Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$(0, N)$ spaces

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1. Introduction

$\mathcal{W}$-entropy on Riem. mfd

M: m-dim. cpt. Riem. mfd, $t > 0$, $f \in C^\infty(M)$

$u := \dfrac{\mathrm{e}^{-f}}{(4\pi t)^{m/2}}$, $\displaystyle\int_M u \, \mathrm{d}\,\mathrm{vol} = 1$

$$\mathcal{W}(f,t) := \int \left[t|\nabla f|^2 + f - m \right] u \, \mathrm{d}\,\mathrm{vol}$$

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$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{W}(f,t) \leq 0$$

Entropy formula and rigidity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(f,t) \le 0$$

$$\Uparrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = -2\int_M t\left(\left|\nabla^2 f - \frac{g}{2t}\right|^2 + \mathrm{Ric}(\nabla f, \nabla f)\right) u \,\mathrm{d}\,\mathrm{vol}$$

[L. Ni '04]

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$\rightarrow$ Extension to weighted Riem. mfds [X.-D. Li '12]

Reformulation of $\mathcal{W}$-entropy

$$\mu = \rho \,\mathrm{vol} \in \mathcal{P}(M),\ \rho =: \frac{\mathrm{e}^{-f}}{(4\pi t)^{m/2}}$$

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- $I(\mu) := \displaystyle\int \frac{|\nabla \rho|^2}{\rho} \, \mathrm{dvol} = I_{\mathrm{vol}}(\mu)$
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$$\begin{aligned}\mathcal{W}(\mu, t) &:= \int_X \left[t|\nabla f|^2 + f - m\right] \rho \,\mathrm{dvol} \\ &= tI(\mu) - \mathrm{Ent}(\mu) - \frac{m}{2}\log t + c_1\end{aligned}$$

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$$= \frac{\mathrm{d}}{\mathrm{d}t}[t\,\mathrm{Ent}(\mu_t) - t\,\mathrm{Ent}^{\mathrm{Eucl}}(\mathrm{Gauss}(t))]$$

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Connection with (log) Sobolev ineq.'s

- $\inf_{\mu} \mathcal{W}(\mu, t) > -\infty$
  $\Rightarrow$ (defective) LSI: $\mathbf{Ent}(\mu) \leq tI(\mu) + C(t)$

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  $\Leftrightarrow$ Sobolev ineq. $\|f\|_{\frac{2m}{m-2}} \leq {}^{\exists}C' \|\nabla f\|_2$

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  $\Leftrightarrow \mathfrak{m}(B_r(x)) \geq C'' r^m$ (under $\mathrm{Ric} \geq 0$)

Purpose

Q.

Can one extend the monotonicity/rigidity of $\boldsymbol{\mathcal{W}}$ on "Riemannian" metric measure spaces with "$\mathbf{Ric} \geq \mathbf{0}$ & $\mathbf{dim} \leq \boldsymbol{N}$" (**RCD**$(0, N)$ spaces)?

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  $\leftsquigarrow$ optimal transport approach

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- Weaken ass'n(s) even on (weighted) Riem. mfds
- Without the entropy formula
  - $\leftsquigarrow$ optimal transport approach
- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Motivations

- Better understanding of $\mathcal{W}$
- **RCD** cond'n is stable under mGH conv.
  $\rightarrow$ Almost rigidity
- $\{$**RCD** sp.'s$\} \supset \{(\mathbf{lim}$ of) (weighted) Riem. mfds$\}$
- Rigidity results on **RCD** sp.'s admit singular sp.'s

Outline of the talk

Bochner inequality

$\mathfrak{m} := \mathrm{e}^{-V}\mathrm{vol}$, $(M, d, \mathfrak{m})$: weighted Riem. mfd

$\mathcal{L} := \Delta - \nabla V \cdot \nabla$: self-adj. on $L^2(\mathfrak{m})$

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For $N \in (m, \infty)$,

$$\mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq 0$$

$$\Updownarrow$$

$$\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle \geq \frac{1}{N}|\mathcal{L} f|^2$$

(Bakry-Émery's curv.-dim. cond./Bochner ineq.)

(Cond'ns corresponding to "$\mathrm{Ric} \geq 0$ & $\dim \leq N$")

Characterization by heat semigroups

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$$|\nabla P_t f|^2 + \frac{2t}{N}|\mathcal{L} P_t f|^2 \leq P_t(|\nabla f|^2)$$

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$$W_2(P_s\mu, P_t\nu)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

$$W_2(\mu, \nu) := \inf_{\pi} \left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l} \pi(A \times X) = \mu(A) \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

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Heat flow as a gradient flow of Ent

$\mu_t := P_t \mu$

$$\Rightarrow \boxed{\text{“}\dot{\mu}_t = -\nabla \mathrm{Ent}_{\mathfrak{m}}(\mu_t)\text{”}} \text{ on } (\mathcal{P}(M), W_2)$$

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- $\dfrac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ent}_{\mathfrak{m}}(\mu_t) = -I_{\mathfrak{m}}(\mu_t)$ (de Brujin identity)
- Rigorously,

$$\boxed{\overline{\lim_{\delta \downarrow 0}} \frac{1}{\delta^2} W_2(\mu_t, \mu_{t+\delta})^2 = I_{\mathfrak{m}}(\mu_t)} \text{ a.e. } t$$

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$$W_2(\mu_t, \mu_{(1+\delta)t})^2 \le W_2(\mu_s, \mu_{(1+\delta)s})^2 + \cdots$$

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$$t^2 I(\mu_t) \le s^2 I(\mu_s) - \frac{N}{2}(t - s)$$

$\Rightarrow t\,h(t) \searrow$ in t (cf. [Topping '09] on Ricci flow)

Met. meas. sp. & heat flow on it

$(X, d, \mathfrak{m})$: Polish geod. met. meas. sp.

($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

$P_t = \mathrm{e}^{t\Delta} \leftrightarrow$ Cheeger's L^2-energy

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$$2\mathsf{Ch}(f) := \inf \left\{ \varliminf_{n} \int_X \mathrm{lip}(f_n)^2 \mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$

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$$= \int_X {}^{\exists}|Df|_w^2 \, \mathrm{d}\mathfrak{m}$$

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Definition 1 ([Ambrosio, Gigli & Savaré '14])

$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form ($\Leftrightarrow P_t$: linear $\Leftrightarrow \Delta$: linear)

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$\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form ($\Leftrightarrow$ P_t: linear $\Leftrightarrow$ Δ: linear)

$\Rightarrow {}^{\exists}\langle D\cdot, D\cdot\rangle_w$ bilinear s.t. $\langle Df, Df\rangle_w = |Df|_w^2$

Characterizations of RCD cond.

RCD$(0,N)$: infin. Hilb., some regularity ass'ns &

$$\text{“}\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \geq \frac{1}{N}|\Delta f|^2\text{”}$$

(weakly)

★ Equiv. cond'ns (up to reg. assn's)

- $$W_2(P_s f\,\mathfrak{m}, P_t g\,\mathfrak{m})^2 \leq W_2(f\,\mathfrak{m}, g\,\mathfrak{m})^2 + 2N(\sqrt{t} - \sqrt{s})^2$$
- On $(\mathcal{P}_2(X), W_2)$, $^\forall \mu_0$, $^\exists (\mu_t)_{t\geq 0}$ sol. to $(0,N)$-evolution variational inequality of Ent (a (metric) formulation of "$\dot{\mu}_t = -\nabla \mathrm{Ent}(\mu_t)$")

[Erbar, K. & Sturm '15]

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$$\text{“}\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \geq \frac{1}{N}|\Delta f|^2\text{”}$$

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- $W_2(P_s f\,\mathfrak{m}, P_t g\,\mathfrak{m})^2 \leq W_2(f\,\mathfrak{m}, g\,\mathfrak{m})^2 + 2N(\sqrt{t} - \sqrt{s})^2$
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Main thm

Theorem 2 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: RCD$(0, N)$, $N \geq 2$, $\mu_t := P_t \mu$

(1) $^{\forall} u \geq 0$, $\mathcal{W}(\mu_t, t+u) \searrow$ *in* $t \in (0, \infty)$

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Definition 3 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Longleftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2$
  $:= r^2 + s^2 - 2rs\cos(d_Y(x, y) \wedge \pi)$
- $\mathfrak{m}(\mathrm{d}r\mathrm{d}x) := r^N \mathrm{d}r\mathfrak{m}_Y(\mathrm{d}x)$

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- [Jiang & Zhang '16] X: cpt. $\Rightarrow$ Theorem 2 (1)
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
- $(0, N)$-cone of Y is a (smooth) Riem. mfd
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Almost rigidity

Theorem 4 ([X.-D. Li & K.])

Let $r_n : (0, T) \to (-\infty, 0) \searrow$ *s.t.* $r_n(t) \nearrow$ *in* n *&* $\lim_{t \downarrow 0} \lim_{n \to \infty} r_n(t) = 0$.

Fix $s > 0$.

$(X_n, d_n, \mathfrak{m}_n, x_n)$*: seq. of pointed* $\mathbf{RCD}(0, N)$ *sp.'s,*

$$\mathcal{W}(P^{(n)}_{s+t'} \delta_{x_n}, s + t') - \mathcal{W}(P^{(n)}_{s} \delta_{x_n}, s) \ge r_n(t) t'$$

for $0 < t' \le t$.

$\Rightarrow (X_n, d_n, \mathfrak{m}_n, x_n)$ *is close to a* $(0, N)$*-cone if* $n \gg 1$.

Geometric properties of $\mathsf{RCD}(0, N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $\Delta d(x_0, \cdot)^2 \leq 2N$ in the distributional sense

- Volume rigidity [Gigli & De Philippis]: $N \geq 2$,

$$\exists x_0, \forall r, R > 0, \ \mathfrak{m}(B_R(x_0)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(x_0))$$

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Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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$\Rightarrow \Delta d(x_0, \cdot)^2 = 2N$ ($\because$ Varadhan-type asymptotic) □

5.2. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0, \cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0, x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0, x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0, x)^2}{4t}\right)$$

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RHS enjoys the energy dissipation identity for Ent $\square$

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In particular, X *is non-compact*

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Heat flow is a W_2-geodesic

Proposition 2

Suppose $\Delta d(x_0, \cdot)^2 = 2N$ *and* $\mu_t = P_t \delta_{x_0}$.

$\Rightarrow (\mu_{t^2/(2N)})_{t \geq 0}$*:* W_2*-min. geod.*

$\because$

- $$\frac{N}{2t} = I(\mu_t) = \frac{1}{4t^2} \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$$
- $$W_2(\mu_0, \mu_t)^2 = \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$$
- $$\|\dot{\mu}_t\|^2 = I(\mu_t)$$

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- $$\|\dot{\mu}_t\|^2 = I(\mu_t)$$

$\therefore \mu_t^* := \mu_{t^2/(2N)} \Rightarrow W_2(\mu_0^*, \mu_t^*) = t \;\&\; \|\dot{\mu}_t^*\| = 1 \quad \square$

Questions

- $\mathcal{W}$-entropy for other (non-linear) diffusions?
- $\mathcal{W}$-entropy for W_2-geodesics
  - † [X.-D. Li & K.]: "Monotonicity" holds
  - † Rigidity?
- $\mathcal{W}$-entropy for $\mathrm{Ric} \geq K$ and $\dim \leq N$?
- Rigidity for $\dfrac{\mathrm{d}}{\mathrm{d}t} \inf_{\mu} \mathcal{W}(\mu, t) = 0$?