Analysis of heat distributions on metric measure spaces with a lower Ricci curvature bound

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PDE workshop at Sendai (May 26–27, 2016)

1. Introduction

Heat flow and Ricci curvature

M: complete Riemannian manifold

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), \\ u(0,\cdot) = f \end{cases}$$ heat eq. on M

$\Rightarrow u = P_t f$

- $P_t = \mathrm{e}^{t\Delta}$: heat semigroup
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★ P_t characterizes "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$"

(curvature-dimension cond.)

★ Extensions to metric measure spaces $(X, d, \mathfrak{m})$

$\longleftarrow$ Optimal transport

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Bakry-Émery's approach

Bochner-Weitzenböck formula

$$\Gamma_2(f,f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\nabla^2 f\|_{\mathsf{HS}}^2,$$
$$\Gamma_2(f,f) := \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle$$

★ "Ric $\geq K$ & dim $\leq N$",

$\Leftrightarrow$ **BE**(K,N): $\boxed{\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2}$

(Bakry-Émery's curv.-dim. cond.[Bakry-Émery '85])

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(Bakry-Émery's curv.-dim. cond.[Bakry-Émery '85])

$\Leftrightarrow |\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2)$

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- Bring a mass $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(X)$
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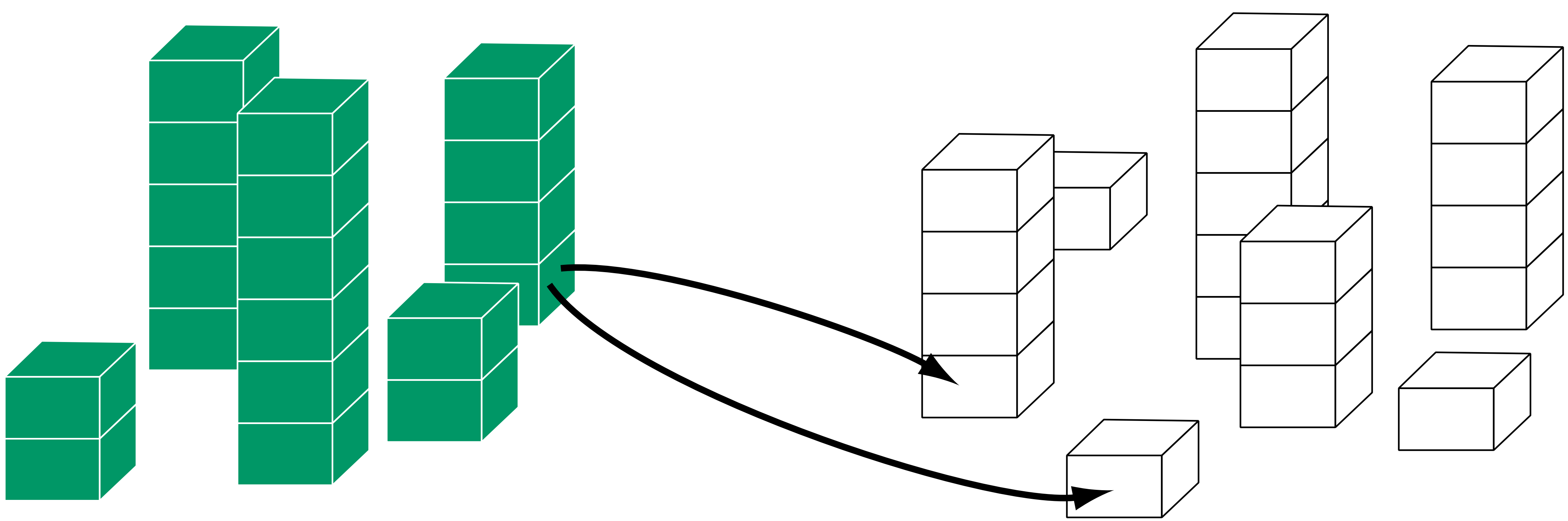

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Optimal transportation cost

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} c(x, y)\, \pi(dxdy)$$

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X^2) \,\middle|\, \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

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- W(K, ∞):
  $W_2(\mu P_t, \nu P_t)^2 \leq \mathrm{e}^{-2Kt} W_2(\mu, \nu)^2$
- W$(0, N)$:
  $W_2(\mu P_t, \nu P_s)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$

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Questions

- Estimate(s) for other costs than d^2
- Another bound of $W_2(\mu P_t, \mu P_t)$ when $N < \infty$

Outline of the talk

1. **Introduction**

2. **Basics of optimal transport**

3. **RCD spaces**

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Wasserstein distance

(X, d): complete separable geodesic metric space

L^p-Wasserstein distance ($p \in [1, \infty)$)

$$W_p(\mu, \nu) := \mathcal{T}_{d^p}(\mu, \nu)^{1/p} \left(= \inf_{\pi \in \Pi(\mu, \nu)} \|d\|_{L^p(\pi)} \right)$$

- W_p: (pseudo-)distance
- W_p-conv. $\Leftrightarrow$ weak conv. & conv. of p-th moment
- Property of $(X, d) \Rightarrow$ the same for $(\mathcal{P}(X), W_p)$

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- Property of $(X, d) \Rightarrow$ the same for $(\mathcal{P}(X), W_p)$ (e.g. separability/completeness ($p < \infty$))

Superposition principle ($p < \infty$)

$^\forall \mu_0, \mu_1 \in \mathcal{P}_p(X)$, $^\exists (\mu_r)_{r\in[0,1]}$: W_p-min. geod.
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Kantorovich duality

$$\mathcal{T}_c(\mu, \nu) = \sup_{g,f} \left[\int_X g \, d\mu - \int_X f \, d\nu \right]$$

where $f, g \in C_b(X)$,

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$$= \sup_{f}\left[\int_X \hat{f}\,d\mu - \int_X f\,d\nu\right],$$

where $f, g \in C_b(X)$,

$$g(x) - f(y) \leq c(x,y),$$

$$\hat{f}(x) := \inf_{y \in X}\left[f(y) + c(x,y)\right]$$

(inf-convolution)

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$$\frac{1}{p}W_p(\mu,\nu)^p = \sup_{g,f}\left[\int_X g\,d\mu - \int_X f\,d\nu\right]$$
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$\rightsquigarrow$ Hamilton-Jacobi eq.'s for $Q_s f$

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Energy f'nal and its density

$(X, d, \mathfrak{m})$: met. meas. sp. ($\mathfrak{m}$: loc.-finite meas.)

$$\boxed{P_t = \mathrm{e}^{t\Delta} \leftrightarrow \text{Cheeger's } L^2\text{-energy}}$$

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★ $|f(\gamma_1) - f(\gamma_0)| \le \int_0^1 |\nabla f|_w(\gamma_s)|\dot{\gamma}_s| \, ds$

for a.e. trajectories $(\gamma_s)_{s\in[0,1]}$ of "nice" transports $\Gamma \in \mathcal{P}(C([0,1]; X))$ [Ambrosio, Gigli & Savaré '13]

$\mathsf{RCD}^*(K, N)$ space

Definition 1 ([Ambrosio, Gigli & Savaré '14])

$(X, d, \mathfrak{m})$: Riemannian $\mathbf{CD}(K, \infty)$ sp.

$\overset{\text{def}}{\Leftrightarrow}$ "$\nabla^2 \operatorname{Ent} \geq K$"

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(Basic) properties

- "$\partial_t(\mu P_t) = -\nabla \mathrm{Ent}(\mu P_t)$" on $(\mathcal{P}_2(X), W_2)$
- **Ch**: str. local quasi-reg. Dirichlet form admitting carré du champ ($\rightsquigarrow$ Brownian motion $(B(t), \mathbb{P}_x)$)
- Bakry-Émery's cond. $\mathbf{BE}(K, N)$

  "$\frac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla \Delta f \rangle_w \geq K|\nabla f|_w^2 + \frac{1}{N}(\Delta f)^2$"

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$\mathsf{RCD}^*(K,N)$ space

Definition 1 ([Erbar, K. & Sturm '15])

$(X,d,\mathfrak{m})$: $\mathsf{RCD}^*(K,N)$ sp.

$$\Leftrightarrow \text{“}\nabla^2 \operatorname{Ent} - \frac{1}{N}\nabla \operatorname{Ent}^{\otimes 2} \geq K\text{”} \ \ \& \ P_t\text{: linear}$$

(Basic) properties

- “$\partial_t(\mu P_t) = -\nabla \operatorname{Ent}(\mu P_t)$” on $(\mathcal{P}_2(X), W_2)$
- **Ch**: str. local quasi-reg. Dirichlet form admitting carré du champ ($\rightsquigarrow$ Brownian motion $(B(t), \mathbb{P}_x)$)
- Bakry-Émery's cond. $\mathsf{BE}(K,N)$

  “ $\frac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla \Delta f\rangle_w \geq K|\nabla f|_w^2 + \frac{1}{N}(\Delta f)^2$ ”

$\mathsf{RCD}^*(K, N) \Rightarrow \mathsf{BE}(K, N)$

$\underline{N = \infty}$ [AGS '13–'15 / AG, Mondino & Rajala '15]

$\mathsf{RCD}(K, \infty)$: $\nabla^2 \mathrm{Ent} \geq K$ & P_t: linear

$\Rightarrow$

$$\mathsf{BE}(K, \infty)\text{: } \frac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla \Delta f \rangle_w \geq K|\nabla f|_w^2$$

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$\Rightarrow \mathsf{W}_2(K,\infty)$: $\boxed{W_2(\mu P_t, \nu P_t) \leq \mathrm{e}^{-Kt} W_2(\mu,\nu)}$

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$\Downarrow$ [K. '10, '13 / AGS '14 / $\cdots$]

$\mathsf{G}_2(K,\infty)$: $\boxed{|\nabla P_t f|_w \leq \mathrm{e}^{-Kt} P_t(|\nabla f|_w^2)^{1/2}}$

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- $\mathsf{G}_2(K,\infty)$ "$\Rightarrow$" $\mathsf{RCD}(K,\infty)$ [AGS '15]

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- (All) extensions to $N<\infty$ [EKS '15]

Analytic properties of $\mathsf{RCD}^*(K, N)$

- A moment bound of $\mathfrak{m}$ [Sturm '06]
- (Global) f'nal ineq.'s (e.g. log Sobolev)
  [$\cdots$/Cavalletti & Mondino]
- L^∞/Lip. regularization of P_t [AGS '14/AGMR '15]
- Self-improvement of $\mathbf{BE}(K, \infty)$ [Savaré '14]
  $\rightsquigarrow$ (Almost) full-strength of Bakry-Émery
- Li-Yau's ineq./Gaussian heat kernel estimate/
  L^p-bddness of Riesz transf. [R. Jiang/$\cdots$]
- Regularity of (loc.) harm. fn's [Jiang '14 / Kell]
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1. Introduction

2. Basics of optimal transport

3. RCD spaces

4. Transportation costs between heat flows

Self-improvement

$$\mathsf{W}_p(K,\infty):\ W_p(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_p(\mu, \nu)$$

$$\mathsf{G}_q(K,\infty):\ |\nabla P_t f|_w \le \mathrm{e}^{-Kt} P_t(|\nabla f|_w^q)^{1/q}$$

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$$\mathsf{W}_2(K,\infty)$$

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$$\begin{array}{ccc} \mathsf{W}_2(K,\infty) & & \\ \Updownarrow & & \\ \mathsf{G}_2(K,\infty) & & \mathsf{G}_1(K,\infty) \\ \Updownarrow & & \Updownarrow \\ \mathsf{BE}(K,\infty) & \Rightarrow & \mathsf{BE}^*(K,\infty) \end{array}$$

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Applications

★ $|\nabla P_t f| \le \mathrm{e}^{-Kt} P_t(|\nabla f|)$

$\Rightarrow$ (P_t-)(reversed) log Sobolev ineq.,

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Coupling by parallel transport [Sturm '15]

$^\forall x_0, x_1 \in X$ $^\exists(B_t^{(0)}, B_t^{(1)})$: coupling of BMs on X s.t.

- $(B_0^{(0)}, B_0^{(1)}) = (x_0, x_1)$
- $\mathrm{e}^{Kt} d(B_t^{(0)}, B_t^{(1)}) \searrow$ a.s.

Total variation bound

Theorem 2 ([K.] cf. [K. & Sturm '13])

Suppose $\mathsf{RCD}(K, \infty)$. *Then*

$$\frac{1}{2}\|\mu P_t - \nu P_t\|_{var} \leq \mathcal{T}_{\varphi_t(d)}(\mu, \nu),$$

where $\varphi_t(r) := 2\Phi\left(\frac{r}{2\sqrt{\sigma(t)}}\right) - 1$,

$$\sigma(t) := 2\int_0^t e^{Ks} ds, \qquad \Phi(r) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r} e^{-u^2/2} du$$

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$\because$ Use P_t-reversed Gaussian isoperimetric ineq.

$$I(P_t f)^2 - P_t(I(f))^2 \ge \sigma(t)|\nabla P_t f|^2 \;\; (I := \Phi' \circ \Phi^{-1})$$

L^p/L^q-extension when $N < \infty$

Theorem 3 (Dimensional self-improvement [K.])

Suppose **RCD***(K, N). *For* $p \geq 2$ & $q = p/(p-1)$,

$$\Gamma_2(f, f) - K|\nabla f|_w^2 - \frac{1}{N + p - 2}(\Delta f)^2 \geq \frac{2-q}{4}\frac{|\nabla|\nabla f|_w^2|_w^2}{|\nabla f|_w^2}$$

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Ex. $\mathbf{W}_p(0,N)$: $W_p(\mu P_t, \nu P_s)^2 \leq W_p(\mu,\nu)^2 + (N+p-2)(\sqrt{t}-\sqrt{s})^2$

L^p/L^q-extension when $N < \infty$

Theorem 4 (A reverse dim'nal isop. ineq. [K.])

Suppose **RCD**$^*(K, N)$, $K > 0$. *Then*

$$(\mathrm{e}^{2Kt} - 1)|\nabla P_t f|^2 \le I_N(P_t f)^2 - P_t({I_N}^{N/(N-1)}(f))^{2(N-1)/N} - \frac{1}{N-1}\int_0^t (\mathrm{e}^{2Ks} - 1)ds(\Delta P_t f)^2,$$

$I_N := \Phi'_N \circ \Phi_N^{-1}$, $\Phi_N(r) := \overline{\mathrm{vol}}_{K,N}(B_r(o))$,

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$\because$ Use $\mathbf{G}_{N/N-1}(K, N)$

New dimensional W_2-contraction

$\underline{\widehat{\mathsf{W}}(K,N)}$

$$W_2(\mu P_t, \nu P_t)^2 \leq \mathrm{e}^{-2Kt} W_2(\mu,\nu)^2 - \frac{2}{N}\int_0^t \mathrm{e}^{-2K(t-s)} \big(\,\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s)\big)^2 ds$$

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Theorem 5 ([Bolley, Gentil, Guillin & K.])

On Riemannian energy measure spaces,
for $K \in \mathbb{R}$ and $N > 0$,

$$\mathsf{BE}(K,N) \Leftrightarrow \widehat{\mathsf{W}}(K,N)$$

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- "$\Rightarrow$" for Markov transportation dist. [BGG '14]
- "$\Leftrightarrow$" on Riem. mfds [Gentil '15]