Equivalence between dimensional contractions in Wasserstein distance and the curvature-dimension condition

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1. Introduction

Heat flow and Ricci curvature

M: complete Riemannian manifold

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), \\ u(0,\cdot) = f \end{cases} \quad \text{heat eq. on } M$$

$\Rightarrow u = P_t f$

- $P_t = \mathrm{e}^{t\Delta}$: heat semigroup

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★ P_t characterizes "$\mathrm{Ric} \geq K$ & $\dim \leq N$"

(curvature-dimension cond.)

Bakry-Émery's approach

Bochner-Weitzenböck formula

$$\Gamma_2(f,f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess}\, f\|_{\mathrm{HS}}^2,$$
$$\Gamma_2(f,f) := \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle$$

★ "$\mathrm{Ric} \geq K$ & $\dim \leq N$"

$\Leftrightarrow$ **BE**(K,N): $\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$

(Bakry-Émery's curv.-dim. cond.[Bakry-Émery '85])

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(**BE**(K,N) is formulated only in terms of Δ)

Optimal transport approach

$$W_2(\mu,\nu) := \inf\left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l}\pi\text{: coupling}\\ \text{of } \mu \,\&\, \nu\end{array} \right\}$$

$$\mathrm{Ent}(\rho\,\mathrm{vol}) := \int_M \rho \log \rho \, d\,\mathrm{vol}$$

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- **W**(K, ∞):
  $W_2(\mu P_t, \nu P_t)^2 \leq \mathrm{e}^{-2Kt} W_2(\mu, \nu)^2$
- **W**$(0, N)$:
  $W_2(\mu P_t, \nu P_s)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$

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Q.
Estimate of $W_2(\mu P_t, \nu P_t)^2$
characterizing "$\mathrm{Ric} \ge K$ & $\dim \le N$"
$\rightsquigarrow$ Sharper est. of $W_2(\mu P_t, \nu)^2$ when $\nu P_s = \nu$

Outline of the talk

1. **Introduction**

2. **Framework and main results**

3. **Idea of the proof**

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$$\bigstar \; |f(\gamma_1) - f(\gamma_0)| \le \int_0^1 |\nabla f|_w(\gamma_s)|\dot{\gamma}_s| \, ds$$

for a.e. trajectories $(\gamma_s)_{s \in [0,1]}$ of "nice" transports

Framework

Assumptions

- **Ch** is a quadratic form (infinitesimally Hilbertian)
- $\int_X \exp\left(-{}^{\exists}c d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty$
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★ Extension of $\boldsymbol{P_t}$ to a map $\mathcal{P}_2(X) \to \mathcal{P}_2(X)$

$$(\mathcal{P}_2(X) := \{\mu \in \mathcal{P}(X) \mid W_2(\delta_{x_0}, \mu) < \infty)$$

Bakry-Émery's curvature-dimension cond.

BE(K, N)

$^\forall f \in \mathcal{D}(\Delta)$ with $\Delta f \in \mathcal{D}(\mathsf{Ch})$ &

$g \in D(\Delta) \cap L^\infty$ with $g \geq 0$ & $\Delta g \in L^\infty$

$$\int_X \left(\frac{1}{2} \Delta g |\nabla f|_w^2 - g \langle \nabla f, \nabla \Delta f \rangle \right) d\mathfrak{m} \geq \int_X g \left(K |\nabla f|_w^2 + \frac{1}{N} |\Delta f|^2 \right) d\mathfrak{m}$$

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- "Localized version" is also known [Ambrosio, Mondino & Savaré '16]/[Hua, Kell & Xia]

Another dimensional W_2-contractions

$\widehat{\mathsf{W}}(K,N)$

$$\left(\frac{W_2(\mu P_t,\nu P_t)}{2}\right)^2 \leq \mathrm{e}^{-2Kt} \qquad \left(\frac{W_2(\mu,\nu)}{2}\right)^2$$

Another dimensional W_2-contractions

$\widehat{\mathsf{W}}(K, N)$

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu P_t, \nu P_t)}{2}\right)^2 \le \mathrm{e}^{-2Kt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu, \nu)}{2}\right)^2$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}\right)$$

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Another dimensional W_2-contractions

$\widehat{\mathsf{W}}'(K, N)$

$$
\begin{aligned}
W_2(\mu P_t, \nu P_t)^2 &\le \mathrm{e}^{-2Kt} W_2(\mu, \nu)^2 \\
&\quad - \frac{2}{N} \int_0^t \mathrm{e}^{-2K(t-s)} \big(\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s) \big)^2 ds
\end{aligned}
$$

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Another dimensional W_2-contractions

$\underline{\widehat{\mathsf{W}}'(K,N)}$

$$W_2(\mu P_t, \nu P_t)^2 \le e^{-2Kt} W_2(\mu,\nu)^2 - \frac{2}{N}\int_0^t e^{-2K(t-s)}\big(\operatorname{Ent}(\mu P_s) - \operatorname{Ent}(\nu P_s)\big)^2 ds$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}} \right)$$

$$\left(\mathfrak{s}_\kappa(r) \approx r \ (r \ll 1), \sinh(r) \ge r \quad \rightsquigarrow \widehat{\mathsf{W}}(K,N) \Rightarrow \widehat{\mathsf{W}}'(K,N) \right)$$

Another dimensional W_2-contraction

Theorem 1 ([Bolley, Gentil, Guillin & K.])

For $K \in \mathbb{R}$ *and* $N > 0$*, TFAE:*

(i) $\mathsf{BE}(K, N)$

(ii) $\widehat{\mathsf{W}}(K, N)$

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- "**(i)** $\Rightarrow$ **(iii)**" for Markov transportation dist. [Bolley, Gentil & Guillin '14]
- "**(i)** $\Leftrightarrow$ **(iii)**" on cpt. Riem. mfd [Gentil '15]

Remarks

- $\mathsf{BE}(K,N) \Leftrightarrow \mathsf{(R)CD}^e(K,N)$

$$\left(\text{“}\nabla^2 \operatorname{Ent} - \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \geq K\text{”} \quad \text{on } (\mathcal{P}_2(X), W_2) \right)$$

[Ambrosio, Gigli & Savaré '15]($N = \infty$)

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- $\textbf{(R)CD}^e(K,N) \Rightarrow \textbf{BE}(K,N)$ via non-linear diffusion is also possible

[Ambrosio, Mondino & Savaré]

1. Introduction

2. Framework and main results

3. Idea of the proof

4. Applications and further problems

For simplicity, we assume $\boldsymbol{K} = \mathbf{0}$ in the sequel

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(We still use "$\boldsymbol{K}$" if it holds for general $\boldsymbol{K}$)

Proof of

$\mathsf{BE}(K,N) \Rightarrow \widehat{\mathsf{W}}(K,N)$

Formal derivation of $\widehat{\mathsf{W}}'(0, N)$

† $\mathsf{BE}(0, N) \Leftrightarrow \nabla^2 \operatorname{Ent} - \dfrac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \geq 0$

† $\mu_t^{(i)} := \mu^{(i)} P_t$ solves $\dot{\mu}_t^{(i)} = -\nabla \operatorname{Ent}(\mu_t^{(i)})$

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$(\sigma_r)_{r \in [0,1]}$: W_2-min. geod. in $\mathcal{P}_2(X)$ from $\mu_t^{(0)}$ to $\mu_t^{(1)}$

$$\frac{1}{2}\frac{d}{dt} W_2(\mu_t^{(0)}, \mu_t^{(1)})^2 = [\langle \dot{\mu}_t^{(r)}, \dot{\sigma}_r \rangle]_{r=0}^1$$

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$(\sigma_r)_{r\in[0,1]}$: W_2-min. geod. in $\mathcal{P}_2(X)$ from $\mu_t^{(0)}$ to $\mu_t^{(1)}$

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$$= - \int_0^1 \nabla^2 \operatorname{Ent}(\sigma_r)(\dot{\sigma}_r, \dot{\sigma}_r) dr \quad (\because \ddot{\sigma}_r = 0)$$

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$$\Rightarrow \boxed{\begin{aligned} W_2(\mu_t^{(0)}, \mu_t^{(1)})^2 &\leq W_2(\mu_0^{(0)}, \mu_0^{(1)})^2 \\ &\quad - \frac{2}{N}\int_0^t (\operatorname{Ent}(\mu_s^{(1)}) - \operatorname{Ent}(\mu_s^{(0)}))^2 ds \end{aligned}}$$

Evolution variational inequality

- $\mathbf{BE}(K, N) \Rightarrow \mathbf{(R)CD}^e(K, N)$
- $\mathbf{RCD}^e(K, N)$ "$\Leftrightarrow$" $^\forall$initial data, $^\exists$sol. to (K, N)-**EVI** of Ent

[Ambrosio, Gigli, Mondino & Rajala '15/$\cdots$] ($N = \infty$)

[Erbar, K. & Sturm '15]

What's **EVI**?

- A formulation of grad. flow for (K, N)-convex pot. & "Riemannian"
- In this case, sol. to **EVI** of $\mathrm{Ent} = \mu P_t$
- (K, N)-**EVI** $\Rightarrow$ $\mathbf{W}(K, N)$ & $\widehat{W}(K, N)$

Proof of

$$\widehat{\mathsf{W}}'(\boldsymbol{K},\boldsymbol{N}) \Rightarrow \mathsf{BE}(\boldsymbol{K},\boldsymbol{N})$$

($\widehat{\mathsf{W}}(\boldsymbol{K},\boldsymbol{N}) \Rightarrow \widehat{\mathsf{W}}'(\boldsymbol{K},\boldsymbol{N})$ is already explained in §2)

Review: $\mathsf{W}(K,N) \Rightarrow \mathsf{BE}(K,N)$

$$\mathsf{W}(K,N) \;\; \overset{\Rightarrow}{\Leftarrow} \;\; \mathsf{G}(K,N) \;\; \overset{\Rightarrow}{\Leftarrow} \;\; \mathsf{BE}(K,N)$$

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$\mathsf{G}(0,N)$ (Gradient estimate for P_t) :

$$|\nabla P_t f|^2 \le P_t(|\nabla f|^2) - \frac{2}{N}(\Delta P_t f)^2$$

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$$W_2(\mu P_t, \nu P_s)^2 \le W_2(\mu,\nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

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$$\mathsf{W}(K,N) \;\underset{\Leftarrow}{\overset{\Rightarrow}{}}\; \mathsf{G}(K,N) \;\underset{\Leftarrow}{\overset{\Rightarrow}{}}\; \mathsf{BE}(K,N)$$

[Bakry & Émery '84 ($N = \infty$)/ Bakry & Ledoux '06]
[K. '10, '13 ($N = \infty$)/K. '15/...]

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★ $\mathsf{W}(K,N)$ for Dirac meas.'s $\Rightarrow G(K,N)$

Our problem: $\widehat{\mathsf{W}}'(K,N) \Rightarrow \mathsf{BE}(K,N)$

Remarks

- No use of $\mathsf{G}(K,N)$:

$$\widehat{\mathsf{W}}'(K,N) \Rightarrow \mathsf{BE}(K,N) \text{ directly}$$

- On $\mathbb{R}^m$,

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  $\because$ Recall $\widehat{\mathbf{W}}(0,N)$:

$$W_2(\mu P_t, \nu P_t)^2 \le W_2(\mu,\nu)^2 - \frac{2}{N}\int_0^t (\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s))^2 ds$$

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Formal differential calc. on $(\mathcal{P}_2(X), W_2)$ (Otto calc.)

$$\boxed{\text{“}\dot{\nu}_0 = \nabla f \text{ in } T_{\nu_0}\mathcal{P}_2(X)\text{”}} \text{ for } (\nu_r)_r \subset \mathcal{P}_2(X)$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \forall\varphi, \boxed{\partial_r \int_X \varphi\, d\nu_r \Big|_{r=0} = \int_X \langle \nabla f, \nabla\varphi\rangle d\nu_0}$$

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$$\Rightarrow \dot{\mu}_0 = \nabla f$$

Overview of the proof

$\mu = \mu_0$, $\nu = \mu_r$

& consider $\mathbf{W}(0, N)$ at $r \approx 0$, $t \approx 0$

(Use Kantorovich duality & Hopf-Lax semigroup)

$$\frac{1}{2r} W_2(\mu, \nu)^2 = \sup_{\varphi \in \mathrm{Lip}_b(X)} \left[\int Q_r \varphi \, d\nu - \int \varphi \, d\mu \right],$$

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$$\frac{(3)}{r^2} \ge \left(\int P_s(g \Delta^g f) \log P_s g \, d\mathfrak{m} \right)^2 + o(1)$$

Overview (cont.'d)

$$\Downarrow$$

$$\int \big(-P_t(|\nabla f|^2) + 2\langle \nabla f, \nabla P_t f\rangle\big) g \, d\mathfrak{m}$$

$$\leq \int |\nabla f|^2 g \, d\mathfrak{m} - \frac{2}{N} \int_0^t ds \left| \int P_s(g \Delta^g f) \log P_s g \, d\mathfrak{m} \right|^2$$

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$$\Downarrow \partial_t|_{t=0}$$

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$$-2\int \Gamma_2(f,f) g \, d\mathfrak{m} \leq -\frac{2}{N}\left| \int (g\Delta^g f) \log g \, d\mathfrak{m} \right|^2$$

$$= -\frac{2}{N}\left| \int g\langle \nabla f, \nabla \log g \rangle \, d\mathfrak{m} \right|^2$$

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$$g \to \delta_x \qquad = -\frac{2}{N}\left|\int (\Delta f) g \, d\mathfrak{m}\right|^2 \qquad \square$$

1. Introduction

2. Framework and main results

3. Idea of the proof

4. **Applications and further problems**

Applications to functional inequalities

- New proof of Entropy-energy inequality ($K > 0$):

$$\mathrm{Ent}(\rho\mathfrak{m}) \leq \frac{N}{2}\log\left(1 + \frac{1}{NK}\int \frac{|\nabla\rho|^2}{\rho}d\mathfrak{m}\right).$$

$\Rightarrow$ Sharp Sobolev ineq. (e.g. [Profeta '15])

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$\Rightarrow$ Sharp Sobolev ineq. (e.g. [Profeta '15])

- An Entropy regularization bound: When $K = 0$,

$$\mathrm{Ent}(\mu P_t) \le \frac{N}{2}\max\left\{C, \log\frac{W_2(\mu,\mathfrak{m})^2}{Nt}\right\}$$

Questions

- Applications of $\widehat{\mathsf{W}}(K,N)$ to a sharper rate of conv. (when $K>0$): e.g.,

$$W_2(\mu P_t, \mathfrak{m}) \le \exp\left(-\frac{NK}{N-1}t\right) W_2(\mu, \mathfrak{m})$$

- A self-improvement of $\widehat{\mathsf{W}}(K,N)$ or $\widehat{\mathsf{W}}'(K,N)$

  e.g. a self-improvement of $\mathsf{BE}(K,\infty)$

  $$\Rightarrow W_p(\mu P_t, \nu P_t) \le W_p(\mu, \nu) \ ({}^{\forall} p \ge 2)$$

- $\widehat{\mathsf{W}}(K,N)$ by coupling method

  for diffusion processes (even on Riem. mfd)

- In another framework? (e.g. b.w.(super-)Ricci flow)

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$\mathcal{W}$-entropy

- **W**$(0, \infty) \Rightarrow I(\mu P_t) \searrow$

  $\left(I(\rho v) := \int \frac{|\nabla \rho|^2}{\rho} dv\right.$: Fisher information)

- **W**$(0, N) \Rightarrow \mathcal{W}(t) \searrow$ [Jiang & Zhang '16]/[K.]

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Q. $\widehat{\mathbf{W}}(K,N) \Rightarrow$ Monotonicity of some (nice) f'nal?