Radial processes on RCD(K, N) spaces

Kazumasa Kuwada

(Tohoku University)

joint work with K. Kuwae (Fukuoka University)

Geometric Analysis on Smooth and Nonsmooth Spaces
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1. Introduction

Radial process on $\mathbb{R}^m$

$(Z_t)_{t\geq 0}$: a stochastic process on a metric space (X, d)
$\to d(x_0, Z_t)$: radial process

Example

$(B_t)_{t\geq 0}$: Brownian motion on $\mathbb{R}^m$ generated by Δ
$m \geq 2 \Rightarrow r_t := d(0, B_t)$ solves

$$r_t = r_0 + \sqrt{2}\beta_t + (m-1)\int_0^t \frac{\mathrm{d}s}{r_s}$$

(β_t: 1-dim. standard BM)

("m-dimensional" Bessel process)

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B_t: m-dim. proc. $\leftrightarrow$ r_t: 1-dim. proc.

Local time

Example

$(B_t)_{t\geq 0}$: BM on $\mathbb{R}$ generated by Δ

$\Rightarrow r_t := d(0, B_t)$ solves

$$r_t = \sqrt{2}\beta_t + L_t$$

★ $L_t =$ "$2\int_0^t \delta_0(r_s)\mathrm{d}s$": local time at 0 of r_t

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$L_t \geq 0$, $\nearrow$, increases only on $\{s \geq 0 \mid r_s = 0\}$

Radial proc. on Riem. mfds

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$(B_t)_{t\geq 0}$: BM on a cpl. Riem. mfd of $\dim = m \geq 2$

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[Kendall '87/Cranston, Kendall & March '93]

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Expression of r_t

Goal

A corresponding expression of r_t on **RCD(K, N)** sp.'s (met. meas. sp.'s with "$\mathbf{Ric} \geq \boldsymbol{K}$ and $\mathbf{dim} \leq \boldsymbol{N}$")

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Outline of the proof on Riem. mfd

- $d(B_t, \mathrm{cut}(x_0)) > \varepsilon$: Itô formula to $d_{x_0}(B_t)$
- $d(B_t, \mathrm{cut}(x_0)) \leq \varepsilon$: Change the ref. pt. from x_0
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Difficulty

- Lack of usual differentiable structure
- Δd_{x_0} & $\mathrm{cut}(x_0)$ can be wilder

Outline of the talk

1. **Introduction**

2. **Framework and main result**

3. **Outline of the proof**

4. **Local structure**

Met. meas. sp. & Cheeger energy

$(X, d, \mathfrak{m})$: Polish geod. met. meas. sp.

($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

BM $\leftrightarrow$ $P_t = \mathrm{e}^{t\Delta}$ $\leftrightarrow$ Cheeger's L^2-energy

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$$2\mathsf{Ch}(f) := \inf \left\{ \varliminf_{n} \int_X \mathrm{lip}(f_n)^2 \mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$

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Definition 1 ([Ambrosio, Gigli & Savaré '14])

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$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form ($\Leftrightarrow$ P_t: linear $\Leftrightarrow$ Δ: linear)

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$\mathsf{RCD}(K, N)$ spaces $(N < \infty)$

$\mathsf{RCD}(K, N)$: infin. Hilb., some regularity ass'ns &

$$\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \geq K|Df|_w^2 + \frac{1}{N}|\Delta f|^2$$

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Brownian motion and heat kernel

- (X, d): loc. cpt. ($\because$ Bishop-Gromov ineq.)
- $\mathcal{E}(f, f) := 2\mathsf{Ch}(f)$, $\mathcal{F} := D(\mathsf{Ch})(\subset L^2(\mathfrak{m}))$
  $\Rightarrow (\mathcal{E}, \mathcal{F})$: reg. str. loc. Dirichlet form on $L^2(\mathfrak{m})$
- $(\mathcal{E}, \mathcal{F}) \leftrightarrow ((B_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in X})$: diffusion process
  ("Brownian motion")
- P_t admits a heat kernel $\mathfrak{p}_t(x, y)$,

$$\mathfrak{p}_t(x, y) \leq \frac{c}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x, y)^2}{Ct} + \lambda t\right)$$

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Fukushima decomposition

★ $d_{x_0} \in C^{\mathrm{Lip}}(X) \subset \mathcal{F}_{\mathrm{loc}} \cap C(X)$

$$\Rightarrow \boxed{d_{x_0}(B_t) - d_{x_0}(B_0) = M_t + N_t}$$

P_x-a.s. q.e. $x \in X$

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Q. Can "q.e." be removed?

(q.e.: $^{\exists}$exceptional set of zero capacity)

Revuz correspondence

Positive continuous additive functional A_t

$$\updownarrow$$

Smooth measure $\mu \in S$

For $f, h \geq 0$ & $t > 0$,

$$\int_X \mathbb{E}_x \left[\int_0^t f(B_s) \mathrm{d} A_s \right] h(x) \mathfrak{m}(\mathrm{d}x) = \int_0^t \left(\int_X f P_s h \, \mathrm{d}\mu \right) \mathrm{d}s$$

Example

$$A_t = \int_0^t \varphi(B_s) \mathrm{d}s \leftrightarrow \mu = \varphi \mathfrak{m}$$

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Revuz correspondence II

$$\boldsymbol{r_t - r_0 = M_t + N_t}$$

- $\langle M \rangle_t \leftrightarrow \mu_{\langle d_{x_0} \rangle}$,
  $$\int_X f \, \mathrm{d}\mu_{\langle \varphi \rangle} := 2\mathcal{E}(\varphi, f\varphi) - \mathcal{E}(\varphi^2, f)$$
- $\mu_{\langle d_{x_0} \rangle} = 2|Dd_{x_0}|_w^2 \mathfrak{m} = 2\mathfrak{m}$
- $\mathcal{E}(d_{x_0}, v) = -\int_X v \, \mathrm{d}\tilde{\nu}$

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- $\mathcal{E}(d_{x_0}, v) = -\int_X v \, \mathrm{d}\tilde{\nu}$

Revuz correspondence II

$$r_t - r_0 = M_t + N_t$$

- $\langle M \rangle_t \leftrightarrow \mu_{\langle d_{x_0} \rangle}$,
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Laplacian comparison thm

$$\boldsymbol{\kappa} := \frac{K}{N-1},\ \mathfrak{s}_{\kappa}(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}},\ \mathfrak{ct}_{\kappa} := \frac{\mathfrak{s}_{\kappa}'(r)}{\mathfrak{s}_{\kappa}(r)}$$

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Theorem 2 ([Gigli '15]/[K. & Kuwae])

$\exists$ *Radon meas.* ν *on* $X \setminus \{x_0\}$ *s.t.*

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★ *Under* (**R1**) *below,* $X \setminus \{x_0\} \rightsquigarrow X$ & $\nu(\{x_0\}) = 0$

Main thm

$\boldsymbol{\sigma_{x_0} := \inf\{t \geq 0 \mid B_t = x_0\}}$

Suppose that $\boldsymbol{X}$ is not "1-dimensional" ($\Rightarrow \boldsymbol{N > 1}$)

(cf. [Kitabeppu & Lakzian '16])

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(2) *Under* (**R2**) *below,* $\boldsymbol{X} \setminus \{x_0\} \rightsquigarrow \boldsymbol{X}$ & $\boldsymbol{\sigma}_{x_0} \rightsquigarrow \infty$

Assumptions

$$\textbf{(R1)} \int_{B_1(x_0)} \frac{\mathrm{d}\mathfrak{m}}{d_{x_0}} < \infty$$

$$\textbf{(R2)} \frac{1}{\mathfrak{m}(B_r(x_0))} \int_{B_r(x_0)} \frac{\mathrm{d}\mathfrak{m}}{d_{x_0}} \leq \frac{{}^{\exists}C}{r} \text{ for } r \ll 1$$

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Remarks

- When X: Riem. mfd,
$$A_t^\nu \leftrightarrow -\int_0^t \Delta d_{x_0}(B_s)\,\mathrm{d}s + (N-1)\int_0^t \mathfrak{ct}_\kappa(r_s)\,\mathrm{d}s + L_t^c$$
- $\sigma_{x_0} < \infty$ may happen even under (**R2**), while $\mathbb{P}_x[\sigma_{x_0} = \infty] = 1$ for any x $\mathfrak{m}$-a.e. x_0

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1. Introduction

2. Framework and main result

3. **Outline of the proof**

4. Local structure

Notations

- $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha\langle u,v\rangle_{\mathfrak{m}}$
  $(u,v \in \mathcal{F};\ \alpha > 0)$
- $R_\alpha f := \displaystyle\int_0^\infty e^{-\alpha t} P_t f \,\mathrm{d}t$

★ $\mathcal{E}_\alpha(R_\alpha f, g) = \langle f, g\rangle_{\mathfrak{m}} \quad (f \in L^2(\mathfrak{m}), g \in \mathcal{F})$

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$$\Rightarrow R_\alpha f(x) = \int_X \mathfrak{r}_\alpha(x,y) f(y) \mathfrak{m}(\mathrm{d}y),$$

$R_\alpha \mu$ for a meas. μ can be defined

Smooth measures in the strict sense

- A_t^ν: PCAF in the strict sense (i.e., without "q.e.")
  $\Leftrightarrow \nu \in S_1$
  $\Leftarrow {}^\forall \varphi \in C_{c,+}^{\mathrm{Lip}}(X)$, $\varphi\nu \in S_{00}$
  (similarly for $X \setminus \{x_0\}$)
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  (i) $\mu(X) < \infty$

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$$R_\alpha(\varphi\nu)(x) \le \int_0^\infty \frac{c\mathrm{e}^{-(\alpha-\lambda)t}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \times \int_X \exp\left(-\frac{d(x,y)^2}{Ct}\right)\varphi(y)\nu(\mathrm{d}y)\mathrm{d}t$$

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<u>Idea</u>

$$\int_{B_R(x)} \exp\left(-\frac{d_x^2}{C't}\right)\mathrm{d}\mathfrak{m}$$

$$= \int_X \int_{d_x}^{\infty} \frac{2u}{C't}\exp\left(-\frac{u^2}{C't}\right)\mathrm{d}u\,\mathrm{d}\mathfrak{m}$$

$=$ (integral in u involving $\mathfrak{m}(B_u(x))$)

& use the Bishop-Gromov ineq.

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Suppose (**R2**)*:* $\displaystyle \frac{1}{\mathfrak{m}(B_r(x_0))} \int_{B_r(x_0)} \frac{\mathrm{d}\mathfrak{m}}{d_{x_0}} \leq \frac{{}^\exists C}{r}$

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★ "Excluding 1-dim. sp's" $\leftrightarrow$ $\boxed{\boldsymbol{E}_1 = \emptyset}$

[Kitabeppu & Lakzian '16]

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[Gigli & Pasqualetto] / [Kell & Mondino]

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- $\boldsymbol{x_0 \in E_k}$, $\boldsymbol{k \geq 3 \Rightarrow \sigma_{x_0} = \infty}$ $\mathbb{P}_x$-*a.s.* $(\boldsymbol{x \in X})$
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★ When $k = 2$, $\mathfrak{m}|_{R_j} = \rho \mathcal{H}^2|_{R_j}$ $(k_j = 2)$,

$$\text{a.e. } x_0 \in R_j,\ \varlimsup_{r \downarrow 0} \frac{\mathfrak{m}(B_r(x_0))}{r^2} \leq c_1 \rho(x_0) < \infty$$

Strange example

$$(X, d, \mathfrak{m}) = (D^2_{1/\sqrt{\mathrm{e}}}, d_E, \mathrm{e}^{-V}\mathcal{L}^2),$$

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