Monotonicity and rigidity of the $\mathcal{W}$-entropy on $\mathsf{RCD}^*(0, N)$ spaces

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joint work with X.-D. Li (Chinese Academy of Science)

Dirichlet forms and their geometry
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1. Introduction

$\mathcal{W}$-entropy on Riem. mfd

M: m-dim. cpt. Riem. mfd, $t > 0$, $f \in C^\infty(M)$

$u := \dfrac{\mathrm{e}^{-f}}{(4\pi t)^{m/2}}$, $\displaystyle\int_M u \, \mathrm{d}\,\mathrm{vol} = 1$

$$\mathcal{W}(f, t) := \int \left[t|\nabla f|^2 + f - m \right] u \, \mathrm{d}\,\mathrm{vol}$$

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$$\Rightarrow \frac{d}{dt}\mathcal{W}(f, t) \leq 0$$

Entropy formula and rigidity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(f,t) \leq 0$$

$$\Uparrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = -2\int_M t\left(\left|\nabla^2 f - \frac{g}{2t}\right|^2 + \mathrm{Ric}(\nabla f, \nabla f)\right) u \,\mathrm{d}\,\mathrm{vol}$$

[L. Ni '04]

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⇝ Extension to weighted Riem. mfds [X.-D. Li '12]

Purpose

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Can one extend the monotonicity/rigidity of $\mathcal{W}$ on "Riemannian" metric measure spaces with "$\mathbf{Ric} \geq 0$ & $\mathbf{dim} \leq N$" (**RCD**$(0, N)$ spaces)?

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- Without the entropy formula
  - $\leftsquigarrow$ optimal transport approach
- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof

4.1 Monotonicity

4.2 Rigidity

4.3 Additional remarks

5. Further questions

Examples

- (X, g): m-dim. cpl. Riem. mfd., $\partial X = \emptyset$,
  d: Riem. dist., $\mathfrak{m} = e^{-V} \mathrm{vol}_g$ $\quad (V : X \to \mathbb{R})$
  (Weighted Riem. mfd)

$$\Downarrow$$

$$\mathbf{RCD}^*(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

- (Pointed) measured GH lim. of $\mathbf{RCD}^*(K, N)$ sp.'s
  [$\cdots$/Gigli, Mondino & Savaré '15]

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Met. meas. sp. & heat flow on it

$(X, d, \mathfrak{m})$: Polish geod. met. meas. sp.

($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

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$$2\mathsf{Ch}(f) := \inf \left\{ \varliminf_{n} \int_X \mathrm{lip}(f_n)^2 \mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$

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Definition 1 ([Ambrosio, Gigli & Savaré '14])

$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form ($\Leftrightarrow P_t$: linear $\Leftrightarrow \Delta$: linear)

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RCD$(0, N)$: infin. Hilb., some regularity ass'ns &

$$W_2(P_s f\,\mathfrak{m}, P_t g\,\mathfrak{m})^2 \le W_2(f\,\mathfrak{m}, g\,\mathfrak{m})^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

★ Equiv. cond'ns (up to reg. assn's)

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  (Bakry-Émery's curv.-dim. cond./Bochner ineq.)

- On $(\mathcal{P}_2(X), W_2)$, $^\forall\mu_0$, $^\exists(\mu_t)_{t\ge 0}$ sol. to $(0, N)$-evolution variational inequality of Ent (a (metric) formulation of "$\dot{\mu}_t = -\nabla\,\mathrm{Ent}(\mu_t)$")

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Heat flow

Properties of the heat semigr. P_t under **RCD**$^*(K, N)$

- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
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- $\mu_t = P_t\mu (= \rho_t \mathfrak{m}) \in \mathcal{P}(X)$ satisfies

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(Fisher information)

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- $\mathrm{Ent}(\mu) := \int_X \rho \log \rho\,\mathrm{d}\mathfrak{m}$

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$\mu = \rho\, \mathfrak{m} \in \mathcal{P}(X)$,

$$\mathcal{W}(\mu, t) := tI(\mu) - \mathrm{Ent}(\mu) - \frac{N}{2}\log t + c_1$$
$$= \frac{d}{dt}[t\,\mathrm{Ent}(\mu_t) - t\,\mathrm{Ent}^{\mathrm{Eucl}}(\mathrm{Gauss}(t))]$$

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Main thm

Theorem 2 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: RCD$(0, N)$, $N \geq 2$, $\mu_t := P_t\mu$

(1) ${}^{\forall}u \geq 0$, $\mathcal{W}(\mu_t, t+u) \searrow$ *in* $t \in (0, \infty)$

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Main thm

Theorem 2 ([X.-D. Li & K.])

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Cone

Definition 3 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Leftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2$
  $:= r^2 + s^2 - 2rs\cos(d_Y(x, y) \wedge \pi)$
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- [Jiang & Zhang '16] X: cpt. $\Rightarrow$ Theorem 1 (1)
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
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4.1. Monotonicity

(cf. [Topping '09] on (backward) Ricci flow)

Derivation from RCD$(0, N)$

★ $t\,h(t) := t^2 I(\mu_t) - \dfrac{Nt}{2}$

$$\Rightarrow \boxed{\text{“}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))\text{”}}$$

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$\Rightarrow t\,h(t) \searrow$ in t □

4.2. Rigidity

Geometric properties of $\mathsf{RCD}(0, N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $\Delta d(x_0, \cdot)^2 \leq 2N$ in the distributional sense

- Volume rigidity [Gigli & De Philippis]: $N \geq 2$,

$$\exists x_0,\ \forall r, R > 0,\ \mathfrak{m}(B_R(x_0)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(x_0))$$

$$\Rightarrow X \simeq \begin{array}{l} (0, N-1)\text{-cone of} \\ \text{an } \mathsf{RCD}^*(N-2, N-1) \text{ sp.} \end{array}$$

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★ $\Delta d(x_0, \cdot)^2 = 2N$ "$\Rightarrow$" conclusion

Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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★ "$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 0$" & **RCD**$(0, N)$ cond.

$\Rightarrow h(t) = 0$, i.e. $I(\mu_t) = \frac{N}{2t}$ $(t \in (0, t_*])$

Equality in Laplacian comparison

$$I(\mu_t) = \frac{N}{2t} \; (t \in (0, t_*])$$

Li-Yau: $-\Delta \log p_t^{x_0} = \frac{|Dp_t^{x_0}|^2}{(p_t^{x_0})^2} - \frac{\Delta p_t^{x_0}}{p_t^{x_0}} \le \frac{N}{2t}$

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$$\Downarrow \quad \lim_{t \downarrow 0} 4t \log p_t^{x_0}(x) = -d(x_0, x)^2$$

[Jiang, Li & Zhang '16]

$$\Delta d(x_0, \cdot)^2 = 2N$$

□

4.3. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0, \cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0, x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0, x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0, x)^2}{4t}\right)$$

$\because$ Compute I for "Gaussian kernel" in two ways & RHS enjoys the energy dissipation identity for Ent □

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In particular, X *is non-compact*

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Initial data

Lemma 2

Suppose $I(\mu_t) = \dfrac{N}{2t}$. *Then* $I(p_t^x\,\mathfrak{m}) = \dfrac{N}{2t}$ μ-*a.e.* x.

$\because$ $\mu \mapsto I(\mu)$ convex

$$\Rightarrow \frac{N}{2t} = I(\mu_t)\,\text{“}\le\text{”} \int_X I(p_t^x)\mu(\mathrm{d}x) \overset{\text{Li-Yau}}{\le} \frac{N}{2t} \qquad \square$$

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Suppose $I(\mu_t) = \dfrac{N}{2t}$. *Then* μ *is Dirac.*

$$\because \text{ Reduce to } \mu = \frac{\delta_x + \delta_y}{2} \Rightarrow \frac{|Dp_t^x|}{p_t^x} = \frac{|Dp_t^y|}{p_t^y}$$

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Heat flow is a W_2-geodesic

Proposition 4

Suppose $\Delta d(x_0, \cdot)^2 = 2N$ *and* $\mu_t = P_t \delta_{x_0}$.
$\Rightarrow (\mu_{t^2/(2N)})_{t \geq 0}$: W_2*-min. geod.*

$\because$

- $$\frac{N}{2t} = I(\mu_t) = \frac{1}{4t^2} \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$$
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$\Rightarrow (\mu_{t^2/(2N)})_{t \ge 0}$*:* W_2*-min. geod.*

$\because$

- $$\frac{N}{2t} = I(\mu_t) = \frac{1}{4t^2} \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$$
- $$W_2(\mu_0, \mu_t)^2 = \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$$
- $\|\dot{\mu}_t\|^2 = I(\mu_t)$

$\therefore \mu_t^* := \mu_{t^2/(2N)} \Rightarrow W_2(\mu_0^*, \mu_t^*) = t \ \& \ \|\dot{\mu}_t^*\| = 1 \quad \Box$

Questions

- $\mathcal{W}$-entropy for other (non-linear) diffusions?
- $\mathcal{W}$-entropy for W_2-geodesics
  - † [X.-D. Li & K.]: "Monotonicity" holds
  - † Rigidity?
- $\mathcal{W}$-entropy for $\mathrm{Ric} \geq K$ and $\dim \leq N$?
- Rigidity for $\dfrac{\mathrm{d}}{\mathrm{d}t} \inf_{\mu} \mathcal{W}(\mu, t) = 0$?