Rigidity for the spectral gap on RCD(K, ∞) spaces

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joint work with N. Gigli (SISSA), C. Ketterer (Univ. Freiburg) & S. Ohta (Osaka Univ.)

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1. Introduction

Spec. gap under a positive Ricci curv.

On a cpl. conn. weighted Riem. mfd $(M, g, \mathfrak{m})$,
$(\mathfrak{m} = \mathrm{e}^{-V} \mathrm{vol}_g)$

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(e.g. by the log-Sobolev ineq.)

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$$(M, g) = (M_1 \times \mathbb{R}, g_1 \times g_{\mathbb{R}})$$

$$V(x,t) = V_1(x) + \frac{K}{2}t^2, \qquad \mathrm{Ric}_{V_1}^{\infty} \geq K g_1 \text{ on } M_1$$

$\Rightarrow \boldsymbol{\lambda_1 = K}$, $u(x,t) = t$: eigenfunction

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${}^{\exists}(M_1, g_1)$ & $V_1 : M_1 \to \mathbb{R}$ *s.t.* M *is as in Example 1*

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★ Spherical suspensions appear in the rigidity

Outline of the talk

1. **Introduction**
2. **Proof in the smooth case**
3. **Framework and main result**
4. **Sketch of the proof ($K = 1$)**
5. **Questions**

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$\bigstar$ $-\mathcal{L}u = Ku \;\Rightarrow\; \boxed{\frac{1}{2}\mathcal{L}|\nabla u|^2 \geq \| \mathrm{Hess}\, u\|_{\mathrm{HS}}^2}$

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$\Rightarrow M_1 := u^{-1}(0) \subset M$ totally geodesic, ...

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Definition 2 ([Ambrosio, Gigli & Savaré '14])

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$\Rightarrow {}^{\exists}\langle D\cdot, D\cdot\rangle_w$ bilinear s.t. $\langle Df, Df\rangle_w = |Df|_w^2$

RCD cond.

RCD(K, ∞): infin. Hilb. & either one of the following:
(up to regularity ass.'ns)

- "$\mathrm{Hess}\,\mathrm{Ent}_{\mathfrak{m}} \geq K$" on $(\mathcal{P}_2(X), W_2)$
- $\frac{1}{2}\mathcal{L}|Df|_w^2 - \langle Df, D\mathcal{L}f\rangle_w \geq K|Df|_w^2$ (weakly)

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★ $K > 0$
$\Rightarrow$ $\mathfrak{m}(X) < \infty$ [Sturm '06], $\lambda_1 \geq K$ &
$\mathcal{L}$ has discrete spec. [Gigli, Mondino & Savaré '15]

Main result

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Difficulty (in addition to non-smoothness)

- $\mathfrak{m}$ may not enjoy the volume doubling property
- X may not be locally compact
- The eigenfunction $u \notin L^\infty$

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1st: The lift of eigenfunction is affine

Proposition 3 ([GKKO])

$(X, d, \mathfrak{m})$: RCD$(1, \infty)$ *sp.*, $\lambda_1 = 1$, $-\mathcal{L}u = u$

$\Rightarrow \mathcal{U}(\mu) := \int_X u \, \mathrm{d}\mu$: *affine along* W_2*-geod.* $(\mu_t)_t$

if $\mu_t \ll \mathfrak{m}$: $\mathcal{U}(\mu_t) = (1-t)\,\mathcal{U}(\mu_0) + t\,\mathcal{U}(\mu_1)$

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Goal: $\mathbf{Hess}\, \boldsymbol{u} \geq \mathbf{0} \Rightarrow \boldsymbol{\mathcal{U}}$: convex on $(\boldsymbol{\mathcal{P}_2^{\mathrm{ac}}}(\boldsymbol{X}), \boldsymbol{W_2})$

Idea: Singular perturbation of **RCD** cond'ns

(cf. [Ketterer '15 / Sturm])

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(cf. [Ketterer '15 / Sturm])

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Goal: Construct a "nice" gradient flow of $-u$

(cf. the proof of nonsmooth splitting thm [Gigli])

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$^{\exists}\tilde{F} : \mathbb{R} \times X \to X$ *s.t*

(1) *For* $f \in W^{1,2}(X)$ *and* $\mathfrak{m}$*-a.e.* x,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\tilde{F}_t(x)) = -\langle Df, Du \rangle (\tilde{F}_t(x))$$

in the distributional sense

(2) *For* $^{\forall}t \in \mathbb{R}$, $\tilde{F}_t : X \to X$*: isometry*

(3) *For* $^{\forall}x \in X$, $(\tilde{F}_t(x))_{t \in \mathbb{R}}$*: min. geod. in* X

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$$\frac{\mathrm{d}}{\mathrm{d}t}\int_X f\,\mathrm{d}\mathfrak{m}^t + \int_X \langle Df, Du\rangle\,\mathrm{d}\mathfrak{m}^t = 0$$
- Construct a "regular Lagrangian flow of $-\nabla u$" $F : \mathbb{R} \times X \to X$ (use [Ambrosio & Trevisan '14])
- $(F_t)_*\mu$ solves the 0-evolution variational eq. of $\mathcal{U}$:
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★ u: affine ($\Rightarrow Y = u^{-1}(0) \subset X$: totally geod.)

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2. Proof in the smooth case

3. Framework and main result

4. Sketch of the proof ($K = 1$)

5. **Questions**

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  (pbm: lack of compactness of $\{\mathbf{RCD}(K, \infty)$ sp.'s)