. This follows from Lemma 10.3.4. Indeed, by putting $g = [a,b]$ and $h = b$ we have that $[a,b]^{2n}[b^{2n},a]$ is a product of n commutators. This implies for any $n \in \mathbb{N}$,

$$\mathrm{cl}([a,b]^{2n}) \leq n+1.$$

This shows the desired inequality.

Next, we show the inverse inequality: $\mathrm{scl}([a,b]) \geq 1/2$. Here we use the following fact:

$$\text{there exists } \phi \in HQH(F_2) \text{ such that } D(\phi) = \phi([a,b]) > 0.$$

Here for a construction, consider Brooks' quasi-homomorphism for $w = [a,b]$ and take homogenization. We take this ϕ and put $D = D(\phi) > 0$.

Suppose $[a,b]^n$ is written as m_n commutators $c_1,\dots,c_{m_n}$. Then by applying ϕ and taking absolute values, we have

$$\begin{aligned} nD &= |\phi(c_1\cdots c_{m_n})| \\ &\le (m_n-1)D + \sum_{i=1}^{m_n} |\phi(c_i)| \\ &\le (2m_n-1)D. \end{aligned}$$

Here we use (the easier direction of) item (iv) of Lemma 10.3.5. Therefore,

$$\frac{m_n}{n} \ge \frac{1}{2} + \frac{1}{2n}.$$

This shows $\mathrm{scl}([a,b]) \ge 1/2$, as claimed. □

The argument above indicates that there is connection between homogeneous quasi-homomorphisms and stable commutator length. Ch. Bavard [Bav] revealed the following theorem, which is the fundamental theory in the study of scl. Recall in this chapter we assume all groups are discrete and we allow uncountable groups.

Theorem 10.3.7. (*The Bavard duality theorem,* [Bav]) *Let G be a discrete group.*

(i) *There is the following formula:*

$$\text{for any } g \in [G,G],\ \ \mathrm{scl}(g) = \sup_{\phi\in HQH(G)} \frac{\phi(g)}{2D(\phi)}.$$

(ii) *The following two conditions are equivalent:*

- (a) *The equality $\widetilde{HQH}(G) = 0$ holds. In other words, the comparison map in degree 2,*
$$\Psi^2_{\mathrm{b}}\colon H^2_{\mathrm{b}}(G) \to H^2(G)$$
is injective.
- (b) *The* scl *(on $[G,G]$) vanishes identically.*

Proof. For item (i), the inequality

$$\mathrm{scl}(g) \ge \sup_{\phi\in HQH(G)} \frac{\phi(g)}{2D(\phi)}$$

can be shown in a similar argument to that in the proof of (the latter half) of Lemma 10.3.6. The proof of the inverse inequality

$$\mathrm{scl}(g) \le \sup_{\phi\in HQH(G)} \frac{\phi(g)}{2D(\phi)}$$

is more involved, and here we will not present it. We refer to Section 2.5 in [Cal2].

Item (ii) is a corollary of item (i). Indeed, thanks to item (i), condition (b) is equivalent to the condition that every $\phi \in HQH(G)$ is zero on $[G,G]$. Item (iv) of Lemma 10.3.5 implies that this condition is equivalent to one that every $\phi \in HQH(G)$ is a homomorphism. This is the statement of condition (a). □

10.4 Question of Abért and Monod

We state a question of Abért and Monod, which is for instance stated in the ICM proceeding paper [Mon2] of Monod. The first solution to this question is given by A. Muranov [Mur], and we will see the statement of the Muranov theorem.

10.4.1 The question

First, recall the definition of bounded generation in Definition 4.2.4.

We see in Theorem 10.2.7 (or Theorem 7.2.3) that specially for $G = \mathrm{SL}_3(\mathbb{Z})$, $\widetilde{HQH}(G) = 0$ holds. However this specific result directly follows from the bounded generation of Carter–Keller for $\mathrm{SL}_3(\mathbb{Z})$ as follows: recall Cater–Keller's bounded generation (Theorem 4.2.5) states the following:

$$G = \mathrm{SL}_3(\mathbb{Z}) \text{ is boundedly generated by elementary matrices.}$$

By commutator relation, every elementary matrix in $\mathrm{SL}_3(\mathbb{Z})$ is a single commutator. Thus the statement above implies the following:

$$\text{the commutator length on } [G,G] \text{ is bounded, namely, } \sup_{g \in [G,G]} \mathrm{cl}(g) < \infty.$$

In this case $[G,G] = G$, and hence the statement above means the commutator width is infinite. This certainly implies the vanishing of scl and the Bavard duality theorem 10.3.7 leads us to the conclusion.

In the connection to this argument, Abért asked the following question at his website. Monod exhibited this question in his ICM proceedings paper [Mon2], and now this question is called *a question of Abért–Monod*.

Question 10.4.1. (Problem Q [Mon2])

(i) Does there exist a example of groups G which enjoys both of the following properties?

  (1) The commutator length on $[G,G]$ is unbounded:

$$\sup_{g \in [G,G]} \mathrm{cl}(g) = \infty.$$

(2) The group G has trivial actual (homogeneous) quasi-homomorphism space, namely,

$$\widetilde{HQH}(G) = 0.$$

Equivalently, the stable commutator length vanishes identically on $[G, G]$:

$$\mathrm{scl} \equiv 0.$$

(*ii*) Let K be a (commutative) field of infinite transcendence degree over its subfield. Let $m \geq 3$. Then, is the space

$$\widetilde{HQH}(\mathrm{SL}_m(K[x]))$$

is the zero-space?

The background of question (*ii*) is the following theorem of Dennis–Vaserstein [DeVa], which is based on the van der Kallen theorem of *un*bounded generation (Theorem 4.3.1).

Proposition 10.4.2. ([DeVa]) *Let K a field as in question (ii) in Question* 10.4.1. *Then for any $m \geq 3$, $G = \mathrm{SL}_m(K[x])$ has infinite commutator width:*

$$\sup_{g \in G} \mathrm{cl}(g) = \infty.$$

Note that $[G, G] = G$ follows from commutator relation.

Proof. Suppose $\sup_{g \in G} \mathrm{cl}(g) < \infty$ happens for some $G = \mathrm{SL}_m(K[x])$. Then we will show that $G' = \mathrm{SL}_{2m}(K[x])$ is boundedly generated by elementary matrices. This contradicts Theorem 4.3.1. The key observation is the following formulae: for any $V, W, X \in \mathrm{SL}_m(K[x])$,

$$\begin{pmatrix} [V,W] & 0 \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} 0 & V \\ -V^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -W^{-1} \\ W & 0 \end{pmatrix} \begin{pmatrix} (WV)^{-1} & 0 \\ 0 & WV \end{pmatrix};$$

$$\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -X^{-1} & I_m \end{pmatrix} \begin{pmatrix} I_m & X - I_m \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_m & 0 \\ I_m & I_m \end{pmatrix} \begin{pmatrix} I_m & -I_m \\ 0 & I_m \end{pmatrix};$$

$$\begin{pmatrix} 0 & X \\ X^{-1} & 0 \end{pmatrix} = \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_m & 0 \\ -X^{-1} & I_m \end{pmatrix} \begin{pmatrix} I_m & X \\ 0 & I_m \end{pmatrix}.$$

These formulae show for each single commutator $[V, W] \in \mathrm{SL}_{2m}(K[x])$, the image of it in the left upper corner of $\mathrm{SL}_{2m}(K[x])$ is a product of elementary matrices.

Finally, recall the definition of stable range (Definition 4.1.4). Proposition 4.1.5 means that the following: if a commutative ring A has stable range r, then for any $m \geq r$, $\mathrm{SL}_m(A)$ is boundedly generated by $\mathrm{SL}_r(A)$, sitting in the left upper corner, together with the set of elementary matrices. Since $\mathrm{sr}(K[x]) \leq 2$ (because this ring is euclidean), this result in particular implies that G' is boundedly generated by G (in the left upper corner) and elementary matrices. Therefore, G' must be boundedly generated by elementary matrices. This leads us to a contradiction, and ends our proof. $\square$

10.4.2 A theorem of Muranov

The answer to question (i) of the Abért–Monod question is positive. The first example is brought by A. Muranov in his paper [Mur].

Theorem 10.4.3. (*Muranov* [Mur]) *There exists a simple group G generated by* 2 *elements which satisfies*

$$\sup_{g \in G} \mathrm{cl}(g) = \infty.$$

together with

$$\mathrm{scl} \equiv 0.$$

The proof of this theorem is involved (in fact Muranov shown much more properties on the group G above), and we refer to his original paper [Mur]. One idea of the construction is to use small cancellation theory on group presentations, and to define presentations recursively. This theory has a background in the theory of hyperbolic groups, and the group G above is a direct limit of a sequence of hyperbolic groups (note that a hyperbolic group itself cannot have the properties in the theorem by Theorem 10.2.5).

Note that in finite group theory, it is known that any simple finite group has commutator width 1. This is a deep theorem, and called the Ore conjecture. We note that in contrast, Caprace–Fujiwara [CapFu] constructed nontrivial quasi-homomorphisms on some class of Kac–Moody groups which are simple. By the Bavard duality theorem (Theorem 10.3.7), this provides with examples of (infinite) simple groups on which scl does *not* vanish identically.

10.5 Quasi-homomorphisms on elementary groups

In this section, we study quasi-homomorphisms on the elementary group over certain (commutative) rings. Recall from Chapter 0 we assume all rings are associative and unital. Also recall in this chapter groups are assumed to be discrete, and we allow uncountable discrete groups. We show Theorem E, which is the main result of the paper [Mim2] of the author. In particular, this results answers the second question (question (ii) of Question 10.4.1) of Abért and Monod, and thus provide with the second class of examples in of question (i). This class has completely different background from one appears in Muranov's theorem.

10.5.1 Precise statement of Theorem D

In the paper [Mim2] of the author, we investigated quasi-homomorphisms on elementary groups over certain rings. Here we restate Theorem E with full generality:

Theorem 10.5.1. ([Mim2]) *Let A be a commutative ring and $m \geq 6$ be a natural number. Suppose the group $G = \mathrm{E}_m(A)$ is boundedly generated by $\mathrm{E}_2(A)$ (in the left upper corner) and the set of single commutators in G. Then*

$$\widetilde{HQH}(\mathrm{E}_m(A)) = 0$$

holds true.

In particular, the following hold true:

(*i*) *Suppose A satisfies GE_2-condition in the sense of P. M. Cohn; and $G = \mathrm{SL}_m(A)$ is boundedly generated by $\mathrm{SL}_2(A)$ (in the left upper corner) and the set of single commutators in $\mathrm{SL}_m(A)$. Then*

$$\widetilde{HQH}(\mathrm{SL}_m(A)) = 0.$$

(*ii*) *Suppose A is a commutative principal ideal ring. If A satisfies GE_2-condition, then*

$$\widetilde{HQH}(\mathrm{SL}_m(A)) = 0.$$

(*iii*) *Suppose A is a euclidean domain. Then*

$$\widetilde{HQH}(\mathrm{SL}_m(A)) = 0.$$

Here the GE_2-condition for a commutative ring A is the following condition:

$$\mathrm{E}_2(A) = \mathrm{SL}_2(A).$$

We note that the original definition of GE_2-condition for a ring R is the following condition

$$\mathrm{GE}_2(R) = \mathrm{GL}_2(R),$$

and this makes sense even for noncommutative rings. Here $\mathrm{GE}_2(R)$ denotes the multiplicative group in $\mathrm{M}_2(R)$ generated by elementary matrices and diagonal invertible matrices. We refer to [Coh] for details and the proof of equivalence.

Item (*iii*) of Theorem 10.5.1 answers question (*ii*) of Abért–Monod (Question 10.4.1), with the assumption that $m \geq 6$. The following corollary follows from Proposition 10.4.2 (and the Bavard duality).

Corollary 10.5.2. ([Mim2]) *Let K be a (commutative) field of infinite transcendence degree over its subfield. Then for $m \geq 6$, the group $G = \mathrm{SL}_m(K[x])$ satisfies*

$$\sup_{g \in G} \mathrm{cl}(g) = \infty.$$

together with

$$\mathrm{scl} \equiv 0.$$

Recall from Subsection 4.3.1 for the examples of such K. Main examples are $\mathbb{C}, \mathbb{R}$ and some countable field, for instance,

$$K = \mathbb{Q}(e^{\sqrt{p_1}}, e^{\sqrt{p_2}}, \ldots),$$

where $(p_n)_n$ is a strictly increasing sequence of primes, and the symbol above means the field generated by these elements.

In addition, we note that it is not known whether $\mathrm{SL}_{m\geq 3}(\mathbb{Q}[x])$ is boundedly generated by elementary matrices. Theorem 10.5.1 shows that, at least the stable commutator length vanishes if $m \geq 6$.

We will prove Theorem 10.5.1 in the next subsection, and here we state some needed lemma.

Firstly, we state the following lemma, which follows directly from Lemma 10.3.5:

Lemma 10.5.3. *Let G a group. Let $g \in G$ be a element which is conjugate to its inverse. Then for any $\phi \in HQH(G)$, $\phi(g) = 0$.*

Let R be a ring and $m \geq 2$. We call an element $g \in \mathrm{E}_m(R)$ a *unit upper* (respectively *lower*) *triangular matrix* if all diagonal entries are 1 and all of the entries below (respectively above) the diagonals are 0. We define $U_m R$ (respectively $L_m R$) as the group of all unit upper (respectively lower) triangular matrices of degree m.

Lemma 10.5.4. *Let ϕ be a homogeneous quasi-homomorphism on $\mathrm{E}_m(R)$. If $n \geq 3$, then the following hold:*

(i) For any elementary matrix $s \in \mathrm{E}_m(R)$, $\phi(s) = 0$.

(ii) For any $h \in (U_m R) \cup (L_m R)$, $\phi(h) = 0$.

Proof. Item (i) follows from Lemma 10.5.3 because for $n \geq 3$, an elementary matrix is conjugate to its inverse. Hence, ϕ is bounded on $U_m R$ and $L_m R$. By the homogeneity of ϕ, we obtain item (ii). □

The following observation plays an important role in this paper.

Lemma 10.5.5. *Let Γ be a group and $H \leqslant \Gamma$ be a subgroup. Let ϕ be a homogeneous quasi-homomorphism on Γ. Suppose ϕ vanishes on H. Then for any $h \in H$ and any $g \in N_\Gamma(H)$, $\phi(hg) = \phi(gh) = \phi(g)$ holds. Here $N_\Gamma(H)$ means the normalizer of H in Γ.*

Proof. We will only show $\phi(hg) = \phi(g)$. Let D be the defect of ϕ. By employing the condition $gHg^{-1} \leqslant H$ repeatedly, we have that for any $n \in \mathbb{N}$, there exists an element $h' \in H$ such that $(hg)^n = h'g^n$. Hence, we obtain that

$$n \cdot |\phi(hg) - \phi(g)| \leq D.$$

This ensures the conclusion. □

10.5.2 Key proposition and proof

The following proposition is the key to proving Theorem 10.5.1:

Proposition 10.5.6. ([Mim2], [Mim4]) *Let R be a (possibly noncommutative) ring. Suppose $g \in \mathrm{E}_2(R)$ and $s \in \mathrm{E}_2(R)$ satisfy the following conditions:*

(i) The matrix $g + g^{-1} \in \mathrm{M}_2(R)$ commutes with s.

(ii) The equality $(s - I_2)^2 = 0$ holds.

Then for any homogeneous quasi-homomorphism ϕ on $\mathrm{E}_6(R)$, we have

$$\phi(gs) = \phi(sg) = \phi(g) + \phi(s).$$

Here we put $\mathrm{E}_2(R)$ in the left upper corner of $\mathrm{E}_6(R)$, and view $g, s \in \mathrm{E}_2(R)$ as elements in $\mathrm{E}_6(R)$ with this identification.

In particular, if a ring A is commutative, the following holds: Let $g \in \mathrm{E}_2(A)$. Let s in $\mathrm{E}_2(A)$ be an elementary matrix. Then for any homogeneous quasi-homomorphism ϕ on $\mathrm{E}_6(A)$, we have

$$\phi(gs) = \phi(sg) = \phi(g) + \phi(s).$$

Here we put $\mathrm{E}_2(A)$ in the left upper corner of $\mathrm{E}_6(A)$.

The latter assersion follows form the following easy fact: if A is commutative, then for any $g \in \mathrm{E}_2(A)$, $g+g^{-1} \in \mathrm{M}_2(A)$ is a diagonal matrix with the same diagonal entries, and is a multiple of the identity element.

For the sake of simplicity, we here give a proof of the latter assertion of the proposition. The basic argument in below also works for general cases. Before proceeding into the proof, we see some facts.

Firstly, we utilize the following result of Dennis–Vaserstein (Lemma 18 of [DeVa]), with $k = 3$.

Lemma 10.5.7. ([DeVa]) *Let J be a (possibly noncommutative) ring and $p, q, r \in \mathrm{GL}_1(J)$ such that $pqr = 1$. Let Δ be the diagonal matrix in $\mathrm{GL}_3(J)$ with the diagonal part $p, q,$ and r. Then $\Delta \in (L_3 J)(U_3 J)(L_3 J)(U_3 J)$.*

Let g and s be as in Proposition 10.5.6. We set $J = \mathrm{M}_2(A)$ in the lemma above and set $p = sg,\ q = g^{-1}$ and $r = s^{-1}$. We need the following explicit form:

$$\begin{pmatrix} I_2 & 0 & 0 \\ p^{-1} & I_2 & 0 \\ 0 & q^{-1} & I_2 \end{pmatrix} \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{pmatrix} = X_1 Y X_2, \text{ where } X_1 = \begin{pmatrix} I_2 & I_2 - p & 0 \\ 0 & I_2 & I_2 - pq \\ 0 & 0 & I_2 \end{pmatrix}^{-1},$$

$$Y = \begin{pmatrix} I_2 & 0 & 0 \\ I_2 & I_2 & 0 \\ 0 & I_2 & I_2 \end{pmatrix}, \text{ and } X_2 = \begin{pmatrix} I_2 & (I_2 - p)q & 0 \\ 0 & I_2 & (I_2 - pq)r \\ 0 & 0 & I_2 \end{pmatrix}.$$

By applying Lemma 10.3.5 and Lemma 10.5.5 (for $H = L_3J$), we conclude that the evaluation of the left-hand side of the first equality in above by ϕ is equal to

$$\phi(p) + \phi(q) + \phi(r) = \phi(sg) - \phi(g) - \phi(s) = \phi(sg) - \phi(g).$$

Here we see p, q, r, sg, g, and s as elements in $\mathrm{E}_2(A)$. It has no problem because each of three diagonal $\mathrm{E}_2(A)$ parts in $\mathrm{E}_6(A)$ can be conjugated to each other by permutation matrices. Hence, in order to show Proposition 10.5.6, it is enough to show that $\phi(X_1YX_2) = 0$ for any g and s as in Proposition 10.5.6.

Proof. (Proposition 10.5.6, the latter assertion) We may assume that $s \in U_2A$ without loss of generality. Note that X_1YX_2 is conjugate to X_2X_1Y. Computation shows that $X_2X_1Y = ZXY$, where

$$X = \begin{pmatrix} I_2 & -(p-I_2)(q-I_2) & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix} \text{ and } Z = \begin{pmatrix} I_2 & 0 & -(p-I_2)(q-I_2)(pq-I_2) \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}.$$

Indeed, the key observation here is that if one regards X_2X_1 as an element in $\mathrm{GL}_3(\mathrm{M}_2(A))$, then the $(2,3)$-th entry of X_2X_1 is $0(=0_2)$. This is because $s + s^{-1} = 2I_2$. We set $x = -(p-I_2)(q-I_2) = -(sg-I_2)(g^{-1}-I_2)$ and $z = -(p-I_2)(q-I_2)(pq-I_2) = -(sg-I_2)(g^{-1}-I_2)(s-I_2)$ as elements in $\mathrm{M}_2(A)$. By definition, g and s can be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (a,b,c,d \in A) \quad \text{and} \quad s = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \quad (f \in A).$$

By substituting these matrix forms of g and s for the expressions of x and z, we continue calculations as follows:

$$x = -\begin{pmatrix} a+fc-1 & b+fd \\ c & d-1 \end{pmatrix}\begin{pmatrix} d-1 & -b \\ -c & a-1 \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix},$$
$$z = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

Here each $*$ respectively represents a certain element in A.

Next we define a (non-unital) subring N of $\mathrm{M}_2(A)$ by

$$N = \left\{ \begin{pmatrix} 0 & l \\ 0 & 0 \end{pmatrix} : l \in A \right\}.$$

Obviously $z \in N$ and the following holds.

Lemma 10.5.8. *In the setting above, the following hold:*

(*i*) *For any $u, v \in N$, $uv = 0$.*

(*ii*) *For any* $u \in N$, $xu \in N$ *and* $ux \in N$.

We also define the following subset in $\mathrm{M}_6(A)$:

$$\Gamma_N = \left\{ \begin{pmatrix} I_2 + * & * & * \\ * & I_2 + * & * \\ * & * & I_2 + * \end{pmatrix} : \text{ each } * \text{ is in } N \right\}.$$

The lemma below is the key observation in this proof.

Lemma 10.5.9. *In the setting above, the following hold:*

(*i*) *The set* Γ_N *is a subgroup of* $\mathrm{E}_6(A)$.

(*ii*) *The matrices* X *and* Y *normalize* Γ_N.

(*iii*) *Any homogeneous quasi-homomorphism* ϕ *on* $\mathrm{E}_6(A)$ *is bounded on* Γ_N. *This means that* ϕ *vanishes on* Γ_N.

Proof. (Lemma 10.5.9) Items (*i*) and (*ii*) are straightforward from Lemma 10.5.8. For item (*iii*), we observe that any element $\gamma \in \Gamma_N$ can be decomposed as

$$\gamma = \begin{pmatrix} I_2 + * & * & * \\ 0 & I_2 + * & * \\ 0 & 0 & I_2 + * \end{pmatrix} \begin{pmatrix} I_2 & 0 & 0 \\ * & I_2 & 0 \\ * & * & I_2 \end{pmatrix}.$$

Here each $*$ is in N. Lemma 10.5.4 ends our proof. $\square$

We also need the following simple fact. Let

$$T = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & I_2 & I_2 \end{pmatrix} \in \mathrm{E}_6(A).$$

Lemma 10.5.10. *In the setting above,* $TXT^{-1} = X^{-1}$ *and* $TYT^{-1} = Y^{-1}$ *hold.*

Now we have all ingredients to complete our proof of Proposition 10.5.6. The essential point is that Lemma 10.5.5 applies to the case of that $H = \Gamma_N$, $h = Z$, and $g = XY$ (this follows from Lemma 10.5.9). Thus, we show that

$$\phi(ZXY) = \phi(XY).$$

By Lemma 10.5.10, we also see that XY can be transformed to $X^{-1}Y^{-1}$ by the conjugation by T. Because $X^{-1}Y^{-1}$ is conjugate to $Y^{-1}X^{-1}$, XY is conjugate to its inverse. Finally, by Lemma 10.5.3, we have

$$\phi(ZXY) = \phi(XY) = 0.$$

This shows

$$\phi(sg) = \phi(g),$$

as desired. $\square$

Proof. (Theorem 10.5.1) The assertion of the first part of the theorem immediately follows from Proposition 10.5.6 because $\mathrm{E}_2(A)$ is generated by elementary matrices. For the latter parts, we only need to check bounded generation properties. For item (*iii*), Proposition 4.1.5 ensures it. For item (*ii*), we use the following Proposition by M. Newman, which can be found in the proof of Theorem 2 in [New].

Proposition 10.5.11. ([New]) *Let A be a (commutative) principal ideal ring and m, l be positive integers with $m \geq 3l$. Then any element in $G = \mathrm{SL}_m(A)$ can be expressed as a product of two commutators in G and some element in $\mathrm{SL}_{m-l}(A)$, where we regard $\mathrm{SL}_{m-l}(A)$ as the subgroup of G in the left upper corner.*

This proposition immediately shows that for a (commutative) principal ideal ring A and $m \geq 3$, $\mathrm{SL}_m(A)$ is boundedly generated by $\mathrm{SL}_2(A)$ (in the left upper corner) and the set of all single commutators (therefore, the case of item (*iii*) also follows from this).

We have accomplished the proof of Proposition 10.5.6. □

10.5.3 Difficulty on universal lattices

In the subsection above, the rings of our main concern are euclidean domains. Here we consider the case of universal lattices.

At the moment, the theorem below seems the best result for universal lattices.

Definition 10.5.12. Let R be a ring and $m \geq 2$. Then we define $\mathrm{U}_m(R)$ as the subgroup of $\mathrm{GL}_m(R)$ generated by all $m \times m$ unipotent matrices.

We warn that $\mathrm{U}_m(R)$ here is completely different from $U_m R$ in the last subsection (the latter denotes the group of all unit upper triangular matrices). We note that $\mathrm{U}_m(R)$ contains $\mathrm{E}_m(R)$ as a (possibly non-proper) subgroup. Also by definition, $\mathrm{U}_m(R)$ is normal in $\mathrm{GL}_m(R)$.

Theorem 10.5.13. ([Mim4]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ with $k \in \mathbb{N}$. Suppose $m \geq 6$. Then for any $\phi \in HQH(\mathrm{SL}_m(A))$, $\phi \equiv 0$ on the subgroup $\mathrm{U}_2(A)$, which sits in the left upper corner.*

Proof. We apply Proposition 10.5.6 to the case of that $g \in \mathrm{U}_2(A)$ and s is a 2×2 unipotent matrix (strictly speaking, $\mathrm{U}_2(A)$ is larger than $\mathrm{E}_2(A)$, but there is no change on the proof). We obtain that

$$\text{for any } \phi \in HQH(\mathrm{SL}_m(A)), \ \ \phi(gs) = \phi(g)\phi(s).$$

Therefore, it suffices to show that for any $\phi \in HQH(\mathrm{SL}_m(A))$, $\phi(s) = 0$ holds.

We use the fact that the ring A is a unique factorization domain. Therefore any 2×2 unipotent matrix s is of the following form:

$$s = \begin{pmatrix} 1 + auv & -au^2 \\ av^2 & 1 - auv \end{pmatrix}.$$

Here $a, u, v \in A$. This matrix is not necessarily an element in $\mathrm{E}_2(A)$ (recall Remark 4.1.8), but the following is known (Mennicke): for any such matrix, it is a product of 8 elementary matrices in $\mathrm{E}_3(A)(=\mathrm{SL}_3(A))$. More precisely, the following is a presentation (see p. 281 of [PaWo]):

$$s = E_{1,3}(-u)E_{2,3}(-v)E_{3,1}(-av)E_{3,2}(au)E_{1,3}(u)E_{2,3}(v)E_{2,1}(av)E_{3,2}(-au).$$

Take any $\phi \in \mathrm{SL}_m(A)$ and consider s as an element of $\mathrm{SL}_m(A)$ in the left upper corner. Recall by Lemma 10.5.4 that ϕ vanishes on the set of elementary matrices. Therefore, $|\phi(s^n)|$ is uniformly bounded for $n \in \mathbb{N}$. This means $\phi(s)$=0. This ends the proof. □

Since we have verified property $(\mathrm{FF}_{\mathcal{C}})/\mathrm{T}$ for universal lattices with degree ≥ 4 for several classes (Theorem B, Theorem C), we have the following corollary.

Corollary 10.5.14. ([Mim4]) *Let $A = \mathbb{Z}[x_1, \ldots, x_k]$ with $k \in \mathbb{N}$. Then for $m \geq 6$, the pair $\mathrm{SL}_m(A) \geqslant \mathrm{U}_2(A)$ has relative (FF_B) for $B = \mathcal{L}_p, C_p$ $(p \in (1, \infty))$ and $B = [\mathcal{H}]$. In particular, $\mathrm{SL}_m(A) \geqslant \mathrm{U}_2(A)$ has relative* (TT). *Here $\mathrm{U}_2(A)$ sits in the left upper corner.*

We warn the following point. In the view of Vaserstein's bounded generation (Theorem 4.3.2), it may seem that Theorem 10.5.13 implies that $HQH(\mathrm{SL}_m(A)) = 0$. However, there is a *big gap* in this guess. The point here is the following (for $k \geq 1$): $\mathrm{E}_2(A)$ *is a very small subgroup in* $\mathrm{SL}_2(A)$*, and for Vaserstein's bounded generation,* $\mathrm{SL}_2(A)$ *is needed.* More precise meaning of *very small* is given by the following theorem of Grunewald–Menniche–Vaserstein [GMV2]:

Theorem 10.5.15. ([GMV2]) *Let F_∞ be the free group of countable (infinite) rank. Then there is a surjection*

$$\mathrm{SL}_2(\mathbb{Z}[x])/\mathrm{U}_2(\mathbb{Z}[x]) \twoheadrightarrow F_\infty.$$

In fact Krstić–McCool [KrMc] later shown that for any euclidean domain A which is not a field, the group quotient $\mathrm{SL}_2(A[x])/\mathrm{U}_2(A[x])$ maps into F_∞.

Therefore, to deal with quasi-homomorphisms on universal lattices, one needs inventional method. We make the following remark that universal lattices are truly universal object in the study of quasi-homomorphisms on elementary groups (over commutative rings).

Lemma 10.5.16. ([Mim4]) *If there exists $m \geq 3$ such that for any $k \geq 1$, $\widetilde{QH}(\mathrm{SL}_m(A_k)) = 0$, then for any* (not *necessarily finitely generated*) *commutative ring C, $\widetilde{QH}(\mathrm{E}_m(C)) = 0$ holds. Here $A_k = \mathbb{Z}[x_1, \ldots, x_k]$.*

Proof. Let $G = \mathrm{E}_m(C)$ for a given commutative ring C. Then for any g, there exist $l \in \mathbb{N}$ and elementary matrices $s_1, \ldots, s_l$ in G such that $g = s_1 \cdots s_l$. Consider the subring C' of C generated by all $r \in C$ which appears some entry of some matrix $s_1, \ldots, s_l$. Then C' is finitely generated, and for sufficiently large k, there is a surjection

$$\mathrm{SL}_m(A_k) \twoheadrightarrow \mathrm{E}_m(C') \ni g.$$

It is easy to see that by homomorphism, scl does not increase. Namely, for any group Γ, Λ and any homomorphism $\Psi\colon \Gamma \to \Lambda$, we have for any $\gamma \in [\Gamma, \Gamma]$

$$\mathrm{scl}_\Gamma(\gamma) \geq \mathrm{scl}_\Lambda(\Psi(\gamma)).$$

Therefore by combining this with the assumption and the Bavard duality theorem, we conclude that $\mathrm{scl}_{\mathrm{E}_m(C')}(g) = 0$. Since $\mathrm{E}_m(C') \leqslant G$, this implies $\mathrm{scl}_G(g) = 0$. This holds for any $g \in G$, and thus we have

$$\mathrm{scl} \equiv 0 \quad \text{on } G.$$

Again by the Bavard duality, we get the conclusion. $\square$

Chapter 11

Applications to group actions

In this chapter, we apply our rigidity results, which have been obtained in this thesis, to group actions. Firstly, we observe the Navas theorem [Nav1], [Nav2] which concern actions on the circle of a discrete group with $(\mathrm{F}_{\mathcal{L}_p})$. As a corollary, we obtain Theorem F.

Secondly, we consider homomorphism rigidity with the target of mapping class groups $\mathrm{MCG}(\Sigma_g)$ (of compact closed oriented surfaces); and of outer automorphism groups $\mathrm{Out}(F_n)$ of (finitely generated) free groups. Since $\mathrm{Out}(F_n)$ (for sufficiently large n) contains the mapping class group of a punctured surface and the automorphism groups of a free group, these two objects above are the essential ones. Here firstly we see known fundamental results on $\mathrm{MCG}(\Sigma_g)$ and $\mathrm{Out}(F_n)$, such as element classifications and subgroup classifications. Secondly, we see quasi-homomorphisms on these groups, respectively due to Bestvina–Fujiwara [BeFu] and Bestvina–Feighn [BeFe]. Thirdly, we see theorems of U. Hamenstädt [Ham]; and Bestvina–Bromberg–Fujiwara on quasi-cocycles on $\mathrm{MCG}(\Sigma_g)$; and on $\mathrm{Out}(F_n)$. These results are one key to establishing Theorem G. With these preliminary facts, we proceed into the proof of Theorem G. Property (TT)/T is the counterpart of theorems of Hamenstädt and Bestvina–Bromberg–Fujiwara. We note that our proof of Theorem G with $\mathrm{Out}(F_n)$ target follows the proof in a paper [BrWa] of Bridson–Wade, in which they shown the homomorphism rigidity theorem for higher rank lattices.

Finally, we consider some shortcut in proving Theorem G in the case of universal lattices. That shortcut also works for noncommutative universal lattices, and this is Theorem H. We argue why this argument may not work in the case of symplectic universal lattices.

In this chapter, we state many theorems on $\mathrm{MCG}(\Sigma_g)$ or $\mathrm{Out}(F_n)$ without providing with proofs. We refer the readers to the original papers. On actions on the circle, there is a survey [Ghy2] by E. Ghys. Concerning Navas' theorem, we also refer to Section 2.9 of [BHV]. For mapping class groups, main references are survey [Iva] of N. V. Ivanov, and a forthcoming book [FaMar] of Farb–Margalit. For automorphism groups of free groups, we refer to an ICM proceedings paper [Vog2] of K.

Vogtmann; surveys [Vog1], [BrVo] of Vogtmann, and Bridson–Vogtmann.

11.1 Actions on the circle

Let S^1 denote the unit circle in $\mathbb{R}^2$ and identify S^1 with $[-\pi, \pi)$. We denote by $\mathrm{Diff}_+(\mathrm{S}^1)$ the group of orientation preserving group diffeomorphisms of S^1.

Definition 11.1.1. Let $\alpha > 0$ be a real number. The group $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ is defined as the class of all orientation preserving group diffeomorphisms f of S^1 such that f' and $(f^{-1})'$ are Hölder continuous with exponent α. Here a function g on S^1 is said to be *Hölder continuous with exponent* α if

$$\|g\|_\alpha := \sup_{\theta_1 \neq \theta_2} \frac{|g(\theta_1) - g(\theta_2)|}{|\theta_1 - \theta_2|^\alpha} < \infty$$

holds.

Firstly, we restate Theorem F:

Theorem 11.1.2. *Let* Γ *be a finite index subgroup either of* $\mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ $(m \geq 4)$ *or of* $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])$ $(m \geq 3)$*. Then for any* $\alpha > 0$*, every homomorphism*

$$\Gamma \to \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$$

has finite image.

In particular, the following holds true: Γ *be a finite index subgroup either of* $\mathrm{E}_m(A)$ $(m \geq 4)$ *or of* $\mathrm{Ep}_{2m}(A)$ $(m \geq 3)$ *for a commutative finitely generated ring* A*. Then for any* $\alpha > 0$*, every homomorphism*

$$\Gamma \to \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$$

has finite image.

This theorem is an immediate corollary of theorem(s) of A. Navas, as we will explain in below.

Navas [Nav1] has shown the following theorem: *For any discrete group* Γ *with property* (T)*, every homomorphism from* Γ *into* $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ *has finite image, for any* $\alpha > 1/2$. He has also noted in Appendix of the paper [Nav2] that his theorem can be extended to general L^p cases.

Theorem 11.1.3. (*Navas* [Nav1], [Nav2]) *Let* $1 < p < \infty$ *and* Γ *be a discrete group with property* $(\mathrm{F}_{\mathcal{L}^p})$*. Then for any* $\alpha > 1/p$*, every homomorphism*

$$\Gamma \to \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$$

has finite image.

We note that this theorem is one of the motivations of Bader–Furman–Gelander–Monod for their study of $(\mathrm{F}_{\mathcal{L}_p})$, see Subsection 1.b of [BFGM].

Here we will give an outline of Navas' theorem. The proof below is an variant of one in Section 2.9 of [BHV], and there an idea of D. Witte is also used.

Proof. (Outlined)

We consider the space $B = L^p(\mathrm{S}^1 \times \mathrm{S}^1, \mathbb{R})$, where we endow $\mathrm{S}^1 \times \mathrm{S}^1$ with the Lebesgue measure $d\theta d\phi$. We consider the following isometric representation ρ of $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ on B:

$$(\rho(f)K)(\theta, \phi) = \left((f^{-1})'(\theta)(f^{-1})'(\phi)\right)^{1/p} K(f^{-1}(\theta, \phi)).$$

(Note that we will consider the image of the discrete group Γ into $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$ (or slightly modified one) and the restriction of ρ on this group. Therefore we do not care on topology on $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$.) Consider the following kernel F on S^1:

$$F(\theta, \phi) = \frac{1}{(2\tan((\theta - \phi)/2))^{2/p}}.$$

Set a formal coboundary of $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$,

$$c(f) := F - \rho(f)F.$$

More precisely,

$$c(f^{-1})(\theta_1, \theta_2) = \frac{1}{|2\sin((\theta_1 - \theta_2)/2)|^{2/p}} - \frac{[f'(\theta_1)f'(\theta_2)]^{1/p}}{|2\sin((f(\theta_1) - f(\theta_2))/2)|^{2/p}}.$$

This map is called the *Liouville L^p-cocycle* . The point here is the following:

> *although F does not belong to $L^p(\mathrm{S}^1 \times \mathrm{S}^1)$,*
> *this formal coboundary belongs to $L^p(\mathrm{S}^1 \times \mathrm{S}^1)$ for each $f \in \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$.*

We omit the proof of this fact: see [Nav1] or Lemma 2.9.2 in [BHV] in the case of $p = 2$. We note that for the proof of the latter, we need the assumption $\alpha > p^{-1}$. This *formal* coboundary (this is said to be formal because $F \notin B$) becomes an ρ-cocycle.

Now consider a homomorphism $\Gamma \to \mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$. Take a triple cover of S^1, and take a corresponding cover of Γ. We name this group G, and with identify with its image in $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$. Since G is a extension of Γ by a finite group $\mathbb{Z}/3\mathbb{Z}$, G also has $(\mathrm{F}_{\mathcal{L}_p})$. Consider the restriction of ρ and c on G, and rewrite them as ρ and c. Here we employ property $(\mathrm{F}_{\mathcal{L}_p})$ for G. Then it indicates that this c is indeed a ρ coboundary. Namely, there exists a kernel ξ on S^1 *which belongs to B* such that

$$\text{for all } g \in G, \quad c(g) = \xi - \rho(g)\xi.$$

This means

$$\text{for all } g \in G, \quad F - \xi = \rho(g)(F - \xi).$$

Therefore the (formal) vector $F - \xi$ is $\rho(G)$-invariant.

We define a (positive) measure μ on $(\mathrm{S}^1 \times \mathrm{S}^1) \setminus \Delta$ by the following formula:

$$d\mu(\theta, \phi) := (F - \xi)^2(\theta, \phi) d\theta d\phi.$$

Here Δ is the diagonal part (recall that F diverges on Δ). By the construction of μ, the following properties can be seen:

1. The measure μ is $\rho(G)$-invariant.

2. For any pairwise distinct and cyclically ordered points a, b, c, d in S^1,

$$\mu([a, b] \times [c, d]) = \mu([c, d] \times [a, b]).$$

3. For any pairwise distinct and cyclically ordered points a, b, c in S^1,

$$\mu([a, a] \times [b, c]) = 0.$$

4. For any pairwise distinct and cyclically ordered points a, b, c, d in S^1,

$$\mu([a, b) \times (b, c]) = \infty.$$

Then it can be also shown that these four properties imply the following:

$$\text{any } g \in G \setminus \{e\} \text{ does not fix three points of } \mathrm{S}^1$$

(see Proposition 2.9.6 in [BHV]). Then in terms of Γ (we also identify Γ with its image in $\mathrm{Diff}_+^{1+\alpha}(\mathrm{S}^1)$), this exactly means the following:

$$\text{any } \gamma \in \Gamma \setminus \{e\} \text{ does not fix any point of } \mathrm{S}^1.$$

This means the action of Γ on S^1 is free.

However, it is a well-known theorem that any such group (even inside $\mathrm{Homeo}_+(\mathrm{S}^1)$) must be abelian. The proof uses ordering on a group. See Corollary 2.9.10 in [BHV]. Since Γ (recall that we identify with the image) has $(\mathrm{F}_{\mathcal{L}_p})$, this implies (the image) Γ is finite, as desired. $\square$

For the proof of Theorem 11.1.2, we just need to recall that $(\mathrm{F}_{\mathcal{L}_p})$ passes to p-integrable lattices, in particular to finite index subgroups.

Remark 11.1.4. (*i*) In the case of higher rank lattices of $\mathbb{Q}$-rank ≥ 2, there is a simple argument by Witte [Wit] focusing on ordering on a group, and it shows that in fact every homomorphism from these into $\mathrm{Homeo}_+(\mathrm{S}^1)$ has finite image (the author acknowledges A. Furman for this reference). However, in the process of deducing finiteness of the image, we need to appeal to the Margulis finiteness theorem. More precisely, in argument on orders, firstly we show that this homomorphism cannot be injective. This means the kernel of this homomorphism cannot be finite. Then we appeal to the Margulis finiteness theorem, and ends the proof.

However, as we have seen in Lemma 4.1.12, universal lattices and symplectic universal lattices *fail* to have the Margulis finiteness property. Therefore arguments [Wit] does not directly apply to these cases (it is shown in the argument in [Wit] that a homomorphism into $\mathrm{Homeo}_+(\mathrm{S}^1)$ is never injective).

(*ii*) The argument in Theorem 11.1.3 does not give any information for the case of $\alpha = 0$. Navas pointed out in [Nav3] that a property valid for $\mathrm{Diff}^{1+\alpha}_+(\mathrm{S}^1)$ for all $\alpha > 0$ is *not* necessarily valid for $\mathrm{Diff}^1_+(\mathrm{S}^1)$. His example is the following: for any $\alpha > 0$, $\mathrm{Diff}^{1+\alpha}_+(\mathrm{S}^1)$ does not contain a finitely generated group of intermediate growth, but $\mathrm{Diff}^1_+(\mathrm{S}^1)$ *does* contain.

Nevertheless, Burger–Monod [BuMo1], [BuMo2] have succeeded in establishing finiteness results for homomorphisms from higher rank lattices, which is based on the study of Ghys [Ghy1] on the euler class $e \in H^2(\mathrm{Homeo}_+(\mathrm{S}^1), \mathbb{Z})$. Burger–Monod shown that if a discrete group Γ satisfies $H^2_{\mathrm{b}}(\Gamma) = 0$ (this symbol denotes the second bounded cohomology with trivial real coefficient) and has finite abelianization, then homomorphism finiteness from Γ into $\mathrm{Diff}^1_+(\mathrm{S}^1)$ follows. Also in this point of view, studies of quasi-homomorphisms on universal lattices (and on symplectic universal lattices) seems important.

(*iii*) As we have seen in item (*ii*) of Remark 4.1.11, universal lattices and symplectic universal lattices are linear groups (Definition 2.7.4). A lemma of A. Selberg states every linear group contains a finite index subgroup which is torsion-free. In study of homomorphism rigidity with target of circle diffeomorphism group, it is much easier to obtain finiteness results if the source group has torsions. Therefore the essential case in statement of Theorem 11.1.2 is the case where we take these torsion-free finite index subgroups.

11.2 Basics on MCG(Σ) and Out(F_n)

11.2.1 Definitions

Recall from Chapter 0 that the symbol Σ_g denotes a compact connected oriented surface with g genus, and the symbol $\Sigma_{g,l}$ denotes a compact connected oriented surface with g genus and l punctures. Hence Σ_g is equal to $\Sigma_{g,0}$

Definition 11.2.1. Let $g, l \geq 0$ and $n \geq 2$ (finite numbers).

(*i*) Let $\Sigma = \Sigma_{g,l}$.

(1) The *mapping class group* of Σ is defined as

$$\mathrm{MCG}(\Sigma) := \mathrm{Homeo}_+(\Sigma)/\mathrm{isotopy}.$$

Here a mapping class is allowed to permute punctures, and isotopies can rotate a neighborhood of a puncture.

(2) The surface $\Sigma = \Sigma_{g,l}$ is said to be *non-exceptional* if

$$3g + l \geq 5.$$

Otherwise, it is said to be *exceptional*.

(*ii*) Let F_n denotes the free group of rank n. The symbol $\mathrm{Aut}(F_n)$ denotes the *automorphism group* of F_n. The symbol $\mathrm{Out}(F_n)$ denotes the *outer automorphism group* of F_n, namely,

$$\mathrm{Out}(F_n) := \mathrm{Aut}(F_n)/\mathrm{Inn}(F_n).$$

Here $\mathrm{Inn}(F_n)$ denotes the normal subgroups of $\mathrm{Aut}(F_n)$ consisting all conjugations.

Here are a few words on the terminology "punctures." There is another terminology "boundary component", and these two are distinguished. In the above definition, if we replace punctures with boundary components, then each mapping class must preserve the individual boundary components pointwise, and isotopies must fix each boundary components pointwise. If surface Σ has at least one boundary component, then it is known that MCG(Σ) has no torsion. This case homomorphism rigidity from a finite index group of (symplectic) universal lattices is much easily deduced.

It is a classical fact that MCG(Σ) is a finitely generated group , and in fact it is finitely presented (see for instance Chapter 6 of [FaMar]). Also, Nielsen has given finite presentations for $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$.

We note that there is an injection

$$\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Out}(F_{n+1}).$$

If $\Sigma = \Sigma_{g,l}$ is exceptional, then it is not difficult to see that $\mathrm{MCG}(\Sigma_{g,l})$ is *virtually free*, which means it contains a free group (possibly $\{e\}$ or $\mathbb{Z}$) as a finite index subgroup. For instance, $\mathrm{MCG}(\Sigma_{0,0}) = \{e\}$; and $\mathrm{MCG}(\Sigma_{1,0}) \cong \mathrm{SL}_2(\mathbb{Z})$, whose isomorphism is realized by a action on homology class $H_1(\Sigma_{1,0}, \mathbb{Z}) \cong \mathbb{Z}^2$. Recall from the proof of Corollary 2.5.11 that $\mathrm{SL}_2(\mathbb{Z})$ contains a copy of F_2 with index 12. Virtually free groups are more or less well-understood, and hence we are interested in non-exceptional surfaces case.

Consider a non-exceptional surface $\Sigma = \Sigma_{g,l}$. There is a correspondence:

$$\left\{\begin{array}{c}\text{Free homotopy classes of} \\ \text{(unbased) maps } \Sigma \to \Sigma\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Conjugacy classes of homo-} \\ \text{morphisms } \pi_1(\Sigma) \to \pi_1(\Sigma)\end{array}\right\}.$$

For details, see Proposition 1 B.9 of a book [Hat] of A. Hatcher (here observe that the universal cover of Σ is contractible). With this correspondence, there is a natural homomorphism

$$\mathrm{MCG}(\Sigma) \to \mathrm{Out}(\pi_1(\Sigma)),$$

which is always *injective*. The Dehn–Nielson–Bear theorem states the following:

Theorem 11.2.2. (*The Dehn–Nielsen–Bear theorem*) *For $g \geq 1$, the homomorphism*

$$\mathrm{MCG}^{\pm}(\Sigma_g) \to \mathrm{Out}(\pi_1(\Sigma_g))$$

is an isomorphism. Here $\mathrm{MCG}^{\pm}(\Sigma_g)$ denotes the extended mapping class group, which is defined by allowing orientation-reversing mapping homeomorphisms in the definition of mapping class groups.

In particular, $\mathrm{MCG}(\Sigma_g)$ is a subgroup of index 2 in the outer automorphism of the surface group of genus g,

$$\pi_1(\Sigma_g) = \langle a_1, \ldots, a_g; b_1, \ldots b_g \mid [a_1, b_1] \cdots [a_g, b_g] = e\rangle.$$

If $l \geq 1$, then $\pi_1(\Sigma_{g,l}) = F_{2g+l-1}$. Thus we have an injection:

$$\mathrm{MCG}(\Sigma_{g,l}) \hookrightarrow \mathrm{Out}(F_{2g+l-1}) \qquad (l \geq 1).$$

11.2.2 Linear representations

Consider a closed surface Σ_g. On a pair of two homology classes of Σ_g, there is a concept of the *algebraic intersection number*. Namely, for $\alpha, \beta \in H_1(\Sigma_g; \mathbb{Z})$, $\tilde{i}(\alpha, \beta) \in \mathbb{Z}$ is defined as the sum of the indices of the intersection point of α and β, where an intersection point is of index $+1$ if the orientation of the intersection agrees with the orientation of S, and is -1 otherwise. Also, there exists an ordered basis $\{[\alpha_1], \ldots, [\alpha_g]; [\beta_1], \ldots, [\beta_g]\}$ of $H^1(\Sigma_g; \mathbb{Z})$ which realizes the intersection number as the symplectic form with respect to it. Each element f of $\mathrm{MCG}(\Sigma_g)$ induces a homomorphism $f_*\colon H^1(\Sigma_g; \mathbb{Z}) \to H^1(\Sigma_g; \mathbb{Z})$, and f_* leaves the intersection number (as a bilinear form) invariant. Thus we obtain a homomorphism

$$\mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}),$$

which is known to be surjective, as we have written as $\twoheadrightarrow$.

For $\mathrm{Out}(F_n)$, by considering its action on the abelianization of $F_n \cong \mathbb{Z}^n$, we obtain a surjection,

$$\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}).$$

These linear representations have enormous kernels. It is an important open problem to determine whether $\mathrm{MCG}(\Sigma_g)$ $(g \geq 2)$ and $\mathrm{Out}(F_n)$ $(n \geq 2)$ are linear groups.

Definition 11.2.3. For $g \geq 1$, the normal subgroup $\mathcal{T}_g$ of $\mathrm{MCG}(\Sigma_g)$ is defined as the kernel of

$$\mathrm{MCG}(\Sigma_g) \twoheadrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}).$$

It is called the *Torelli group.*

For $n \geq 2$, the normal subgroup $\overline{\mathrm{IA}}_n$ of $\mathrm{Out}(F_n)$ is defined as the kernel of

$$\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z}).$$

These groups are of high interest in the theory of $\mathrm{MCG}(\Sigma_g)$ and $\mathrm{Out}(F_n)$, but still their structures are mysterious (the symbol IA stands for "identity on the abelianization").

It is known that these groups are torsion-free. Also, it is known that $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{GL}_n(\mathbb{Z})$ respectively have finite index torsion-free subgroups. Indeed, we are able to appeal to Selberg's lemma, but there are explicit constructions. For $m \geq 2$, let $\mathrm{SL}_m(\mathbb{Z})[3]$ be the congruence subgroup of $\mathrm{SL}_m(\mathbb{Z})$ associated with the ideal $(3) \trianglelefteq \mathbb{Z}$. Namely, $\mathrm{SL}_m(\mathbb{Z})[3]$ is the kernel of the surjection

$$\mathrm{SL}_m(\mathbb{Z}) \twoheadrightarrow \mathrm{SL}_m(\mathbb{Z}/3\mathbb{Z}),$$

which sends $\mathbb{Z} \ni a$ to a mod 3. The group $\mathrm{SL}_m(\mathbb{Z})[3]$ is a finite index subgroup of $\mathrm{SL}_m(\mathbb{Z})$, and it is not so difficult to see it is torsion-free (for instance, see Proposition 8.8 in [FaMar]). Thus we obtain the following result:

Theorem 11.2.4. *For $g \geq 1$ and $m \geq 2$, the groups $\mathrm{MCG}(\Sigma_g)$ and $\mathrm{Out}(F_n)$ have finite index subgroups which are torsion-free.*

This result shall be used for the proof of Theorem H, which is the shortcut of Theorem G. For the proof of Theorem G, we need the following theorem of Bass–Lubotzky [BaLu] (another proof is given by Bridson–Wade [BrWa]) on $\overline{\mathrm{IA}}_n$.

Theorem 11.2.5. (*Bass–Lubotzky* [BaLu]) *Any nontrivial subgroup of* $\overline{\mathrm{IA}}_n$ *maps onto* $\mathbb{Z}$.

11.2.3 Element classification

Here we state the element classification theorem for MCG(Σ), which is so-called *the Nielsen–Thurston Classification theorem* (for instance, see Chapter 13 in [FaMar]). To state this theorem, we need some terminologies on measured foliation on Σ, which is invented by W. Thurston. We briefly see them. For details on this topic, we refer to a book [Bir] of J. S. Birman, Chapter 14 of [FaMar], and a paper of McCarthy–Papadopoulos [McPa]. We call a c curve on a surface a *simple closed curve* if c has no self-intersecting point and is closed.

Let $\Sigma = \Sigma_{g,l}$. Let $\mathcal{MF}(\Sigma)$ be the space of equivalence classes of measured foliation (here the equivalence is generated by isotopy and by so-called Whitehead moves). Here we exclude the zero-foliation. Let $\mathcal{S}$ be the set of isotopy classes of simple closed curves in Σ. Then we can regard $\mathcal{MF}(\Sigma)$ as a subset of $\mathbb{R}^{\mathcal{S}}_{\geq 0} \setminus \{0\}$. We define $\mathcal{PMF}(\Sigma)$ as the image of $\mathcal{MF}(\Sigma)$ by the projection $\mathbb{R}^{\mathcal{S}}_{\geq 0} \setminus \{0\}$ to the projective space $P(\mathbb{R}^{\mathcal{S}})$. It is known that $\mathcal{PMF}(\Sigma)$ is endowed with a natural topology with respect to which it is homeomorphic to a sphere of dimension $6g+2l-7$ (in particular, $\mathcal{PMF}(\Sigma)$ is compact).

Definition 11.2.6. Let $\Sigma = \Sigma_{g,l}$ be a surface.

(i) An element $f \in$ MCG(Σ) is said to be *pseudo-Anosov* if the following condition holds: there exists a pair of transverse measured foliations $(L_+, \mu_+), (L_-, \mu_-)$ on Σ and a real number $\lambda > 1$ and a representative ϕ of f such that

$$\phi \cdot (L_+, \mu_+) = (L_+, \lambda\mu_+) \quad \text{and} \quad \phi \cdot (L_-, \mu_-) = (L_-, \lambda^{-1}\mu_-).$$

Therefore, a pseudo-Anosov element f fixes some point in $\mathcal{PMF}(\Sigma)$.

(ii) Two pseudo-Anosov elements $f_1, f_2 \in$ MCG(Σ) are said to be *independent* if their fixed point sets in $\mathcal{PMF}$ are disjoint.

In the above definition, this λ is uniquely determined for a pseudo-Anosov element f, and is called the *dilatation*

Here is the classification theorem. A simple closed curve c on a surface Σ (possibly with punctures) is said to be *essential* if c is neither homotopic to one point nor homotopic to a boundary component.

Theorem 11.2.7. (*the Nielson–Thurston classification theorem*) *Let* $\Sigma = \Sigma_{g,l}$ *be a surface. Then each* $f \in$ MCG(Σ), *either of the following three holds true:*

(i) *the element* f *is a torsion, namely, there exists* $n \geq 1$ *such that* $f^n = e$;

(*ii*) *the element f is* reducible*. This means, f fixes a collection of isotopy classes of essential simple closed curves that are pairwise disjoint;*

(*iii*) *the element f is pseudo-Anosov.*

Moreover, a pseudo-Anosov element is neither a torsion nor reducible.

Example 11.2.8. Consider $\Sigma = \Sigma_1$ and $\Lambda = \mathrm{MCG}(\Sigma) \cong \mathrm{SL}_2(\mathbb{Z})$. With the identification $\Lambda = \mathrm{SL}_2(\mathbb{Z})$, the classification above corresponds as follows:

$\mathrm{MCG}(\Sigma_1)$	$\longleftrightarrow$	$\mathrm{SL}_2(\mathbb{Z})$
f is a torsion	$\longleftrightarrow$	f is a torsion ;
f is reducible	$\longleftrightarrow$	f is unipotent, $f \sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$;
f is pseudo-Anosov	$\longleftrightarrow$	f is semisimple, and for some $\lambda > 1$, $f \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$.

A basic example of reducilbe element is the *Dehn twist* along an (essential) simple closed curve c. Roughly speaking, the Dehn twist along c is a map which twists once along c on the neighborhood of c:

Definition 11.2.9. Let Σ be a surface. Let c be an essential simple closed curve on Σ.

(*i*) Let N be a regular neighborhood of c, and ϕ be an orientation preserving homomorphism from $A = \mathrm{S}^1 \times [0,1]$ to N. A *Dehn twist* along c is the followng map $\Sigma \to \Sigma$,

$$T_c(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in N; \\ x & \text{if } x \in \Sigma \setminus N. \end{cases}$$

Here $T\colon A \to A$ is the following map:

$$T(\theta, t) = (\theta + 2\pi t, t).$$

(*ii*) An element $f \in \mathrm{MCG}(\Sigma)$ is called the *Dehn twist* along c if there exists a representative which is a Dehn twist in the sense of item (*i*) (for some N and ϕ). We again use the symbol T_c for the Dehn twist along c, as an element in $\mathrm{MCG}(\Sigma)$.

As this definition implicitly says, Dehn twists along c are indeed unique up to isotopy.

Next we proceed to the case of $\mathrm{Out}(F_n)$. For $\mathrm{Out}(F_n)$, there is an analogous concept to that of pseudo-Anosov elements for mapping class groups. This is called that of *fully irreducible elements*.

Definition 11.2.10. Let $n \geq 2$.

(i) A subgroup $L \leqslant F_n$ is called a *free factor* of F_n if there exists $L' \leqslant F_n$ such that
$$F_n = L \star L'$$
holds. Here the symbol $L \star L'$ denotes the free product of L and L'.

(ii) An element f in $\mathrm{Out}(F_n)$ is said to be *fully irreducible* if *no* non-zero power of f fixes any conjugacy class of any free factor of F_n.

There exists an analogue of $\mathcal{PMF}$ for $\mathrm{Out}(F_n)$ in terms of which we can define *independence* of two fully irreducible elements. We briefly state this. Let $\mathcal{T} = \mathcal{T}_n$ be the space of free simplicial trees endowed with cocompact left F_n-action, which does not have vertices of valence 1 or 2. For a conjugacy class γ in F_n and $T \in \mathcal{T}$, define $\langle T, \gamma \rangle$ as the translation length of γ in T. Namely, take a representative g of γ and define
$$\langle T, \gamma \rangle := \inf_{t \in T} d(t, gt).$$
This definition is independent of the choice of g. Then the group $\mathrm{Out}(F_n)$ acts by right by "changing markings", this means for $f \in \mathrm{Out}(F_n)$;
$$T \mapsto T \cdot f \quad \text{such that} \quad \langle T \cdot f, \gamma \rangle = \langle T, f(\gamma) \rangle.$$
The group $\mathbb{R}_{>0}$ acts on $\mathcal{T}$ by scaling and these two actions commute. The *Outer space* of Culler–Vogtmann [CuVo1] is the projectivized space
$$\mathcal{PT} := \mathcal{T}/\mathbb{R}_{>0}.$$
We define $\overline{\mathcal{T}}$ as the closure of $\mathcal{T}$ in the space of trees with minimal left F_n action. Both $\mathrm{Out}(F_n)$ and $\mathbb{R}_{>0}$ continues to act on $\overline{\mathcal{T}}$, and we define
$$\overline{\mathcal{PT}} := \overline{\mathcal{T}}/\mathbb{R}_{>0}.$$
This is a compact space and called Culler–Morgan's equivalent compactification of the outer space [CuMo]. For details on outer spaces, we refer to the original paper [CuVo1], and surveys [Vog1], [Vog2], [BrVo]. The following theorem due to Levitt–Lustig [LeLu] shows in the view of actions on $\overline{\mathcal{PT}}$, fully irreducible elements can be seen as an analogue of pseudo-Anosov elements in mapping class groups.

Theorem 11.2.11. (*Levitt–Lustig* [LeLu]) *Every fully irreducible element* $f \in \mathrm{Out}(F_n)$ *acts on* $\overline{\mathcal{PT}}$ *with exactly two fixed point* T_+, T_- *with north-south dynamics.*

Definition 11.2.12. We say two fully irreducible elements in $\mathrm{Out}(F_n)$ are *independent* if their fixed point sets in $\overline{\mathcal{PT}}$ are disjoint.

11.2.4 Subgroup classification

For $\mathrm{MCG}(\Sigma)$, the following subgroup classification theorem, due to McCarthy–Papadopoulos [McPa], plays a fundamental role:

Theorem 11.2.13. (*McCarthy–Papadopoulos* [McPa]) *Let $\Sigma = \Sigma_{g,l}$ be a surface. Then each $\Lambda \leqslant \mathrm{MCG}(\Sigma)$ satisfies either of the following:*

(*i*) *the group Λ is finite;*

(*ii*) *the group Λ is* reducible. *That means, there exists a collection $\mathfrak{C}$ of essential simple closed pairwise disjoint curves such that every element of Λ fixes $\mathfrak{C}$;*

(*iii*) *the group Λ has a pseudo-Anosov element f, and there is no pseudo-Anosov element in Λ which is independent to f. In this case, Λ is virtually $\mathbb{Z}$;*

(*iv*) *the group Λ contains two independent pseudo-Anosov elements.*

Recall that a group is said to be virtually $\mathbb{Z}$ if it contains $\mathbb{Z}$ as a finite index subgroup.

For the case of $\mathrm{Out}(F_n)$, analogous results have been desired, and finally, Handel–Mosher [HaMo] have successfully obtained the theorem in below.

Definition 11.2.14. (Handel–Mosher [HaMo]) Let $n \geq 2$.

(*i*) A subgroup $\Lambda \leqslant \mathrm{Out}(F_n)$ is said to be *reducible* if there exists a free factor $L \leqslant F_n$ with $L \neq \{e\}$ such that each element of Λ preserves the set of conjugacy classes in L.

(*ii*) A subgroup $\Lambda \leqslant \mathrm{Out}(F_n)$ is said to be *fully irreducible* if no subgroups of Λ of finite index is reducible.

Theorem 11.2.15. (*Handel–Mosher* [HaMo]) *Let $n \geq 2$. Then each $\Lambda \leqslant \mathrm{MCG}(\Sigma)$ satisfies either of the following:*

(*i*) *the group Λ is* not *fully irreducible;*

(*ii*) *the group Λ has a fully irreducible element f, and there is no fully irreducible element in Λ which is independent to f. In this case, Λ is virtually $\mathbb{Z}$;*

(*iii*) *the group Λ contains two independent fully irreducible elements.*

11.2.5 Curve graph for MCG

For a surface $\Sigma = \Sigma_{g,l}$, we can define an associated natural complex, which is called the *complex of curves*. In the study of quasi-cocycles on the mapping class group, 1-sketlon of this complex plays an important role. This graph is called the *curve graph*.

Definition 11.2.16. Let Σ be a surface. Then the *curve graph* associated with Σ, which is written as $C(\Sigma)$, is defines as follows:

(1) The vertex set V is the set of all homotopy classes of essential simple closed curves on Σ.

(2) For each distinct pair $[a], [b] \in V$, we connect $[a]$ and $[b]$ by an edge if there exist representatives a of $[a]$ and b of $[b]$ (essential simple closed curves) such that

$$a \cap b = \emptyset$$

holds

The following is a fundamental fact on curve graphs. For instance, see chapter 3 of [Iva].

Theorem 11.2.17. *Suppose $\Sigma = \Sigma_{g,l}$ is a non-exceptional surface (, namely, $3g + l \geq 5$). Then $C(\Sigma)$ is a connected graph, and the diameter of $C(\Sigma)$ is infinite.*

Thanks to this theorem, for a nonexceptional surface Σ, we regard $C(\Sigma)$ as a metric space with the shortest path metric.

An extreme importance of a curve graph lies in the following two deep theorems, respectively due to Masur–Minsky [MaMi] and B. Bowditch [Bow]. Before stating these theorems, we note that MCG(Σ) naturally acts on $C(\Sigma)$ by isometries. Also, recall the definition of hyperbolicity of a metric space from Definition 2.6.19.

Definition 11.2.18. (Bowditch [Bow] Let G be a discrete group and (X, d) be a metric space. Let α be an isometric action of G on X. We say the action α is *acylindrical* if the following condition is satisfied: for any $R > 0$, there exists $D \geq 0$ and $L \geq 0$ such that:

$$\text{for any } x, y \in X \text{ with } d(x, y) \geq L,$$
$$|\{g \in G : d(x, gx) \leq R \quad \textit{and} \quad d(y, gy) \leq R\}| \leq D.$$

For example, for any finitely generated group G, the natural isometric action of G on a Cayley graph Cay(G; S) (where S is any finite generating subset) is acylindrical.

Now we state the celebrated theorems of [MaMi] and [Bow]:

Theorem 11.2.19. (*Masur–Minsky* [MaMi]) *For a non-exceptional surface Σ, $C(\Sigma)$ is a hyperbolic metric space.*

Theorem 11.2.20. (*Bowditch* [Bow]) *For a non-exceptional surface Σ, the natural isometric action of* $\mathrm{MCG}(\Sigma)$ *on* $C(\Sigma)$ *is acylindrical.*

Remark 11.2.21. One of the main difficulities to deal with a curve graph is that it is *not* locally finite. Namely, the valence of some vertex is infinite. Because of this, much care is needed in studies of mapping class groups, even though the two theorems above are valid.

11.2.6 Homomorphism rigidity from higher rank lattices

There was a question whether $\mathrm{MCG}(\Sigma)$ for a nonexeptional surface can be isomorphic to a lattice in a semi-simple algebraic group, and it was answered in negative (see Chapter 9 of [Iva]). Later, Farb–Masur [FaMas] shown that in fact there is homomorphism rigidity from higher rank lattices to mapping class groups. Recall our terminology of “higher rank lattice” from Chapter 0.

Theorem 11.2.22. (*Farb–Masur* [FaMas]) *Let Γ be a higher rank lattice. Then for any surface, every homomorphism*

$$\Gamma \to \mathrm{MCG}(\Sigma)$$

has finite image.

Their proof uses the study of the Poisson boundary of $\mathrm{MCG}(\Sigma)$ of Kaimanovich–Masur [KaMa], and the theorem above is called the *Farb–Kaimanovich–Masur superrigidity.*

It is natural to expect an analogous result holds with $\mathrm{Out}(F_n)$ target. Recently, Bridson–Wade [BrWa] have succeeded in establishing it:

Theorem 11.2.23. (*Bridson–Wade* [BrWa]) *Let Γ be a higher rank lattice. Then for any $n \geq 2$, every homomorphism*

$$\Gamma \to \mathrm{Out}(F_n)$$

has finite image.

We note that, as we mentioned in Chapter 1, in proofs of both theorems, the Margulis finiteness theorem plays a key role. However it is *not* valid for (symplectic) universal lattices (Lemma 4.1.12). Also, we recall from Chapter 1 another difficulty, which we will explain in below. Before stating it, we note that Bridson–Wade in fact defined and shown the following:

Definition 11.2.24. ([BrWa]) A discrete group G is said to be $\mathbb{Z}$-*averse* if G has *no* finite index subgroup G_0 that contains a normal subgroup $N \trianglelefteq G_0$ which maps onto $\mathbb{Z}$.

Theorem 11.2.25. (*Bridson–Wade* [BrWa]) *Let G be a discrete group. Suppose G is $\mathbb{Z}$-averse. Then for any surface $\Sigma = \Sigma_{g,l}$ and any $n \geq 2$, every homomorphism*

$$\phi\colon \Gamma \to \mathrm{MCG}(\Sigma)$$

and every homomorphism

$$\psi\colon \Gamma \to \mathrm{Out}(F_n)$$

have finite image.

However, we observe the following:

Lemma 11.2.26. *Let $A_k = \mathbb{Z}[x_1, \ldots, x_k]$. If $k \geq 1$, then universal lattices $\mathrm{SL}_m(A_k)$ $(m \geq 3)$ and symplectic universal lattices $\mathrm{Sp}_{2m}(A_k)$ $(m \geq 2)$ are* not *$\mathbb{Z}$-averse.*

Proof. We will only show that $\mathrm{SL}_m(\mathbb{Z}[x])$ is not $\mathbb{Z}$-averse. Consider the congruence subgroup H associated with the ideal $(x) \trianglelefteq \mathbb{Z}[x]$, namely, H is the kernel of a map

$$\mathrm{SL}_m(\mathbb{Z}[x]) \twoheadrightarrow \mathrm{SL}_m(\mathbb{Z}), \text{ which sends } x \text{ to } 0.$$

Define a map on H by the following formula:

$$\sigma\colon H \to \mathbb{Z}; h \mapsto (h' \mid_{x=0})_{1,1}.$$

Here h' is the derivative matrix by the variable x, and $(\cdot)_{1,1}$ means the $(1,1)$-th entry.

We show that this map is a group homomorphism. Indeed, let $h, l \in H$. Then by the definition of the group H, $h \mid_{x=0} = l \mid_{x=0} = I_m$, Therefore we have

$$(hl)' \mid_{x=0} \;=\; (h'l) \mid_{x=0} + (hl') \mid_{x=0} \;=\; h' \mid_{x=0} + l' \mid_{x=0} .$$

This shows our assertion. It is clear that there exists $h \in H$ with $\sigma(h) \neq 0$. □

11.3 Quasi-homomorphisms on MCG(Σ) and Out(F_n)

In this section, we see quasi-homomorphisms on MCG(Σ) and Out(F_n). For the former part, Bestvina–Fujiwara [BeFu] shown that $\widetilde{QH}(\mathrm{MCG}(\Sigma)) \neq 0$ (in fact, the left hand side is infinite dimensional) by employing theorems on curve graphs. The method they use has its origin in [EpFu], which had extended the construction of Brooks' quasi-homomorphisms (see Subsection 10.2.2). For the latter part, some analogue of a curve graph for Out(F_n) is needed. This is done by Bestvina–Feighn [BeFe]. Although, this construction is not as powerful as that of curve graph, it is sufficient to obtain nonvanishing (and infinite dimensionality) of $\widetilde{QH}(\mathrm{Out}(F_n))$. In both cases, also subgroup cases are considered.

11.3.1 A theorem of Bestvina–Fujiwara

Recall from Remark 10.2.6 that (Epstein–Fujiwara [EpFu] and) Calegari–Fujiwara [CalFu] observed the following properties for a group G is essential to deduce that $\widetilde{QH}(G) \neq 0$ (in fact, that $\widetilde{QH}(G)$ is infinite dimensional).

(a) The group G acts on a hyperbolic space X isometrically $\alpha\colon G \curvearrowright X$.

(b) The action α is properly discontinuous, namely, for any $x \in X$ and any $r > 0$, the set
$$\{g \in G : d(x, g \cdot x) < r\}$$
is finite.

(c) There exists $g \in G$ of infinite order such that g acts on X by a hyperbolic isometry, and for any $n > 0$ and any $h \in H$, $hg^nh^{-1} \neq g^{-n}$. In addition, G is not virtually $\mathbb{Z}$.

In [BeFu], Bestvina–Fujiwara extended this machinery, and observed the following. Instead of stating this, we shall state some improved version, which is implicitly in [CalFu].

Theorem 11.3.1. (*Bestvina–Fujiwara* [BeFu], *Calegari–Fujiwara* [CalFu]) *Let G be a group. Suppose there exists a hyperbolic metric space (X, d) such that following hold true:*

(a) *The group G acts on a hyperbolic space X isometrically $\alpha\colon G \curvearrowleft X$.*

(b') *The action α is acylindrical, in the sense in Definition* 11.2.18.

(c) *There exists $g \in G$ of infinite order such that g acts on X by a hyperbolic isometry, and for any $n > 0$ and any $h \in H$, $hg^nh^{-1} \neq g^{-n}$. In addition, G is not virtually $\mathbb{Z}$.*

Then we have
$$\widetilde{QH}(G) = 0.$$
Moreover, this space is infinite dimensional.

More precisely, firstly Bestvina–Fujiwara shown that if a group G acts on a hyperbolic space X isometrically which satisfies the *WPD condition* (*weak proper continuity*), then $\widetilde{QH}(G)$ is infinite dimensional. This condition is weaker than condition (b') (the condition of the action being acylindrical) together with condition (c). For details of the WPD condition, we refer to Section 3 of [BeFu]. Later Calegari–Fujiwara essentially shown that if the action is moreover acylindrical (and satisfies condition (c) above), then there exists a uniform positive lower bound for scl of certain elements of $[G, G]$ (recall Theorem 10.3.7: the Bavard duality theorem).

By combining this theorem together with Theorem 11.2.15, Theorem 11.2.20 and Theorem 11.2.13, Bestvina–Fujiwara obtained the following result on $\widetilde{QH}(\Lambda)$ for each subgroup of mapping class groups. Recall that a group is said to be virtually abelian if it contains an abelian subgroup with finite index.

Theorem 11.3.2. (*Bestvina–Fujiwara* [BeFu]) *Let* Σ *be a surface and* Λ *be a subgroup of* $\mathrm{MCG}(\Sigma)$. *Then* $\widetilde{QH}(\Lambda) \neq 0$ *if and only if* Λ *is not virtually abelian. Moreover, in this case, this space is infinite dimensional.*

Remark 11.3.3. By employing this theorem, Bestvina–Fujiwara gave another proof of Theorem 11.2.22 *without appealing to the Margulis finiteness theorem.* The following is the outline of their argument: let $\phi\colon \Gamma \twoheadrightarrow \Lambda \leqslant \mathrm{MCG}(\Sigma)$ be a homomorphism, where Γ is a higher rank lattice. The Burger–Monod theorem (Theorem 10.2.7) tells that

$$\widetilde{QH}(\Gamma) = 0.$$

Therefore by pulling back argument, we have

$$\widetilde{QH}(\Lambda) = 0.$$

Then Theorem 11.3.2 applies and Λ must be virtually abelian. However, any higher rank lattice Γ (and hence its finite index subgroup) is known to have finite abelianization (if Γ is totally higher rank lattice in the sense in Chapter 0, then it has (T) and this fact follows). Therefore, Λ is finite, as desired.

In the same argument, Theorem G will follow if universal lattices and symplectic universal lattices satisfy $\widetilde{QH} = 0$. However, as we discussed in Subsection 10.5.3, it will be very standing to determine whether $\widetilde{QH} = 0$ holds in these cases.

11.3.2 An Out(F_n) complex of Bestvina–Feighn

In the proof of Theorem 11.3.2, a curve graph plays a key role. In [BeFe], Bestvina–Feighn provided with some analogue for $\mathrm{Out}(F_n)$. Recall the definition of $\overline{\mathcal{PT}}$ and translation length from Subsection 11.2.3.

Theorem 11.3.4. (*Bestvina–Feighn* [BeFe]) *Let* $n \geq 2$. *For any finite collection* $f_1, \ldots, f_k$ *of fully irreducible elements of* $\mathrm{Out}(F_n)$, *there exists a connected hyperbolic graph* X *equipped withan isometric action of* $\mathrm{Out}(F_n)$ *such that the following hold:*

(*i*) *The stabilizer in* $\mathrm{Out}(F_n)$ *of a simplicial tree in* $\overline{\mathcal{PT}}$ *has bounded orbits.*

(*ii*) *The stabilizer in* $\mathrm{Out}(F_n)$ *of a proper free factor* $L < F_n$ *has bounded orbits*

(*iii*) *The elements* $f_1, \ldots f_k$ *have nonzero translation lengths.*

(iv) *The elements* $f_1, \ldots f_k$ *satisfy the* weak proper discontinuity condition*: that means, for every* $1 \leq i \leq k$*, every* $x \in X$ *and every* $R > 0$*, there exists* $N > 0$ *such that the following holds:*

$$\{g \in \mathrm{Out}(F_n) : d(x, gx) \leq R \ \ \textit{and} \ \ d(f_i^N x, g f_i^N x) \leq R\} < \infty.$$

They observed that this construction is enough to deduce nonvanishing of $\widetilde{QH}(\mathrm{Out}(F_n))$. Thus they obtained the following theorem. For details, see Corollary 4.30 in [BeFe] and Section 3 of [BeFu].

Theorem 11.3.5. (*Bestvina–Feighn* [BeFe]) *Let* $n \geq 2$ *be a surface and* Λ *be a subgroup of* $\mathrm{Out}(F_n)$*. If* Λ *contains two independent fully irreducible elements, then we have*

$$\widetilde{QH}(\Lambda) \neq 0.$$

Moreover, in this case, this space is infinite dimensional.

Bestvina–Feighn stated that this construction of the graph is not ideal (compared with that of the curve graph for a mapping class group), because the graph X depends on the finite collection $f_1, \ldots, f_k$.

11.4 Quasi-cocycles on $\mathrm{MCG}(\Sigma)$ and $\mathrm{Out}(F_n)$

In the view of application to homomorphism rigidity from (symplectic) universal lattices to $\mathrm{MCG}(\Sigma)$ or $\mathrm{Out}(F_n)$, nonvanishing of actual quasi-homomorphism spaces for (certain) $\Lambda \leqslant \mathrm{MCG}(\Sigma), \mathrm{Out}(F_n)$seems insufficient. This is because vanishing of this space is not verified at the present for (symplectic) universal lattices Γ (Remark 11.3.3). However vanishing of $\widetilde{QH}(\Gamma; \pi, \mathfrak{H})$ is established for any unitary Γ-representation $(\pi, \mathfrak{H})$ with $\pi \not\supseteq 1_\Gamma$. This is a corollary of Theorem B and Theorem D.

Thus results on $\widetilde{QH}(\Lambda; \pi, \mathfrak{H})$ for certain $\Lambda \leqslant \mathrm{MCG}(\Sigma)$ or $\Lambda \leqslant \mathrm{Out}(F_n)$ is ideal forthe proof of Theorem G. Here $(\pi, \mathfrak{H})$ is a unitary Λ-representation with $\pi \not\supseteq 1_\Gamma$. Such example is given by U. Hamenstädt [Ham], for the case of $\Lambda \leqslant \mathrm{MCG}(\Sigma)$. She consider the actual quasi-cocycle space with *left regular representation* coefficient. In this section, we see her theorem, which is a key to completing the proof of Theorem G.

For the case of $\Lambda \leqslant \mathrm{Out}(F_n)$, a work of Bestvina–Bromberg–Fujiwara in progress studies isometric actions of a group on *quasi-trees*. By means of this they have obtained nonvanishing result in the setting above.

11.4.1 A theorem of Hamenstädt

In [Ham], Hamenstädt examined the space $\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda))$ (recall Definition 7.1.6) for groups acting "nicely" on a hyperbolic space. Here λ is the left regular representation of λ (in fact, she considered any ℓ^p setting). The following is the main theorem of [Ham].

Theorem 11.4.1. (*Hamenstädt* [Ham]) *Let Λ be a discrete group. Suppose Λ admits an isometric action on a hyperbolic space which is non-elementary and weakly acylindrical. Then*

$$\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda)) \neq 0,$$

where λ is the left regular representation of λ. Moreover, in this case, $\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda))$ is infinite dimensional.

The same result holds if $\ell^2(\Lambda)$ is replaced with $\ell^p(\Lambda)$, for any $p \in (1, \infty)$.

We do not explain the definition of non-elementary actions or that of weakly acylindrical actions. However, as the name suggests, all acylindrical (isometric) actions are weakly acylindrical. The proof of this theorem is involved, and employs dynamical properties. We refer to the original paper [Ham] for details. By combining this theorem with Theorem 11.2.15 and Theorem 11.2.20, Hamenstädt obtained the following result.

Theorem 11.4.2. (*Hamenstädt* [Ham]) *Let $\Sigma = \Sigma_{g,l}$ be a surface. Let $\Lambda \leqslant \mathrm{MCG}(\Sigma)$. Suppose Λ contains two independent pseudo-Anosov elements. Then*

$$\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda)) \neq 0,$$

where λ is the left regular representation of λ. Moreover, in this case, $\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda))$ is infinite dimensional.

The same result holds if $\ell^2(\Lambda)$ is replaced with $\ell^p(\Lambda)$, for any $p \in (1, \infty)$.

We note that roughly speaking, $\widetilde{QH}(G; \lambda_G, \ell^2(G))$ easily vanishes. For instance, it is known that if G is a product of infinite groups, then the space above always vanishes (see [MoSh1], [MoSh2]). Therefore, it is not possible to expect the same statement as in Theorem 11.3.2 for $\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda))$.

11.4.2 Another approach – actions on quasi-trees

Definition 11.4.3. Let (X, d_X), (Y, d_Y) be metric spaces.

(i) A (not necessarily continuous) function $f\colon X \to Y$ is called a *quasi-isometry* if there exists constants $A \geq 1$, $B \geq 0$ such that the following two conditions hold:

(1) For any $x_1, x_2 \in X$,

$$\frac{1}{A} d_Y(f(x_1), f(x_2)) - A \leq d_X(x_1, x_2) \leq A d_y(f(x_1), f(x_2)) + A.$$

(2) For any $y \in Y$, there exists $x \in X$ such that

$$d_Y(y, f(x)) \leq B.$$

(ii) The metric space (X, d_X) is said to be *quasi-isometric to* (Y, d_Y) if there exists a quasi-isometry $f\colon X \to Y$.

(iii) A *quasi-tree* is a metric space (X, d) which is quasi-isometric to a tree.

In [Man], J. F. Manning considered a strengthening of Serre's property (FA) (see Definition 2.5.13) and defined property (QFA) as follows: a group G is said to have *property* (QFA) if any isometric action on any quasi-tree has finite orbits. He shown that $\mathrm{SL}_{m\geq 3}(\mathbb{Z})$ enjoys this property.

In a work [BBF] in progress, Bestvina–Bromberg–Fujiwara have been studying in the opposite direction. They shown that the following groups admits an unbounded action on a quasi-trees by isometries, *which also satisfies certain additional condition*: any non-elementary hyperbolic group; any subgroups of $\mathrm{MCG}(\Sigma)$ containing two independent pseudo-Anosov elements; any subgroups of $\mathrm{Out}(F_n)$ containing two independent fully irreducible elements (compare with the fact that $\mathrm{Out}(F_n)$ $(n \geq 3)$ has (FA), [Bog], [CuVo2]). By this result, they in particular show the following theorem:

Theorem 11.4.4. (*Bestvina–Bromberg–Fujiwara* [BBF]) *Let $n \geq 2$. Let $\Lambda \leqslant \mathrm{Out}(F_n)$. Suppose Λ contains two independent fully irreducible elements. Then*

$$\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda)) \neq 0,$$

where λ is the left regular representation of λ. Moreover, in this case, $\widetilde{QH}(\Lambda; \lambda, \ell^2(\Lambda))$ is infinite dimensional.

This result is also a key to ending the proof of Theorem G.

11.5 Homomorphism rigidity into $\mathrm{MCG}(\Sigma)$ and $\mathrm{Out}(F_n)$

In this section, we prove Theorem G, with the aid of all preliminary facts above in this chapter. We restate our goal:

Theorem 11.5.1. ([Mim4]) *Let Γ be a finite index subgroup either of $\mathrm{SL}_m(\mathbb{Z}[x_1, \ldots, x_k])$ $(m \geq 3)$ or of $\mathrm{Sp}_{2m}(\mathbb{Z}[x_1, \ldots, x_k])$ $(m \geq 2)$. Then for any $g \geq 0$ and $n \geq 2$, every homomorphism*

$$\Phi\colon \Gamma \to \mathrm{MCG}(\Sigma_g)$$

and every homomorphism

$$\Psi\colon \Gamma \to \mathrm{Out}(F_n)$$

have finite image. In particular, every homomorphism

$$\Gamma \to \mathrm{MCG}(\Sigma_{g,l}) \quad (l \geq 1)$$

and

$$\Gamma \to \mathrm{Aut}(F_n)$$

also have finite image.

Instead of establishing this theorem, we will prove the following theorem:

Theorem 11.5.2. ([Mim4]) *Let* Γ *be a group with* (TT)/T. *Then all of the conclusions in Theorem* 11.5.1 *hold true.*

Proof. (Theorem 11.5.2) Firstly, recall that (TT)/T (= ($\mathrm{FF}_{\mathcal{H}}$)/T) implies (T) (see Lemma 8.1.2). Also recall that (TT)/T passes to group quotients and to finite index subgroups (see Proposition 8.1.4). In particular, in the statement of the theorem, $\Phi(\Gamma)$ and $\Psi(\Gamma)$ has (TT)/T.

Secondly, the second half of the four assertions follow from the first half. This follows from that $\mathrm{MCG}(\Sigma_{g,l})$ ($l \geq 1$) and $\mathrm{Aut}(F_n)$ are subgroups in $\mathrm{Out}(F_{n'})$ for sufficiently large n' (see Subsection11.2.1). Therefore we deal with the first two statements.

Case1. with target of mapping class groups

Let $\Phi\colon \Gamma \to \mathrm{MCG}(\Sigma_g)$ be a homomorphism. Firstly, we note that for exceptional cases the conclusion holds, because then $\mathrm{MCG}(\Sigma_{g,l})$ is virtually free and has the Haagerup property (see Section 2.5). Therefore, hereafter, we assume every surface which appears in this proof is non-exectional.

Let $\Lambda \leqslant \mathrm{MCG}(S)$ be the image of Φ. We employ Theorem 11.2.13, the subgroup classification result of [McPa]. For convenience we restate here: Λ is either of the following forms:

(*i*) the group Λ is finite;

(*ii*) the group Λ is reducible: there exists a collection $\mathfrak{C}$ of essential simple closed pairwise disjoint curves such that every element of Λ fixes $\mathfrak{C}$;

(*iii*) the group Λ is virtually $\mathbb{Z}$;

(*iv*) the group Λ contains two independent pseudo-Anosov elements.

Then we appeal to Theorem 11.4.2 of Hamenstädt, and exclude option (*iv*). This part is the key in this proof. More precisely, Λ has property (TT)/T. However, if option (*iv*) occurs, then by Theorem 11.4.2 of Hamenstädt,

$$\widetilde{QH}(\Lambda;\lambda,\ell^2(\Lambda)) \neq 0.$$

Since option (*iv*) (or the condition above already) implies Λ is infinite. Therefore $\lambda \not\supseteq 1_\Lambda$. This contradicts (TT)/T for Λ.

Property (T) excludes option (*iii*). Therefore for the proof, it suffices to show that $\Phi(\Gamma)$ must be virtually abelian in the case of option (*ii*). If it is shown, then property (T) forces Λ to be finite.

Suppose option (*ii*) occurs. Take a maximal curve system $\mathfrak{C}$ preserved by Λ. Cut Σ open along $\mathfrak{C}$ and replace each boundary circle of the resulting bordered surface with a puncture. Then we get a possibly disconnected surface Σ'. Let $\Sigma'_1,\ldots,\Sigma'_n$ be connected components of Σ'. Then there is a homomorphism:

$$\begin{aligned}\Lambda \twoheadrightarrow \Lambda' &\leqslant (\mathrm{MCG}(\Sigma'_1)\times\cdots\times\mathrm{MCG}(\Sigma'_n))\rtimes\mathfrak{S}\\ &\longrightarrow \mathfrak{S}.\end{aligned}$$

Here $\mathfrak{S}$ is a subgroup of $\mathfrak{S}_n$, the symmetric group of degree n, and $\mathfrak{S}$ acts by permutations on mapping class groups of homeomorphic surfaces among $\Sigma'_1,\ldots,\Sigma'_n$. And the second homomorphism is the projection to $\mathfrak{S}$. It is known the kernel of the map $\Lambda\to\Lambda'$ is a free abelian group, generated by multiple Dehn twists associated to the curves of the curve system $\mathfrak{C}$ (for instance, see Chapter 4 of [Iva]). Take the kernel $\Lambda'_0 \trianglelefteq \Lambda'$ of the map $\Lambda'\to\mathfrak{S}$. Then there is a natural map from Λ'_0 to each $\mathrm{MCG}(\Sigma'_i)$, which takes the i-th component.

Note that by definition Λ'_0 is a finite index subgroup of Λ'. Therefore again Theorem 11.4.2 tells us that each image of Λ'_0 inside $\mathrm{MCG}(\Sigma'_i)$ is either finite or reducible. However it cannot be reducible. Indeed, if an image of Λ'_0 fixes a curve system, then by translation by $\mathfrak{S}$ we have a curve system on $\Sigma'_1\cup\cdots\cup\Sigma'_n$ which is preserved by Λ'. This contradicts the maximality of the curve system $\mathfrak{C}$, which we took at the beginning. Therefore Λ'_0 must be finite, and thus Λ is virtually abelian. This ends our proof.

Case2. with target of outer automorphism groups

This part is based on the argument in the work of Bridson–Wade [BrWa], who have utilized study of $\overline{\mathrm{IA}}_n$ to show homomorphism rigidity result. (Recall from Definition 11.2.3 that $\overline{\mathrm{IA}}_n$ denotes the kernel of the map $\mathrm{Out}(F_n)\to\mathrm{GL}_n(\mathbb{Z})$ induced by the abelianization of F_n. First we note that the conclusion holds for $n=2$ because then $\mathrm{Out}(F_n)$ has the Haagerup property.

Let $\Lambda\leqslant\mathrm{Out}(F_n)$ be the image of Γ by Ψ. Firstly, we appeal to Theorem 11.2.15, the classification of subgroups in $\mathrm{Out}(F_n)$ by Handel–Mosher [HaMo]: a subgroup $\Lambda<\mathrm{Out}(F_n)$ is either of the following forms:

(i) the group Λ is not fully irreducible: there exists a finite index subgroup which preserves each conjugacy class of some proper free factor of F_n;

(ii) the group Λ s virtually $\mathbb{Z}$;

(iii) the group Λ contains two independent fully irreducible elements.

Secondly, we appeal to Theorem 11.4.4 of Bestvina–Bromberg–Fujiwara and exclude option (ii). This is done in a similar manner to one in Case 1. Property (T) excludes option (i). We shall show that a finite index subgroup $H \leqslant \Lambda$ as in option (2) must be finite, by induction on n.

If $n = 2$, then we have already seen it. Suppose the assertion holds true for every natural number $< n$, and we will verify the case of n. We take $H \leqslant \Lambda$ and a free factor L as in option (ii). Then for any $[f] \in H$, one can choose an element $g_f \in F_n$ such that $f(L) = g_f^{-1} L g_f$ holds. Then the map $L \ni h \mapsto g_f h g_f^{-1} \in L$ is an element of $\mathrm{Aut}(L)$, and the image in $\mathrm{Out}(L)$ is uniquely determined by $[f] \in H$. This induces a well-defined group homomorphism $H \to \mathrm{Out}(L)$. Likewise, the action on $F_n/\langle\langle L\rangle\rangle$ induces a homomorphism $H \to \mathrm{Out}(L')$. Because the ranks of L and L' are strictly less than n, the assumption of induction applies. Hence both images are finite. By taking abelianization of $L * L'$, with respect to the union of basis for L and L' the action of H is of the following form:

$$\begin{pmatrix} G & 0 \\ * & G' \end{pmatrix},$$

where both G and G' are finite. Hence the image of the homomorphism $H \to GL_n(\mathbb{Z})$ is virtually abelian. Since H is a finite index subgroup of Λ, H in particular has property (T). By combining these two, we have the image is in fact finite. Therefore, the kernel H' of the map above is a finite index subgroup of H.

Note that H' is in $\overline{\mathrm{IA}}_n$. Finally we appeal to Theorem 11.2.5 of Bass–Lubotzky and Bridson–Wade that every nontrivial subgroup in $\overline{\mathrm{IA}}_n$ surjects onto $\mathbb{Z}$. On the other hand, H' is a finite index subgroup of H and hence has property (T). Therefore H' must be trivial, and thus Λ is finite. This ends our proof.

□

In the last part of this section, we make a remark. It is not known whether property (T) itself does imply homomorphism rigidity above. However, for instance, homomorpshim rigidity for $\mathrm{Sp}_{m,1}$ into MCG(Σ) is obtained by S.-K. Yeung [Yeu]. Also, J. E. Andersen [Ande] has announced that mapping class groups do *not* have (T).

Proof. (Theorem 11.5.1) Theorem 11.5.2, together with Theorem B and Theorem D completes the proof. □

11.6 A shortcut for noncommutative universal lattice cases

Here we see one shortcut of Theorem 11.5.1, for universal lattices. The key is study on distorted elements. We prove Theorem H. Finally, we explain why this argument may not work in symplectic lattices case.

11.6.1 Distorted elements in a finitely generated group

Definition 11.6.1. Let G be a finitely generated group. An element $g \in G$ is called a *distorted element* if

$$\lim_{n\to\infty} \frac{l_S(g^n)}{n} = 0.$$

Here S is a finite generating set (the choice of such S does not affect the definition above), and l_S denotes the word length with respect to S. The element g is said to be *undistorted* otherwise.

Remark 11.6.2. In the literatures, it is more common to ask a distorted element *not* to be a torsion. In this thesis, however, we allow the case of g being a torsion because it fits our purpose.

An obvious example of distorted elements (in the sense in this thesis, see the remark above) is a torsion. The following theorems resprectively Farb–Lubotzky–Minsky [FLM] (and L. Mosher [Mos]); and E. Alibegović [Ali] state these are only examples in $\mathrm{MCG}(\Sigma)$; $\mathrm{Out}(F_n)$.

Theorem 11.6.3. ([FLM], [Mos]; [Ali]) *Let Σ be a surface and $n \geq 2$. Then any element respectively in $\mathrm{MCG}(\Sigma)$; and in $\mathrm{Out}(F_n)$ which is not a torsion is undistorted.*

This is a key to the shortcut as we are mentioning.

11.6.2 The proof

We restate Theorem H:

Theorem 11.6.4. ([Mim4]) *Let Γ be a finite index subgroup either of $\mathrm{E}_m(\mathbb{Z}\langle x_1, \ldots, x_k\rangle)$ $(m \geq 3)$. Then for any $g \geq 0$ and $n \geq 2$, every homomorphism*

$$\Phi\colon \Gamma \to \mathrm{MCG}(\Sigma_g)$$

and every homomorphism

$$\Psi\colon \Gamma \to \mathrm{Out}(F_n)$$

have finite image.

In particular, every homomorphism

$$\Gamma \to \mathrm{MCG}(\Sigma_{g,l}) \quad (l \geq 1)$$

and

$$\Gamma \to \mathrm{Aut}(F_n)$$

also have finite image.

The shortcut has its origin in the proof of Theorem 11.2.22 for the case of that the higher rank lattice is non-cocompact, which was shown by Ivanov before the general case of [FaMas]. We explain (some variant of) the proof shortly. Suppose it is known by a theorem of Lubotzky–Mozes–Raghunathan [LMR] that any such lattice has a distorted element of infinite order. Then by Theorem 11.6.3, this implies the homomorphism has infinite kernel. Finally, the Margulis finiteness theorem ends the proof. We also note that with $\mathrm{Out}(F_n)$ target, Bridson–Farb [BrFa] observed a similar result before the general result Theorem 11.2.23.

In our case (universal lattices), this group does not have the Margulis finiteness property. However, we can make a way along this argument. That is the shortcut.

Proof. (Theorem 11.6.4) We only consider the first case because the argument below works without any change. Set $R = \mathbb{Z}\langle x_1, \ldots, x_k\rangle$. Set $H = \mathrm{MCG}(\Sigma)$. By Theorem 11.2.4, there exists a finite index subgroup of H which is torsion-free. We choose one and name it H_0.

Let $\Psi\colon \Gamma \to H$ be a homomorphism. Then $\Phi(\Gamma) \cap H_0$ is a finite index subgroup in $\Phi(\Gamma)$ and hence $\Gamma_0 := \Phi^{-1}(\Psi(\Gamma) \cap H_0)$ is a finite index subgroup of Γ. For distinct pair (i, j) with $1 \leq i, j \leq m$, define a subset of R as $R^0_{i,j}$:=$\{r \in R : E_{i,j}(r) \in \Gamma_0\}$. Since for fixed i, j $\{E_{i,j}(r) : r \in R\} \cong R$ as additive groups, $R^0_{i,j}$ is a finite index subgroup in the additive group R.

Note that if $\mathrm{E}_{i,j}(r) \in \Gamma$, then it is a distorted element in Γ. This follows form the commutator relation:

$$[E_{i,j}(r), E_{j,l}(s)] = E_{i,l}(rs) \quad (i \neq j, j \neq l, l \neq i; r, s \in R).$$

By the construction of H_0, we conclude that for any (i, j) and any $r \in R^0_{i,j}$, $\Phi(E_{i,j}(r)) = e_H$. Indeed, the relation above implies each $\Phi(E_{i,j}(r))$ $(r \in R^0_{i,j})$ is distorted, and Theorem 11.6.3 shows that only distorted element in H_0 is e_H. We next claim that $R^0 := \bigcap_{i,j} R^0_{i,j}$ is a *subring* of R. Indeed, the commutator relation above implies is R is closed under multiplication. Therefore R^0 is a subring of R of finite index. We use the following theorem of J. Lewin (Lemma 1 in [Lew]): *for any finitely generated* (*possibly noncommutative*) *ring* Q, *any finite index subring* Q_0 *of* Q *contains a* (*two sided*) *ideal* J *of* Q *which is a finite index subring of* Q_0. Thus we have a ideal $\mathcal{I}$ of R of finite index, which is included in in R_0.

Finally, we employ the following folklore result (the proof can be found in Lemma 17 of [KaSa]): *for a finite ring S the Steinberg group* $\mathrm{St}_{m\geq 3}(S)$ *is finite.* Here recall Definition 9.2.4 for the Steinberg groups. Now we are done because Φ factors through a subgroup of $\mathrm{St}_m(R/\mathcal{I})$, and this group itself is finite. □

11.6.3 Difficulty for symplectic universal lattices

In fact, the finiteness of the Steinberg group over a finite ring is extended to general cases (, more precisely, twisted Steinberg groups, see Subsection 9.2.2). For the proof of this fact, we refer to Proposition 4.5 of a paper [Rap] of I. A. Rapinchuk. This implies that part can be extended to the case of symplectic universal lattices.

However, as we have seen in Subsection 9.1.1, the commutator relation among elementary symplectic groups are much complicated. The gap lies in the point that we have deduced R_0 in the proof above is multiplication closed.

Appendix I

Relative Kazhdan constant for uniformly bounded representations

In this appendix, we shall prove Proposition I. For convenience, we restate it here.

Proposition I.0.5. *Let* $A_k = \mathbb{Z}[x_1, \ldots, x_k]$ *and set* $G = \mathrm{E}_2(A_k) \ltimes A_k^2$ *and* $N = A_k^2 \trianglelefteq G$. *Set* S *be the set of all unit elementary matrices in* G $(\subset \mathrm{SL}_3(A_k))$. *Then there is an inequality*

$$\overline{\mathcal{K}}(G, N; S; M) > (15k + 100)^{-1} M^{-6}.$$

In the case of $k = 0$, *one has* $\overline{\mathcal{K}}(\mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2; F; M) > (21M^6)^{-1}$. *Here the symbol* $\overline{\mathcal{K}}(G, N; S; M)$ *denotes the relative Kazhdan constant for uniformly bounded representations, which is defined in Definition* 3.5.2.

For the proof, we firstly observe the following. Let Γ be a group and $M \geq 1$. Let $(\rho, \mathfrak{H}) \in \mathcal{A}_M$, namely, ρ is a uniformly bounded Γ-representation on a Hilbert space with $|\rho| \leq M$. For this given ρ, we define the norm $\|\cdot\|_\rho$ on $\mathfrak{H}$ as the dual norm of the following norm $\|\cdot\|_{\rho *}$:

$$\text{for } \phi \in \mathfrak{H}^*,\ \|\phi\|_{\rho *} := \sup_{g \in \Gamma} \|\rho^\dagger(g)\phi\|_{\mathfrak{H}^*}.$$

Here $\rho^\dagger$ denotes the contragredient representation of ρ on $\mathfrak{H}$ (see Definition 3.1.6). This norm $\|\cdot\|_\rho$ then satisfies the following three properties:

(1) $\|\cdot\|_\rho$ is compatible with $\|\cdot\|_{\mathfrak{H}}$ with the norm ratio $\leq M$. More precisely,

$$M^{-1}\|\cdot\|_{\mathfrak{H}} \leq \|\cdot\|_\rho \leq \|\cdot\|_{\mathfrak{H}}.$$

(2) ρ is isometric with respect to $\|\cdot\|_\rho$.

(3) $(\mathfrak{H}, \|\cdot\|_\rho)$ is us.

For item (3) in fact we have some estimate of modulus of smoothness. Indeed, thanks to Lemma 3.1.3, we have the following:

for any $\tau > 0$, the inequality $r_{\|\cdot\|_\rho}(\tau) \leq \sqrt{1+M^2\tau^2} - 1 \leq M^2\tau^2/2$ holds.

Proof. (Proposition I.0.5) Let $\epsilon > 0$. We stick to the notation in the proof of Theorem 6.1.1. Suppose that there exists a G-representation $(\rho, \mathfrak{H}) \in \mathcal{A}_M$ such that the following holds: ρ admits a non-zero vector ξ in $\mathfrak{H}'_{\rho(N)}$ which satisfies $\sup_{s\in S} \|\xi - \rho(s)\xi\|_{\mathfrak{H}} \leq \epsilon\|\xi\|_{\mathfrak{H}}$. We may assume that $\mathfrak{H}^{\rho(N)} = 0$. For this $(\rho, \mathfrak{H})$, we take the us norm $\|\cdot\|_\rho$ defined in the paragraph above. Thus by applying Lemma 3.1.5 and (precise estimate of) item (3) above, we can assume that there exists $\xi \in \mathfrak{H}$ with $\|\xi\|_\rho = 1$ such that the following two inequalities hold:

(a) $\sup_{s\in S} \|\xi - \rho(s)\xi\|_\rho \leq M\epsilon$.

(b) $\sup_{s\in S} \|\xi^* - \rho^\dagger(s)\xi^*\|_{\rho*} \leq 4M^3\epsilon$.

Here $\xi \mapsto \xi^*$ is a duality mapping.

Thanks to Dixmier's unitarization (Proposition 6.1.2), we have an invertible operator $T \in \mathbb{B}(\mathfrak{H})$ with

$$\|T\|_{\mathbb{B}(\mathfrak{H})}\|T^{-1}\|_{\mathbb{B}(\mathfrak{H})} \leq M^2$$

such that $\pi := \mathrm{Ad}(T) \circ \rho\mid_N$ is unitary. Let $\hat{N}$ denote the unitary dual of N. By general theory of Fourier analysis, one obtains a standard unital $*$-hom $\sigma : \mathrm{C}(\hat{N}) \to \mathbb{B}(\mathfrak{H})$ from the unitary operators $\pi(N)$. Indeed, for $i \in \{1, 2\}$, let $z_i \in \mathrm{C}(\mathbb{T}^2)$ be the map $t \mapsto e^{2\pi\sqrt{-1}t_i}$ (t_i is the i-th component of $t \in \mathbb{T}^2$) and let $l_i \in S_0$ be $E_{i,3}(1)$. We define σ by setting $\sigma(z_i) = T\rho(l_i)T^{-1}$ for each i.

Then from Riesz–Markov–Kakutani theorem, from this σ one obtains the complexed-valued regular Borel measure ν on $\hat{N}$ satisfying the following: for any $f \in \mathrm{C}(\hat{N})$, $\int_{\hat{N}} f\, d\nu = \langle T^{-1}\sigma(f)T\xi, \xi^*\rangle$ (we note that $T = I$ in our proof of Theorem 6.1.1). We take the Jordan decomposition of $\mathrm{Re}\nu = \nu_+ - \nu_-$. Here $\nu_+ \perp \nu_-$ (this means they are singular to each other) and both of them are positive regular Borel measure. Then the inequality $\nu_+(\hat{N}) \geq 1$ holds.

For the proof of the proposition, first we discuss the case of that $k = 0$. We take the following well-known decomposition of $\hat{N} = \mathbb{T}^2 \cong \left\{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} : t_1,\, t_2 \in \left[-\frac{1}{2}, \frac{1}{2}\right)\right\}$:

$$\begin{aligned}
&\{0\},\ D_0 = \{|t_1| \geq 1/4 \text{ or } |t_2| \geq 1/4\},\\
&D_1 = \{|t_2| \leq |t_1| < 1/4 \text{ and } t_1t_2 > 0\},\ D_2 = \{|t_1| < |t_2| < 1/4 \text{ and } t_1t_2 \geq 0\},\\
&D_3 = \{|t_1| \leq |t_2| < 1/4 \text{ and } t_1t_2 < 0\},\ D_4 = \{|t_2| < |t_1| < 1/4 \text{ and } t_1t_2 \leq 0\}.
\end{aligned}$$

We consider the natural $\mathrm{SL}_2(\mathbb{Z})$-action on $\mathbb{T}^2$ defined as follows: for any $h \in \mathrm{SL}_2(\mathbb{Z})$, the action map $\hat{h}$ of $h : t \mapsto \hat{h}t$ is the left multiplication of the matrix $\hat{h} = {}^th^{-1}$. . Then one can check the following equality: for any $g \in \mathrm{SL}(2, \mathbb{Z})$ and any $f \in \mathrm{C}(\mathbb{T}^2)$, $\sigma(\hat{h}f) = T\rho(h)T^{-1}\sigma(f)T\rho(h^{-1})T^{-1}$. With some calculation, one can also obtain the following two estimations:

(i) The inequality $\nu_+(D_0) \leq 4M^7\epsilon^2$ holds.

(ii) For any Borel subset $Z \subset \mathbb{T}^2$ and any $h \in S_0(\subseteq \mathrm{SL}_2(\mathbb{Z}))$, the inequality $|\nu_+(\hat{h}Z) - \nu_+(Z)| \leq 5M^6\epsilon$ holds.

Indeed, for instance, inequality (i) follows from the argument below. For $i = 1, 2$, set $D_0^i = \mathrm{supp}\nu_+ \cap \{|t_1| \geq 1/4\} \subset D_0$. By approximating (pointwisely) $\chi_{D_0^i}$ by continuous functions and obtaining an associated projection $P \in \mathbb{B}(\mathfrak{H})$, one can make estimate as follows: for each $i \in \{1, 2\}$,

$$
\begin{aligned}
2\nu_+(D_0^i) \leq &\left| \int_{D_0^i} |1 - z_i|^2 \, d\nu \right| = \left| \int_{\mathbb{T}^2} \overline{(1 - z_i)} \chi_{D_0^i} (1 - z_i) \, d\nu \right| \\
= &\left| \left\langle T^{-1} \sigma \, (1 - z_i)^* \, P \, \sigma \, (1 - z_i) \, T\xi, \xi^* \right\rangle \right| \\
= &\left| \left\langle T^{-1} \left(I - T\rho(l_i)T^{-1} \right)^* P \left(I - T\rho(l_i)T^{-1} \right) T\xi, \xi^* \right\rangle \right| \\
= &\left| \left\langle T^{-1} \left(I - T\rho(l_i^{-1})T^{-1} \right) PT \left(I - \rho(l_i) \right) \xi, \xi^* \right\rangle \right| \\
&\text{(Recall } \mathrm{Ad}(T) \circ \rho\,|_N \text{ is unitary),} \\
= &\left| \left\langle \left(I - \rho(l_i^{-1}) \right) T^{-1} PT \left(I - \rho(l_i) \right) \xi, \xi^* \right\rangle \right| \\
= &\left| \left\langle T^{-1} PT \left(\xi - \rho(l_i)\xi \right), \xi^* - \rho^\dagger(l_i)\xi^* \right\rangle \right| \\
\leq &\left\| T^{-1} PT \left(\xi - \rho(l_i)\xi \right) \right\|_\rho \left\| \xi^* - \rho^\dagger(l_i)(\xi^*) \right\|_{\rho*} \\
\leq &M \left\| T^{-1} PT \right\|_{\mathbb{B}(\mathfrak{H})} \left\| \xi - \rho(l_i)\xi \right\|_\rho \left\| \xi^* - \rho^\dagger(l_i)\xi^* \right\|_{\rho*} \\
\leq &4M^7\epsilon^2 \quad \text{(by item } (a) \text{ and item } (b)\text{).}
\end{aligned}
$$

In above, we remark that for $V \in \mathbb{B}(\mathfrak{H})$, V^* means the adjoint operator of V. Also, by item (1) in appendix, we note that

$$\|V\eta\|_\rho \leq M\|V\|_{\mathbb{B}(\mathfrak{H})}\|\eta\|_\rho,$$

holds in general.

Thanks to these two estimations, one can verify $\nu_+(D_i) < 5M^6\epsilon + 4M^7\epsilon^2$ for $1 \leq i \leq 4$ (use for instance, $\widehat{h_{1,-}}(D_1 \cup D_2) \subset D_2 \cup D_0$, where $h_{1,-} = E_{1,2}(-1)$). Hence the inequality $\nu_+(\mathbb{T}^2 \setminus \{0\}) \leq 20M^6\epsilon + 20M^7\epsilon^2$ holds. If $\epsilon \leq (21M^6)^{-1}$, then there must exist a non-zero $\rho(N)$-invariant vector. It is a contradiction.

For the general case, let us recall Kassabov's argument in [Kas1]. We identify $\widehat{A_k}$ with the set of all formal power series of variables x_l^{-1} $(1 \leq l \leq k)$ over $\widehat{\mathbb{Z}} \cong \mathbb{T}$. Here the pairing is defined by

$$\langle a x_1^{i_1} \cdots x_k^{i_k} | \phi x_1^{-j_1} \cdots x_k^{-j_k} \rangle = \phi(a)\delta_{i_1,j_1} \cdots \delta_{i_k,j_k}.$$

We define the *valuation* v on $\widehat{A_k}$ as the minimum of the total degrees of all terms. Here we naturally define $v(0) = +\infty$. We decompose $\hat{N} \setminus \{0\} = \widehat{A_k^{\,2}} \setminus \{0\}$ as follows:

$$
\begin{aligned}
&A = \{(\chi_1, \chi_2) : v(\chi_1) > v(\chi_2) > 0\}, \ B = \{(\chi_1, \chi_2) : v(\chi_1) = v(\chi_2) > 0\}, \\
&C = \{(\chi_1, \chi_2) : v(\chi_2) > v(\chi_1) > 0\}, \ D = \{(\chi_1, \chi_2) : v(\chi_1)v(\chi_2) = 0\}.
\end{aligned}
$$

Then from an argument similar to one in [Kas1], we have the following inequalities:

$$\begin{aligned} \nu_+(A) &\le \nu_+(D) + 5(k+1)M^6\epsilon, \\ \nu_+(B) &\le \nu_+(D) + 5kM^6\epsilon, \\ \text{and} \quad \nu_+(C) &\le \nu_+(D) + 5(k+1)M^6\epsilon. \end{aligned}$$

We naturally define the restriction map res: $\hat{N} \to \hat{\mathbb{Z}}^2$ and obtain that $\nu_+(D) = \nu_+(\hat{N} \setminus \mathrm{res}^{-1}\{0\}) \le 20M^6\epsilon + 20M^7\epsilon^2$. Finally, by combining these inequalities we conclude that

$$1 \le \nu_+(\hat{N}) = \nu_+(\hat{N} \setminus \{0\}) \le (15k + 90)M^6\epsilon + 80M^7\epsilon^2.$$

Here the middle equality in above follows from the assumption that $\mathfrak{H}^{\rho(N)} = 0$. Hence in particular ϵ must be more than $(15k + 100)^{-1}M^{-6}$. □

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