Rigidity for the spectral gap on $\mathsf{RCD}(K,\infty)$ spaces

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joint work with N. Gigli (SISSA), C. Ketterer (Univ. Freiburg) & S. Ohta (Osaka Univ.)

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1. Introduction

On a cpl. conn. weighted Riem. mfd
$$(M,g,\mathfrak{m})$$
, $(\mathfrak{m}=\mathrm{e}^{-V}\operatorname{vol}_g)$

$$\mathrm{Ric}_V^\infty := \mathrm{Ric} + \mathrm{Hess}\,V$$

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(e.g. by the log-Sobolev ineq.)

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Example 1

$$(M,g)=(M_1 imes {\mathbb R}, g_1 imes g_{\mathbb R})$$

$$V(x,t)=V_1(x)+rac{K}{2}t^2$$
, $\mathrm{Ric}_{V_1}^\infty \geq Kg_1$ on M_1

$$\Rightarrow$$
 $\lambda_1 = K$, $u(x,t) = t$: eigenfunction

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$$\dim M = {\color{red}N} \; \& \; V = 0$$

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★ Spherical suspensions appear in the rigidity

Outline of the talk

- 1. Introduction
- 2. Proof in the smooth case
- 3. Framework and main result
- 4. Sketch of the proof (K=1)
- 5. Questions

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 $\| \geq \| \operatorname{Hess} f \|_{\operatorname{HS}}^2 + K(\nabla f, \nabla f) \|$

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$$\Rightarrow M_1 := u^{-1}(0) \subset M$$
 totally geodesic, ...

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Cheeger's L^2 -energy functional

$$2\mathsf{Ch}(f) := ext{``relaxation'' of } f \mapsto \int_X \mathsf{lip}(f)^2 \,\mathrm{d}\mathfrak{m}$$

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$$2\mathsf{Ch}(f) :=$$
 "relaxation" of $f \mapsto \int_X \mathsf{lip}(f)^2 \, \mathrm{d}\mathfrak{m}$ $= \int_X {}^{\exists} |Df|_w^2 \, \mathrm{d}\mathfrak{m}$

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Definition 2 ([Ambrosio, Gigli & Savaré '14])

 (X,d,\mathfrak{m}) : infinitesimally Hilbertian

 $\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form (o generator $\mathcal{L}/2$, $P_t=\mathrm{e}^{t\mathcal{L}}$)

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 $\exists \langle D\cdot,D\cdot
angle_w$ bilinear s.t. $\langle Df,Df
angle_w = |Df|_w^2$

RCD cond.

 $\mathsf{RCD}(K,\infty)$: infin. Hilb. & either one of the following: (up to regularity ass.'ns)

- "Hess $\operatorname{Ent}_{\mathfrak{m}} \geq K$ " on $(\mathcal{P}_2(X), W_2)$
- $ullet rac{1}{2}\mathcal{L}|Df|_w^2 \langle Df, D\mathcal{L}f
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$$\bigstar K > 0$$

 $\Rightarrow \mathfrak{m}(X) < \infty$ [Sturm '06], $\lambda_1 \geq K$ &

L has discrete spec. [Gigli, Mondino & Savaré '15]

Theorem 2 ([Gigli, Ketterer, K. & Ohta])

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Difficulty (in addition to non-smoothness)

- m may not enjoy the volume doubling property
- ullet X may not be locally compact
- ullet The eigenfunction $u
 otin L^{\infty}$

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 \bigstar Require a cut-off of u (:: $u \notin L^{\infty}$)

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 s.t

(1) For $f\in W^{1,2}(X)$ and ${\mathfrak m}$ -a.e. x, $rac{\mathrm{d}}{\mathrm{d}t}f(ilde{F}_t(x))=-\langle Df, Du
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in the distributional sense

- (2) For $\forall t \in \mathbb{R}$, $\tilde{F}_t : X \to X$: isometry
- (3) For $\forall x \in X$, $(\tilde{F}_t(x))_{t \in \mathbb{R}}$: min. geod. in X

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• $\mathfrak{m}^t := e^{-tu-t^2/2}\mathfrak{m}$ solves the conti. eq. $\leftrightarrow -u$:

$$rac{\mathrm{d}}{\mathrm{d}t}\int_X f\,\mathrm{d}\mathfrak{m}^t + \int_X \langle Df, extbf{ extit{D}}u
angle\,\mathrm{d}\mathfrak{m}^t = 0$$

ullet Construct a "regular Lagrangian flow of abla u" $F: \mathbb{R} imes X o X$ (use [Ambrosio & Trevisan '14])

$$rac{\mathrm{d}}{\mathrm{d}t}rac{W_2^2((F_t)_*\mu,
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- $\Rightarrow F_t$ preserves W_2
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1. Introduction

- 2. Proof in the smooth case
- 3. Framework and main result
- 4. Sketch of the proof (K=1)

5. Questions

- Rigidity for the log-Sobolev inequality?
- Rigidity for the Gaussian isoperimetric inequality?
- Almost splitting?

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