Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$(0, N)$ spaces

Kazumasa Kuwada

(Tokyo Institute of Technology)

joint work with X.-D. Li (Chinese Academy of Science)

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1. Introduction

Perelman's $\mathcal{W}$-entropy

(M, g): m-dim. cpt. Riem. mfd, $\tau > 0$,

$f \in C^\infty(M)$, $\displaystyle\int_M \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}}\,\mathrm{d}\,\mathrm{vol} = 1$

$$\mathcal{W}(g, f, \tau) := \int_M \left[\tau(R + |\nabla f|^2) + f - m\right] \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}}\,\mathrm{d}\,\mathrm{vol}$$

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★ $(g(t), f(t), \tau(t))$: $\partial_t \tau = -1$,

$$\partial_t g = -2\,\mathrm{Ric},\ \partial_t f = -\Delta f + |\nabla f|^2 - R + \frac{m}{2\tau}$$

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$$\Rightarrow \frac{d}{dt}\mathcal{W}(g, f, \tau) \geq 0$$

Entropy formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g, f, \tau) \geq 0$$

$$\Uparrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 2\int_M \tau \left| \mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}} \,\mathrm{d}\,\mathrm{vol}$$

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- $\partial_\tau \,\mathrm{vol} = R\,\mathrm{vol} \Rightarrow \partial_\tau (u\,\mathrm{vol}) = \Delta(u\,\mathrm{vol})$

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[L. Ni '04]

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⇝ Extension to weighted Riem. mfds [X.-D. Li '12]

Purpose

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Can one extend the monotonicity/rigidity of $\mathcal{W}$ on metric measure spaces with "$\mathbf{Ric} \geq \mathbf{0}$ & $\mathbf{dim} \leq N$" ($\mathbf{RCD}(0, N)$ spaces)?

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- Without the entropy formula
  $\leftsquigarrow$ optimal transport approach
- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof

4.1 Monotonicity
4.2 Rigidity
4.3 Additional remarks

Met. meas. sp. & heat flow on it

$(X, d, \mathfrak{m})$: Polish geod. met. meas. sp.

($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

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$$2\mathsf{Ch}(f) := \inf \left\{ \varliminf_{n} \int_X \mathrm{lip}(f_n)^2 \mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$

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Definition 1 ([Ambrosio, Gigli & Savaré '14])

$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form ($\Leftrightarrow$ P_t: linear $\Leftrightarrow$ Δ: linear)

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$\Rightarrow {}^{\exists}\langle D\cdot, D\cdot\rangle_w$ bilinear s.t. $\langle Df, Df\rangle_w = |Df|_w^2$

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Definition 2 (RCD$(0, N)$ $(N \in [1, \infty))$)

- $(X, d, \mathfrak{m})$: infin. Hilb.
- $\displaystyle\int_X \exp\left(-{}^{\exists}c d({}^{\exists}x_0, x)^2\right) \mathfrak{m}(\mathrm{d}x) < \infty$
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★ **RCD**$^*(K, N)$ $(K \neq 0)$ can be defined similarly

Examples

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  d: Riem. dist., $\mathfrak{m} = \mathrm{e}^{-V}\mathrm{vol}_g$ $\quad (V : X \to \mathbb{R})$
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$$\mathbf{RCD}^*(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

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RCD$(0, N)$: Some regularity ass'ns &

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★ Equiv. cond'ns to **RCD**$(0, N)$ (up to reg. assn's)

- "$\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \ge \frac{1}{N}|\Delta f|^2$"

  (Bakry-Émery's curv.-dim. cond.)
- On $(\mathcal{P}_2(X), W_2)$, $^\forall\mu_0$, $^\exists(\mu_t)_{t\ge 0}$ sol. to $(0, N)$-evolution variational inequality of Ent (a (metric) formulation of "$\dot{\mu}_t = -\nabla\,\mathrm{Ent}(\mu_t)$")

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Heat flow

Properties of the heat semigr. P_t under **RCD**$^*(K, N)$

- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
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- $\mu_t = P_t\mu (= \rho_t \mathfrak{m}) \in \mathcal{P}(X)$ satisfies

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(Fisher information)

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$$\mathcal{W}(\mu, t) := \int_X \left[t|Df|_w^2 + f - N\right] \rho \,\mathrm{d}\mathfrak{m}$$

- $I(\mu) := \int \frac{|D\rho|_w^2}{\rho}\,\mathrm{d}\mathfrak{m}$
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$$\begin{aligned}\mathcal{W}(\mu, t) &:= tI(\mu) - \mathrm{Ent}(\mu) - \frac{N}{2}\log t + c_1 \\ &= \frac{d}{dt}[t\,\mathrm{Ent}(\mu_t) - t\,\mathrm{Ent}^{\mathrm{Eucl}}(\mathrm{Gauss}(t))] + c\end{aligned}$$

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Main thm

Theorem 3 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: RCD$(0, N)$, $N \geq 2$, $\mu_t := P_t \mu$

(1) $\mathcal{W}(\mu_t, t) \searrow$ *in* $t \in (0, \infty)$

(2) *Suppose* $^\exists t_* > 0$ *s.t.*

$$\varlimsup_{t \downarrow t_*} \frac{\mathcal{W}(\mu_t, t) - \mathcal{W}(\mu_{t_*}, t_*)}{t - t_*} = 0$$

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$X \simeq$ $(0, N-1)$*-cone of* *an* $\mathsf{RCD}^*(N-2, N-1)$ *sp.*

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Cone

Definition 4 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Leftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2$
  $:= r^2 + s^2 - 2rs \cos(d_Y(x, y) \wedge \pi)$
- $\mathfrak{m}(\mathrm{d}r\mathrm{d}x) := r^N \mathrm{d}r \mathfrak{m}_Y(\mathrm{d}x)$

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Remarks

- [Jiang & Zhang '16] X: cpt. $\Rightarrow$ Theorem 1 (1)
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
- $(0, N)$-cone of Y is a (smooth) Riem. mfd
  $\Leftrightarrow Y \simeq \mathbb{S}^{N-1}(1)$
- An "almost rigidity" can be formulated

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4.1. Monotonicity

Optimal transport approach on Ricci flow

$\partial_\tau g_\tau = 2\,\mathrm{Ric}$, μ_τ: $\partial_\tau \mu_\tau = \Delta_\tau \mu_\tau$

$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r}\left(|\dot{\gamma}_r|_r^2 + R(\gamma_r)\right) \mathrm{d}r \right]$$

$\mathcal{T}_{L_s^t}(\mu,\nu) := \inf_\pi \int_{X\times X} L_s^t \,\mathrm{d}\pi$: L-opt. trans. cost

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

$$\Rightarrow \lim_{\tau \downarrow 1} \frac{\Xi_1^\tau(t)}{(\tau - 1)^2} \searrow$$

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

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Toward the time-inhomogeneous case

$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r}\left(|\dot\gamma_r|_r^2 + R(\gamma_r)\right) \mathrm{d}r \right]$$

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$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r}\,|\dot\gamma_r|^2 \,\mathrm{d}r \right]$$

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$$L_s^t(x,y) := \inf_{\gamma_s=x,\,\gamma_t=y}\left[\int_s^t \sqrt{r}|\dot\gamma_r|^2\,\mathrm{d}r\right]$$

★ $\gamma_r^* := \gamma_{\xi(r)}$, $\xi(r) := ((1-r)\sqrt{s} + r\sqrt{t})^2$

$$\Rightarrow 2(\sqrt{t}-\sqrt{s})\int_s^t \sqrt{r}|\dot\gamma_r|^2\,\mathrm{d}r = \int_0^1 |\dot\gamma_u^*|^2\,\mathrm{d}u$$

Toward the time-inhomogeneous case

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$$\Rightarrow 2(\sqrt{t} - \sqrt{s}) L_s^t(x,y) = d(x,y)^2,$$

$$2(\sqrt{t} - \sqrt{s}) \mathcal{T}_{L_s^t}(\mu,\nu) = W_2(\mu,\nu)^2$$

Toward the time-inhomogeneous case

$$2(\sqrt{t}-\sqrt{s})\mathcal{T}_{L_s^t}(\mu,\nu)=W_2(\mu,\nu)^2$$

$$\Xi_{\tau_0}^{\tau_1}(t) := 2(\sqrt{\tau_1 t}-\sqrt{\tau_0 t})\mathcal{T}_{L_{\tau_0 t}^{\tau_1 t}}(\mu_{\tau_0 t},\mu_{\tau_1 t}) - 2N(\sqrt{\tau_1 t}-\sqrt{\tau_0 t})^2$$

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Toward the time-inhomogeneous case

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$$\Downarrow$$

$$\Xi_1^{\tau}(t) \searrow$$

$$\Updownarrow$$

$$\begin{aligned}W_2(\mu_t,\mu_{\tau t})^2 \le W_2(\mu_s,\mu_{\tau s})^2 \\ +2N(\sqrt{\tau(t-s)}-\sqrt{t-s})^2\end{aligned}$$

Derivation from $\mathsf{RCD}(0, N)$

$$W_2(P_{t'}\mu', P_{s'}\nu')^2 \le W_2(\mu', \nu')^2 + 2N(\sqrt{t'} - \sqrt{s'})^2$$

Derivation from RCD$(0, N)$

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$\Downarrow$

$\mu' = \mu_s$, $\nu' = \mu_{s+s^\alpha\delta}$

$t' = t - s$, $s' = (t^\alpha - s^\alpha)\delta + t - s$

& "$\overline{\lim_{\delta \downarrow 0}} \dfrac{1}{\delta^2}$" with $\|\dot{\mu}_t\|^2 = I(\mu_t)$

Derivation from $\mathrm{RCD}(0, N)$

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$$t^{2\alpha} I(\mu_t) \le s^{2\alpha} I(\mu_s) - \frac{N}{2} \cdot \frac{(t^\alpha - s^\alpha)^2}{t - s} \qquad (F(\alpha))$$

Derivation from $\mathsf{RCD}(0, N)$

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$\overset{\alpha=1}{\Rightarrow}$ $t\,h(t) := t^2 I(\mu_t) - \frac{Nt}{2} \searrow$ in t

Derivation from $\mathsf{RCD}(0, N)$

$$W_2(P_{t'}\mu', P_{s'}\nu')^2 \le W_2(\mu',\nu')^2 + 2N(\sqrt{t'} - \sqrt{s'})^2$$

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$\bigstar$ "$\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))$"

Derivation from $\mathsf{RCD}(0, N)$

$$W_2(P_{t'}\mu', P_{s'}\nu')^2 \le W_2(\mu', \nu')^2 + 2N(\sqrt{t'} - \sqrt{s'})^2$$

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$\bigstar$ "$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))$" $\Rightarrow \mathcal{W}(\mu_t, t) \searrow$ $\square$

4.2. Rigidity

Geometric properties of $\mathsf{RCD}(0, N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $$\Delta d(x_0, \cdot)^2 \le 2N \text{ in the distributional sense}$$

- Volume rigidity [Gigli & De Philippis]: $N \ge 2$,

$$^\exists x_0, {}^\forall r, R > 0,\ \mathfrak{m}(B_R(x_0)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(x_0))$$

$$\Rightarrow X \simeq \begin{array}{l} (0, N-1)\text{-cone of} \\ \text{an } \mathsf{RCD}^*(N-2, N-1) \text{ sp.} \end{array}$$

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Analytic properties of RCD$(0, N)$ sp.

- Li-Yau ineq. [Garofalo & Mondino '14/Jiang '15]:

$$-\Delta \log P_t f = \frac{|DP_t f|_w^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \le \frac{N}{2t}$$

- Sharp heat kernel estimate [Jiang, Li & Zhang '16]:

$^\forall \varepsilon > 0$, $^\exists C_\varepsilon > 0$ s.t.

$$\frac{C_\varepsilon^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x,y)^2}{(4-\varepsilon)t}\right) \le p_t(x,y)$$

$$\le \frac{C_\varepsilon}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right)$$

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$\Rightarrow$ $\lim_{t \downarrow 0} 4t \log p_t(x,y) = -d(x,y)^2$ cpt. unif. in y

Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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Equality in Laplacian comparison

$$\boldsymbol{I(\mu_t) = \frac{N}{2t}\ (t \in (0, t_*])}$$

Li-Yau: $\boldsymbol{-\Delta \log p_t^{x_0} = \frac{|Dp_t^{x_0}|^2}{(p_t^{x_0})^2} - \frac{\Delta p_t^{x_0}}{p_t^{x_0}} \le \frac{N}{2t}}$

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$$\Delta d(x_0, \cdot)^2 = 2N$$

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$$\Delta d(x_0, \cdot)^2 = 2N$$

$\Rightarrow$ Volume rigidity [Gigli & De Philippis] is applicable □

4.3. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0, \cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0, x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0, x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0, x)^2}{4t}\right)$$

$\because$ Compute I for "Gaussian kernel" in two ways &
RHS enjoys the energy dissipation identity for Ent $\square$

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In particular, X *is non-compact*

$\because$ Compute I for "Gaussian kernel" in two ways &

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Initial data

Lemma 2

Suppose $I(\mu_t) = \dfrac{N}{2t}$. *Then* $I(p_t^x\,\mathfrak{m}) = \dfrac{N}{2t}$ μ*-a.e.* x.

$\because\ \mu \mapsto I(\mu)$ convex

$$\Rightarrow \frac{N}{2t} = I(\mu_t) \text{``}\leq\text{''} \int_X I(p_t^x)\mu(\mathrm{d}x) \overset{\text{Li-Yau}}{\leq} \frac{N}{2t} \qquad \square$$

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Lemma 3

Suppose $I(\mu_t) = \frac{N}{2t}$*. Then* μ *is Dirac.*

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Heat flow is a W_2-geodesic

Proposition 4

Suppose $\Delta d(x_0, \cdot)^2 = 2N$ *and* $\mu_t = P_t \delta_{x_0}$.
$\Rightarrow (\mu_{t^2/(2N)})_{t \geq 0}$: W_2*-min. geod.*

$\because$

- $\dfrac{N}{2t} = I(\mu_t) = \dfrac{1}{4t^2} \int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$
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$\Rightarrow \mu_t^* := \mu_{t^2/(2N)}$ satisfies
$W_2(\mu_0^*, \mu_t^*) = t$ & $\|\dot{\mu}_t^*\| = 1$ $\square$