Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$(0, N)$ spaces

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1. Introduction

$\mathcal{W}$-entropy on Riem. mfd

M: m-dim. cpl. Riem. mfd with "bdd. geom."

$f \in C^\infty(M)$, $u := \dfrac{e^{-f}}{(4\pi t)^{m/2}}$, $\int_M u \, d\,\mathrm{vol} = 1$

$$\mathcal{W}(f,t) := \int \left[t|\nabla f|^2 + f - m \right] u \, d\,\mathrm{vol}$$

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[Ni '04]/[X.-D. Li '12]

Purpose

Q.

Can one extend the monotonicity/rigidity of $\mathcal{W}$ on "Riemannian" met. meas. sp.'s $(X, d, \mathfrak{m})$ with "$\mathrm{Ric} \geq 0$ & $\dim \leq N$" (**RCD**$(0, N)$ spaces)?

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- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

1. Introduction

2. Curvature-dimension conditions

3. Main results

4. Proof

4.1 Monotonicity

4.2 Rigidity

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"R" CD condition

"R": the canonical heat semigroup $P_t = \mathrm{e}^{t\Delta}$ is linear.

CD$(0, N)$: up to some regularity ass'ns, either/both of

- "$\frac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla \Delta f \rangle_w \geq \frac{1}{N}|\Delta f|^2$"

  (Bakry-Émery's curv.-dim. cond.)

- $W_2(P_s\mu, P_t\nu)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$

[Erbar, K. & Sturm '15]

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Examples

- (X, g): m-dim. cpl. Riem. mfd., $\partial X = \emptyset$,
  d: Riem. dist., $\mathfrak{m} = \mathrm{e}^{-V} \mathrm{vol}_g$ $\quad (V : X \to \mathbb{R})$
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$$\textbf{RCD}^*(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

- (Pointed) measured GH lim. of $\textbf{RCD}^*(K, N)$ sp.'s
  [$\cdots$ /Gigli, Mondino & Savaré '15]

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Main thm

Theorem 1 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: RCD$(0, N)$, $N \geq 2$, $\mu_t := P_t \mu$

(1) $\mathcal{W}(\mu_t, t) \searrow$ *in* $t \in (0, \infty)$

(2) *Suppose* $^\exists t_* > 0$ *s.t.*

$$\varlimsup_{t \downarrow t_*} \frac{\mathcal{W}(\mu_t, t) - \mathcal{W}(\mu_{t_*}, t_*)}{t - t_*} = 0$$

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$X \simeq$ $(0, N-1)$*-cone of* *an* RCD$^*(N-2, N-1)$ *sp.*

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Definition 2 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Longleftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2 := r^2 + s^2 - 2rs \cos(d_Y(x, y) \wedge \pi)$
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Remarks

- [Jiang & Zhang '16] X: cpt. $\Rightarrow$ Theorem 1 (1)
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
- $(0, N)$-cone of Y is a (smooth) Riem. mfd
  $\Leftrightarrow Y \simeq \mathbb{S}^{N-1}(1)$
- An "almost rigidity" can be formulated

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Heat flow as a gradient flow of Ent

$\mu_t := P_t \mu$

$$\Rightarrow \boxed{\text{“}\dot{\mu}_t = -\nabla \operatorname{Ent}(\mu_t)\text{”}} \text{ on } (\mathcal{P}(X), W_2)$$

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- Rigorously,
$$\boxed{\overline{\lim_{\delta \downarrow 0}} \frac{1}{\delta^2} W_2(\mu_t, \mu_{t+\delta})^2 = I(\mu_t)} \text{ a.e. } t$$

4.1. Monotonicity

(cf. [Topping '09] on (backward) Ricci flow)

Derivation from RCD$(0, N)$

$$\bigstar\; t\,h(t) := t^2 I(\mu_t) - \frac{Nt}{2} \;\&\; \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}(\mu_t) = -I(\mu_t)$$

$$\Rightarrow \boxed{\text{“}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))\text{”}}$$

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$$W_2(\mu_t, \mu_{(1+\delta)t})^2 \le W_2(\mu_s, \mu_{(1+\delta)s})^2 + \cdots$$

$$\Downarrow \qquad \boxed{\text{“}\overline{\lim_{\delta\downarrow 0}}\,\frac{1}{\delta^2}\text{”} \text{ with } \|\dot{\mu}_t\|^2 = I(\mu_t)}$$

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$$\Downarrow \qquad \boxed{\text{“}\overline{\lim_{\delta\downarrow 0}}\frac{1}{\delta^2}\text{”} \text{ with } \|\dot{\mu}_t\|^2 = I(\mu_t)}$$

$$t^2 I(\mu_t) \le s^2 I(\mu_s) - \frac{N}{2}(t - s)$$

Derivation from $\mathsf{RCD}(0, N)$

$$\bigstar\ t\,h(t) := t^2 I(\mu_t) - \frac{Nt}{2} \ \&\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ent}(\mu_t) = -I(\mu_t)$$

$$\Rightarrow \boxed{\text{``}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))\text{''}}$$

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$\Rightarrow t\,h(t) \searrow$ in t $\square$

4.2. Rigidity

Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

$\Delta d(x_0, \cdot)^2 = 2N \Rightarrow$ concl. [Gigli & de Philippis '17]

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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$\Rightarrow \Delta d(x_0, \cdot)^2 = 2N$ ($\because$ Varadhan-type asymptotic) □