Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$^*(0, N)$ spaces

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joint work with X.-D. Li (Chinese Academy of Science)

Heat kernel, Stochastic Processes and Functional Inequalities
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1. Introduction

$\mathcal{W}$-entropy on Riem. mfd

M: m-dim. cpt. Riem. mfd, $t > 0$, $f \in C^\infty(M)$

$u := \dfrac{\mathrm{e}^{-f}}{(4\pi t)^{m/2}}$, $\displaystyle\int_M u \,\mathrm{d}\,\mathrm{vol} = 1$

$$\mathcal{W}(f,t) := \int \left[t|\nabla f|^2 + f - m\right] u \,\mathrm{d}\,\mathrm{vol}$$

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Entropy formula and rigidity

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$$\Uparrow$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = -2\int_M t\left(\left|\nabla^2 f - \frac{g}{2t}\right|^2 + \mathrm{Ric}(\nabla f, \nabla f)\right) u \,\mathrm{d}\,\mathrm{vol}$$

[L. Ni '04]

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$\rightsquigarrow$ Extension to weighted Riem. mfds [X.-D. Li '12]

Purpose

Q.

Can one extend the monotonicity/rigidity of $\mathcal{W}$ on "Riemannian" metric measure spaces with "$\mathbf{Ric} \geq \mathbf{0}$ & $\mathbf{dim} \leq \boldsymbol{N}$" ($\mathbf{RCD}(\mathbf{0}, \boldsymbol{N})$ spaces)?

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  $\leftsquigarrow$ optimal transport approach

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- Weaken ass'n(s) even on (weighted) Riem. mfds
- Without the entropy formula
  - $\leftsquigarrow$ optimal transport approach
- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

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2. Framework: RCD spaces

3. Main results

4. Proof

4.1 Monotonicity

4.2 Rigidity

4.3 Additional remarks

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Met. meas. sp. & heat flow on it

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$$2\mathsf{Ch}(f) := \inf \left\{ \varliminf_{n} \int_X \mathrm{lip}(f_n)^2 \mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$

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$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ $\mathbf{Ch}$: quadratic form ($\Leftrightarrow P_t$: linear $\Leftrightarrow \Delta$: linear)

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$\Rightarrow {}^{\exists}\langle D\cdot, D\cdot\rangle_w$ bilinear s.t. $\langle Df, Df\rangle_w = |Df|_w^2$

Characterizations of RCD cond.

RCD$^*(0, N)$: infin. Hilb., some regularity ass'ns &

$$W_2(P_s f\,\mathfrak{m}, P_t g\,\mathfrak{m})^2 \le W_2(f\,\mathfrak{m}, g\,\mathfrak{m})^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

★ Equiv. cond'ns (up to reg. assn's)

- "$\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \ge \frac{1}{N}|\Delta f|^2$"

  (Bakry-Émery's curv.-dim. cond.)

- On $(\mathcal{P}_2(X), W_2)$, $^\forall\mu_0$, $^\exists(\mu_t)_{t\ge 0}$ sol. to $(0, N)$-evolution variational inequality of Ent (a (metric) formulation of "$\dot{\mu}_t = -\nabla\,\mathrm{Ent}(\mu_t)$")

[Erbar, K. & Sturm '15]

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Examples

- (X, g): m-dim. cpl. Riem. mfd., $\partial X = \emptyset$,
  d: Riem. dist., $\mathfrak{m} = e^{-V} \mathrm{vol}_g$ $\quad (V : X \to \mathbb{R})$
  (Weighted Riem. mfd)

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$$\mathbf{RCD}^*(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

- (Pointed) measured GH lim. of $\mathbf{RCD}^*(K, N)$ sp.'s
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Properties of the heat semigr. P_t under **RCD**$^*(K, N)$

- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
- P_t admits a continuous kernel (heat kernel) p_t
- $\mu_t = P_t \mu (= \rho_t \mathfrak{m}) \in \mathcal{P}(X)$ satisfies

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$$\begin{aligned}\mathcal{W}(\mu, t) &:= \int_X \left[t|Df|_w^2 + f - N\right] \rho\, \mathrm{d}\mathfrak{m} \\ &= tI(\mu) - \mathrm{Ent}(\mu) - \frac{N}{2}\log t + c_1\end{aligned}$$

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$(X, d, \mathfrak{m})$: $\mathsf{RCD}^*(0, N)$, $N \geq 2$, $\mu_t := P_t\mu$

(1) $\mathcal{W}(\mu_t, t) \searrow$ *in* $t \in (0, \infty)$

(2) *Suppose* $^{\exists} t_* > 0$ *s.t.*

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$X \simeq$ $(0, N-1)$*-cone of* *an* $\mathsf{RCD}^*(N-2, N-1)$ *sp.*

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Cone

Definition 3 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Leftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2$
  $:= r^2 + s^2 - 2rs \cos(d_Y(x, y) \wedge \pi)$
- $\mathfrak{m}(\mathrm{d}r\mathrm{d}x) := r^N \mathrm{d}r\,\mathfrak{m}_Y(\mathrm{d}x)$

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Remarks

- [Jiang & Zhang '16] X: cpt. $\Rightarrow$ Theorem 1 (1)
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
- $(0, N)$-cone of Y is a (smooth) Riem. mfd
  $\Leftrightarrow Y \simeq \mathbb{S}^{N-1}(1)$
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  $\rightsquigarrow$ It is a conclusion in Theorem 1
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1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof
4.1 Monotonicity
4.2 Rigidity
4.3 Additional remarks

4.1. Monotonicity

(cf. [Topping '09] on (backward) Ricci flow)

Derivation from $\mathsf{RCD}^*(0,N)$

★ $t\,h(t) := t^2 I(\mu_t) - \dfrac{Nt}{2}$

$$\Rightarrow \boxed{\text{“}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))\text{”}}$$

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$$W_2(\mu_t, \mu_{(1+\delta)t})^2 \le W_2(\mu_s, \mu_{(1+\delta)s})^2 + \cdots$$

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$\Rightarrow t\,h(t) \searrow$ in t $\square$

4.2. Rigidity

Geometric properties of RCD$^*(0, N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $\Delta d(x_0, \cdot)^2 \le 2N$ in the distributional sense

- Volume rigidity [Gigli & De Philippis]: $N \ge 2$,

$$\exists x_0, \forall r, R > 0, \ \mathfrak{m}(B_R(x_0)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(x_0))$$

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Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

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★ "$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 0$" & **RCD**$^*(0, N)$ cond.

$\Rightarrow h(t) = 0$, i.e. $I(\mu_t) = \dfrac{N}{2t}$ $(t \in (0, t_*])$

Equality in Laplacian comparison

$$\boldsymbol{I(\mu_t) = \frac{N}{2t}\ (t \in (0, t_*])}$$

Li-Yau: $\boldsymbol{-\Delta \log p_t^{x_0} = \frac{|Dp_t^{x_0}|^2}{(p_t^{x_0})^2} - \frac{\Delta p_t^{x_0}}{p_t^{x_0}} \le \frac{N}{2t}}$

[Garofalo & Mondino '14/Jiang '15]

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[Jiang, Li & Zhang '16]

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$\Delta d(x_0, \cdot)^2 = 2N \ \Rightarrow$ Volume rigidity $\square$

4.3. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0,\cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0,x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0,x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0,x)^2}{4t}\right)$$

$\because$ Compute I for "Gaussian kernel" in two ways & RHS enjoys the energy dissipation identity for Ent $\square$

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In particular, X *is non-compact*

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Initial data

Lemma 2

Suppose $I(\mu_t) = \dfrac{N}{2t}$. *Then* $I(p_t^x\,\mathfrak{m}) = \dfrac{N}{2t}$ μ-*a.e.* x.

$\because \mu \mapsto I(\mu)$ convex

$$\Rightarrow \frac{N}{2t} = I(\mu_t) \text{“}\leq\text{”} \int_X I(p_t^x)\mu(\mathrm{d}x) \overset{\text{Li-Yau}}{\leq} \frac{N}{2t} \qquad \square$$

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Heat flow is a W_2-geodesic

Proposition 4

Suppose $\Delta d(x_0, \cdot)^2 = 2N$ *and* $\mu_t = P_t \delta_{x_0}$.
$\Rightarrow (\mu_{t^2/(2N)})_{t \geq 0}$: W_2*-min. geod.*

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$\Rightarrow \mu_t^* := \mu_{t^2/(2N)}$ satisfies
$W_2(\mu_0^*, \mu_t^*) = t$ & $\|\dot{\mu}_t^*\| = 1$ $\square$