Estimates of transportation costs between heat distributions

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1. Introduction

Heat flow

(M, g, v): weighted Riemannian manifold
(complete, $\partial M = \emptyset$, $v = \mathrm{e}^{-V} \mathrm{vol}_g$)

$\mathcal{L} := \Delta - \nabla V \cdot \nabla$: self-adjoint on $L^2(v)$

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  ($\mu = \rho\, v \Rightarrow \mu P_t = P_t \rho\, v$)
  $\Rightarrow \mu P_t \in \mathcal{P}(M)$ (if $P_t 1 \equiv 1$)

Purpose

★ P_t characterizes "$\mathbf{Ric} \geq K$ & $\mathbf{dim} \leq N$"

(curvature-dimension cond.)

★ A bound of $W_2(\mu P_t, \nu P_s)$ characterizes **CD**-cond'n

★ Extensions to metric measure spaces $(X, d, \mathfrak{m})$

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Questions

(1) Another (characteristic) bound of $W_2(\mu P_s, \mu P_t)$

(2) Bounds for other costs than d^2 under **CD**

("Robust" arguments being valid on met. meas. sp.)

Three approaches

Bakry-Émery theory

Gradient estimate of P_t

(Bakry-Émery's **CD**-cond'n)

Stochastic analysis

Coupling method

(2nd variation of d^2)

Estimates of $\mathcal{T}_c(\mu P_t, \nu P_s)$

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Gradient flow interpretation

(Convexity of $\mathbf{Ent}$)

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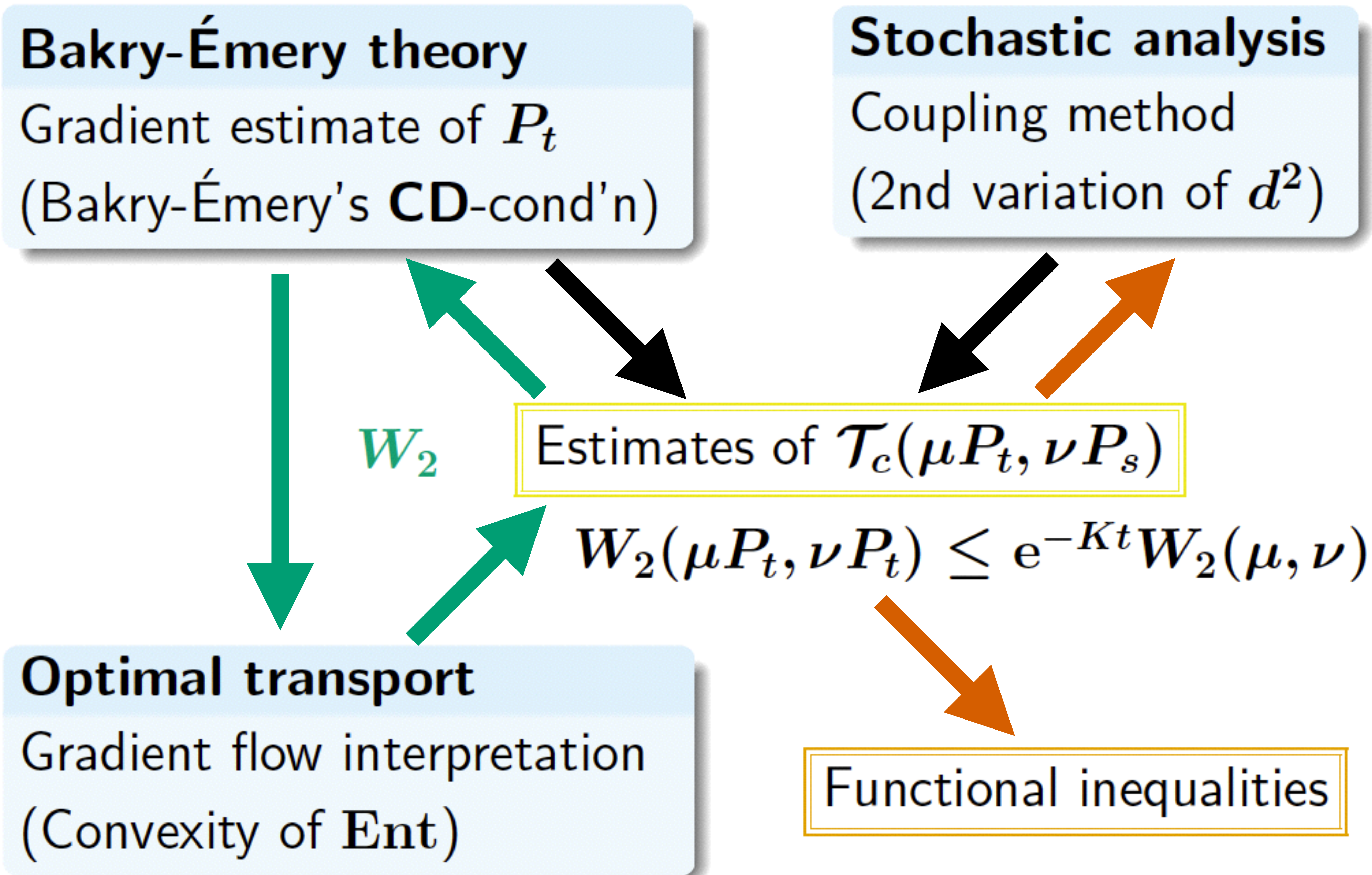

Outline of the talk

1. **Introduction**
2. **Wasserstein distance**
3. **Convexities of** Ent
4. **Bakry-Émery theory**
5. **Self-improvements and their applications**
6. **Stochastic analysis**
7. **Problems**

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5. Self-improvements and their applications

6. Stochastic analysis

7. Problems

Wasserstein distance

(X, d): complete separable geodesic metric space

L^p-Wasserstein distance ($p \in [1, \infty)$)

$$W_p(\mu, \nu) := \mathcal{T}_{d^p}(\mu, \nu)^{1/p} = \inf_{\pi} \|d\|_{L^p(\pi)}$$

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  $\mathcal{P}_p(X) := \{\mu \in \mathcal{P}(X) \mid W_p(\delta_o, \mu) < \infty\}$
- W_p-conv. $\Leftrightarrow$ weak conv. & conv. of p-th moment
- Dual representation (Kantorovich duality)
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  (e.g. sep./compl./geod.)

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Energy f'nal and its density

$(X, d, \mathfrak{m})$: met. meas. sp.

$(\mathfrak{m}$: loc.-finite, $\displaystyle\int \exp\left(-^{\exists}c d(x_0, x)^2\right) \mathfrak{m}(dx) < \infty)$

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$$= \int_X {}^{\exists}|\nabla f|_w^2 \, d\mathfrak{m}$$

$\mathsf{RCD}^*(K, N)$ space

Definition 1 ([Ambrosio, Gigli & Savaré '14])

$(X, d, \mathfrak{m})$: Riemannian $\mathbf{CD}(K, \infty)$ sp. $(K \in \mathbb{R})$

$\overset{\text{def}}{\Leftrightarrow}$ "$\nabla^2 \operatorname{Ent} \geq K$"

& **Ch**: quadratic form ($\Leftrightarrow$ P_t: linear)

$$\operatorname{Ent}(\rho\mathfrak{m}) := \int_X \rho \log \rho \, d\mathfrak{m}$$

$\nabla^2 \operatorname{Ent} \geq K$:

$^\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, $^\exists (\mu_t)_{t\in[0,1]}$: W_2-min. geod. s.t

$$\operatorname{Ent}(\mu_t) \leq (1-t)\operatorname{Ent}(\mu_0) + t\operatorname{Ent}(\mu_1) - \frac{K}{2} t(1-t) W_2(\mu_0, \mu_1)^2$$

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Theorem 1 ([Erbar, K. & Sturm '15])

$(X, d, \mathfrak{m})$: *Riem.* $\mathsf{CD}^*(K,N)$ *sp.* ($K \in \mathbb{R}$, $N > 0$)

$$\Leftrightarrow \text{“}\nabla^2 \mathrm{Ent} - \frac{1}{N} \nabla \mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

& **Ch**: *quadratic form* ($\Leftrightarrow$ P_t: *linear*)

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Example

If $(X,d,\mathfrak{m}) = (M, d_g, \mathrm{e}^{-V}\,\mathrm{vol}_g)$
(n-dim. weighted Riem. mfd),

$\mathsf{(R)CD}^*(K,N)$

$$\Leftrightarrow \mathrm{Ric} + \boldsymbol{\nabla}^2 V - \frac{\boldsymbol{\nabla} V^{\otimes 2}}{N-n} \geq Kg \ \& \ N \geq n$$

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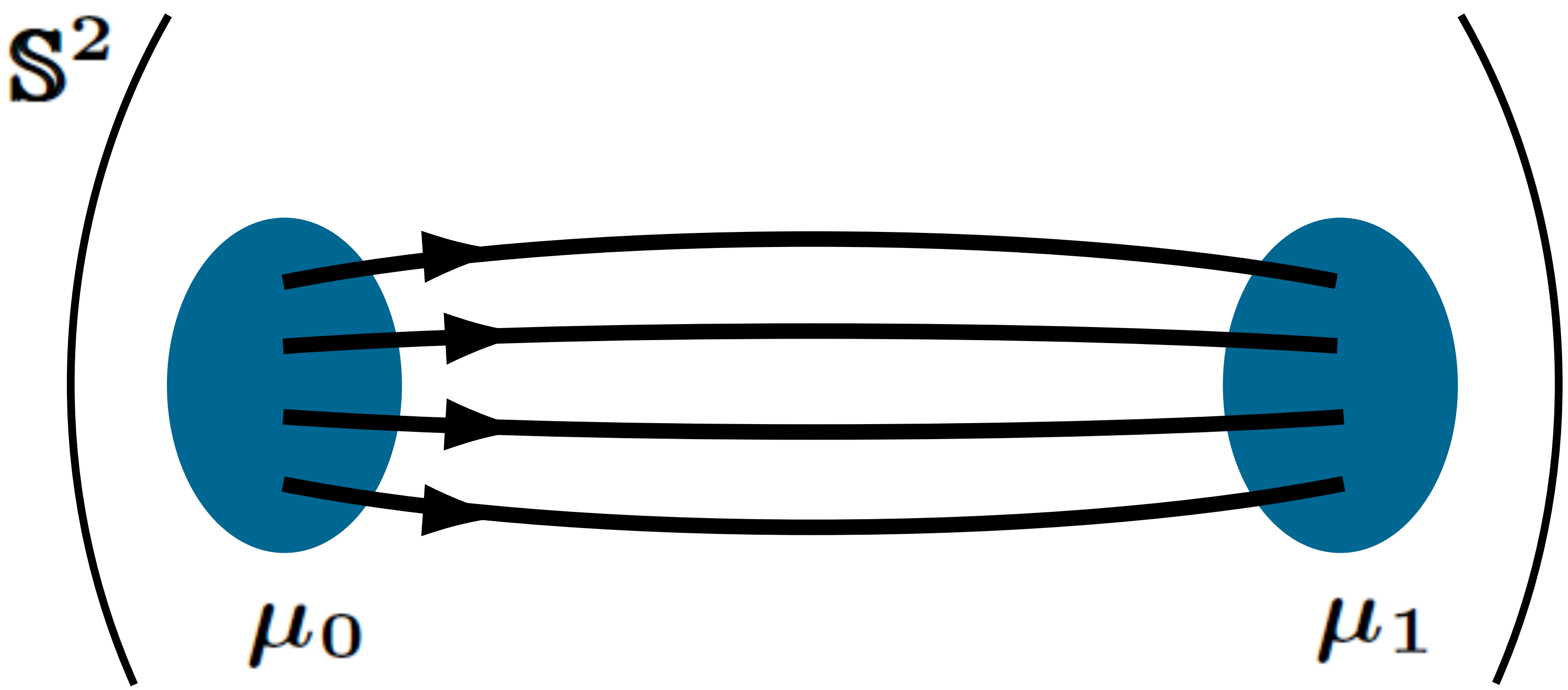

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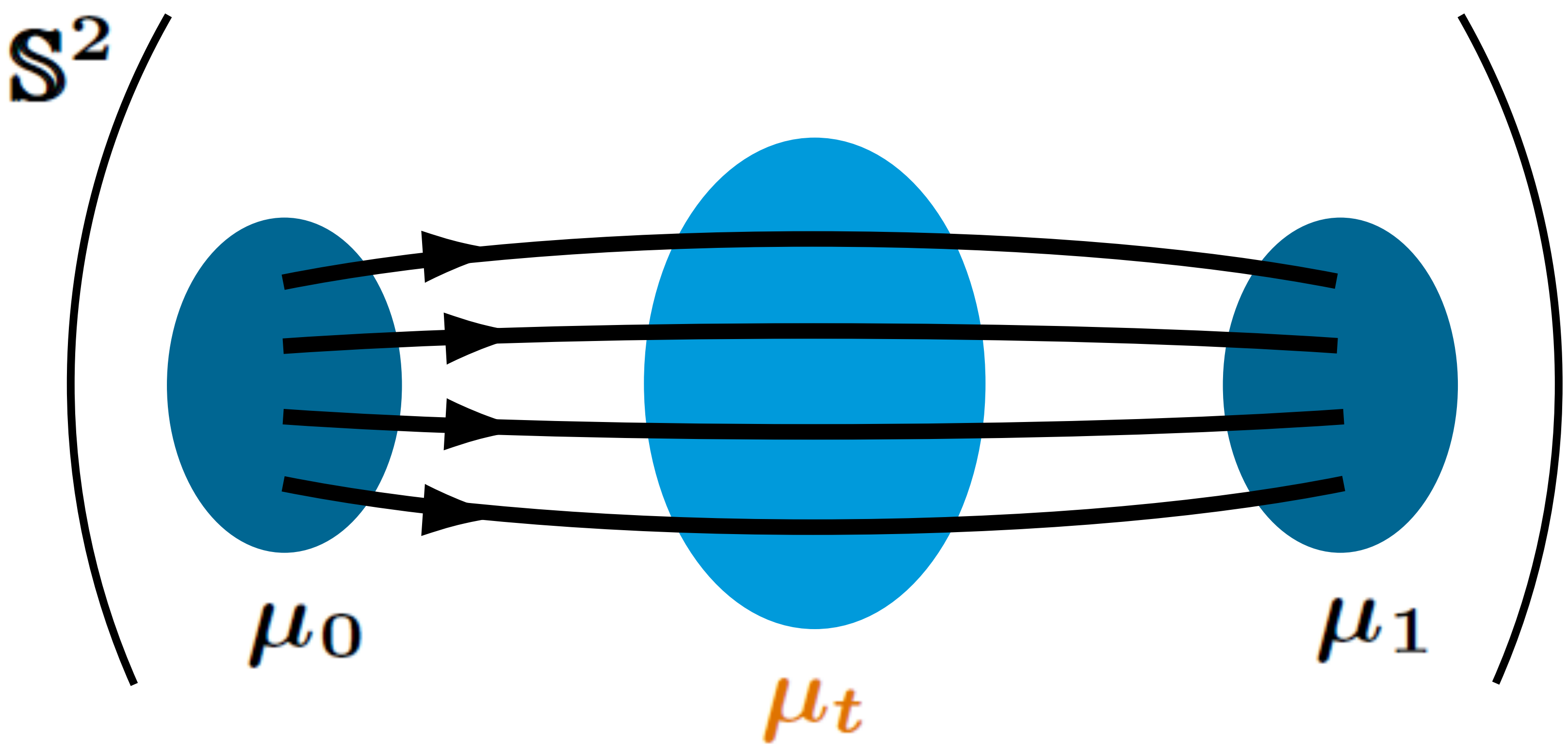

Evolution variational inequality

Formally, $\mu_t = \mu P_t$ solves $\boxed{\dot{\mu}_t = -\nabla \operatorname{Ent}(\mu_t)}$ on $(\mathcal{P}_2(X), W_2)$ (Otto calculus)

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(**RCD**$(K, \infty) \Rightarrow W_2(\mu P_t, \nu P_t) \leq \mathrm{e}^{-Kt} W_2(\mu, \nu)$)

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(K, N)-**E**volution **V**ariational **I**nequality

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- $^\exists(\mu_t)_{t \geq 0}$ for $^\forall \mu_0 \Leftrightarrow (X, d, \mathfrak{m})$: **RCD**$^*(K, N)$
- On **RCD**$^*(K, N)$ sp., μP_t solves (K, N)-**EVI**
- (K, N)-**EVI** $\Rightarrow$ A bound of $W_2(\mu P_t, \nu P_s)$

(K,N)-EVI $\Rightarrow$ W_2-controls

$$\Xi_{s,t} := \mathfrak{s}^2_{K/N}\left(\frac{W_2(\mu P_s, \nu P_t)}{2}\right), \quad \mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}$$

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$\underline{\mathsf{W}_2(K,N)}$ [EKS '15 / K. '15]

$$\Xi_{s,t} \le \mathrm{e}^{-K(s+t)} \Xi_{0,0} + \frac{N}{2} \cdot \frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)} |\sqrt{t} - \sqrt{s}|^2$$

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<u>$\widehat{\mathsf{W}}_2(K,N)$</u> [Bolley, Gentil, Guillin, K.]

$$\Xi_{t,t} \le e^{-2Kt}\Xi_{0,0} - 2N\int_0^t e^{-2K(t-s)} \times \sinh^2\left(\frac{\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s)}{2N}\right) ds$$

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$(0, N)$-EVI $\Rightarrow W_2$-controls (formal)

$$\boxed{\nabla^2 \operatorname{Ent} \geq \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \Rightarrow \mathsf{W}_2(0, N)}$$

$\underline{\mathsf{W}_2(0, N)}$:

$$W_2(\mu P_{t_0}, \nu P_{t_1})^2 \leq W_2(\mu, \nu)^2 + 2N|\sqrt{t_1} - \sqrt{t_0}|^2$$

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$\because)$ $(\sigma_r)_{r \in [0,1]}$: W_2-geod. from $\mu P_{t_0 u}$ to $\nu P_{t_1 u}$,

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$$\le \frac{N}{4}\int_0^1 \frac{\dot t_r^2}{t_r} \, dr$$

$(0, N)$-EVI $\Rightarrow$ W_2-controls (formal)

$\underline{\mathbf{W}_2(0, N)}$:

$$W_2(\mu P_{t_0}, \nu P_{t_1})^2 \le W_2(\mu, \nu)^2 + 2N|\sqrt{t_1} - \sqrt{t_0}|^2$$

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$$\le \frac{N}{4}\int_0^1 \frac{\dot{t}_r^2}{t_r}dr = N(\sqrt{t_1} - \sqrt{t_0})^2$$

$(0, N)$-EVI $\Rightarrow W_2$-controls (formal)

$$\nabla^2 \operatorname{Ent} \geq \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \Rightarrow \widehat{\mathbf{W}}_2'(0, N)$$

$\underline{\widehat{\mathbf{W}}_2'(0, N)}$:

$$\partial_t W_2(\mu P_t, \nu P_t)^2 \leq -\frac{2}{N}(\operatorname{Ent}(\mu P_t) - \operatorname{Ent}(\nu P_t))^2$$

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Bakry-Émery's CD cond'n

Bochner-Weitzenböck formula

On n-dim. weighted Riem. mfd $(M, d_g, \mathrm{e}^{-V}\,\mathrm{vol}_g)$,

$$\Gamma_2(f,f) = (\mathrm{Ric} + \nabla^2 V)(\nabla f, \nabla f) + \|\nabla^2 f\|_{\mathsf{HS}}^2,$$

$$\Gamma_2(f,f) := \frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f\rangle$$

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★ $\mathrm{Ric} + \nabla^2 V \qquad \geq K$

$\Leftrightarrow$ **BE**(K,∞): $\boxed{\Gamma_2(f,f) \geq K|\nabla f|^2}$

(Bakry-Émery's curv.-dim. cond.[Bakry-Émery '85])

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★ $\mathrm{Ric} + \nabla^2 V - \dfrac{\nabla V^{\otimes 2}}{N-n} \geq K$

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(Bakry-Émery's curv.-dim. cond.[Bakry-Émery '85])

$\mathsf{W}_2(K,N) \Leftrightarrow \mathsf{BE}(K,N)$

$\underline{N = \infty}$

$\mathsf{W}_2(K,\infty)$: $$\boxed{W_2(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_2(\mu,\nu)}$$

$\mathsf{BE}(K,\infty)$: $$\boxed{\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f\rangle \ge K|\nabla f|^2}$$

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$\mathsf{G}_2(K,\infty)$: $$\boxed{|\nabla P_t f| \leq \mathrm{e}^{-Kt} P_t(|\nabla f|^2)^{1/2}}$$

$\Updownarrow$ [BÉ '85 / $\cdots$]

$\mathsf{BE}(K,\infty)$: $$\boxed{\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle \geq K|\nabla f|^2}$$

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- Valid even on met. meas. sp.'s $(X, d, \mathfrak{m})$

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- Valid even on met. meas. sp.'s $(X, d, \mathfrak{m})$
- $N < \infty$ [Bakry & Ledoux '06 / K. '15]

$\mathsf{RCD}^*(K,N) \Leftrightarrow \mathsf{BE}(K,N)$

- $\mathsf{RCD}(K,\infty) (\Rightarrow \mathsf{W}_2(K,\infty)) \Rightarrow \mathsf{BE}(K,\infty)$
  [AGS '13–'15 / AG, Mondino & Rajala '15]
- $\mathsf{G}_2(K,\infty)$ "$\Rightarrow$" $\mathsf{RCD}(K,\infty)$ [AGS '15]
- $N < \infty$ [EKS '15 / Ambrosio, Mondino & Savaré]
- $\mathsf{RCD}(K,N) \Rightarrow \widehat{\mathsf{W}}_2(K,N) \Rightarrow \mathsf{BE}(K,N)$
  [Gentil '15 / Bolley, Gentil, Guillin & K.]

$\mathsf{RCD}^*(K, N) \Leftrightarrow \mathsf{BE}(K, N)$

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$\mathsf{RCD}^*(K,N) \Leftrightarrow \mathsf{BE}(K,N)$

- $\mathsf{RCD}(K,\infty)(\Rightarrow \mathsf{W}_2(K,\infty)) \Rightarrow \mathsf{BE}(K,\infty)$
  [AGS '13–'15 / AG, Mondino & Rajala '15]
- $\mathsf{G}_2(K,\infty)$ "$\Rightarrow$" $\mathsf{RCD}(K,\infty)$ [AGS '15]
- $N < \infty$ [EKS '15 / Ambrosio, Mondino & Savaré]
- $\mathsf{RCD}(K,N) \Rightarrow \widehat{\mathsf{W}}_2(K,N) \Rightarrow \mathsf{BE}(K,N)$
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$\mathsf{RCD}^*(K,N) \Leftrightarrow \mathsf{BE}(K,N)$
$\Rightarrow$ Several geometric/analytic applications

Outline of pf: $\mathsf{W}_2(0, N) \Rightarrow \mathsf{G}_2(0, N)$

W_2: $W_2(\mu P_s, \nu P_t)^2 \le W_2(\mu, \nu)^2 + 2N|\sqrt{t} - \sqrt{s}|^2$

G_2: $\dfrac{2t}{N}(\mathcal{L}P_t f)^2 + |\nabla P_t f|^2 \le P_t(|\nabla f|^2)$

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$\mathsf{G}_2 \Leftrightarrow {}^\forall a \in \mathbb{R}$,

$$a\sqrt{\frac{2t}{N}}\mathcal{L} P_t f + |\nabla P_t f| \le \sqrt{P_t(|\nabla f|^2)(a^2 + 1)}$$

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For π: coupling of $\delta_x P_t$ and $\delta_y P_s$,

$$P_t f(x) - P_s f(y) = \int (f(z) - f(w))\pi(dzdw)$$

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Take $t - s = a\sqrt{\frac{2t}{N}}\,d(x,y)$, $\times \frac{1}{d(x,y)}$ & $y \to x$:

$$\frac{\text{(LHS)}}{d(x,y)} \to a\sqrt{\frac{2t}{N}}\mathcal{L}P_t f(x) + |\nabla P_t f|(x)$$

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$$\frac{\text{(RHS)}}{d(x,y)} = \frac{1}{d(x,y)}\int \frac{f(z)-f(w)}{d(z,w)} d(z,w)\pi(dzdw)$$

$$\le \sqrt{P_t(|\nabla f|^2)(x)}\, W_2(\delta_x P_t, \delta_y P_s)/d(x,y) + o(1)$$

$$\le \sqrt{P_t(|\nabla f|^2)(x)(1+a^2)} + o(1)$$

Outline of pf: $\mathbf{G}_2(0, N) \Rightarrow \mathbf{W}_2(0, N)$

Recall the Kantorovich duality for $\boldsymbol{W}_2$:

$$\frac{1}{2}W_2(\mu,\nu)^2 = \sup_f \left[\int Q_1 f \, d\mu - \int f \, d\nu \right],$$

$$Q_s f(x) := \inf_y \left[f(y) + \frac{d(x,y)^2}{2s} \right]$$

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★ $\partial_s Q_s f + \frac{1}{2}|\nabla Q_s f|^2 = 0$

(Hamilton-Jacobi eq. for $Q_s f$)

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For simplicity, suppose $\mu = \delta_x$, $\nu = \delta_y$

$$W_2(\delta_x P_s, \delta_y P_t)^2 = 2\sup_f \left[P_t Q_1 f(y) - P_s f(x)\right]$$

(Kantorovich duality)

Idea: give an upper bound of $[\cdots]$ being uniform in f

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$\gamma : [0,1] \to X$: geod. from x to y

$\alpha(r) := ((1-r)\sqrt{s} + r\sqrt{t})^2$

$$\begin{aligned}\Rightarrow 2P_t Q_1 f(y) - P_s f(x) \\ &= 2\left(P_{\alpha(1)} Q_1 f(\gamma_1) - P_{\alpha(0)} Q_0 f(\gamma_0)\right)\end{aligned}$$

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\Rightarrow 2P_t Q_1 f(y) - P_s f(x) \\
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&= 2\int_0^1 \underline{\partial_r \left(P_{\alpha(r)} Q_r f(\gamma_r)\right)} dr
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$$\alpha(r) := ((1-r)\sqrt{s} + r\sqrt{t})^2$$

$$\begin{aligned} &2\partial_r \left(P_{\alpha(r)} Q_r f(\gamma_r)\right) \\ &\le 4\sqrt{\alpha(r)}(\sqrt{t}-\sqrt{s})\mathcal{L}P_{\alpha(r)} Q_r f(\gamma_r) \\ &\quad + 2|\dot{\gamma}_r||\nabla P_{\alpha(r)} Q_r f|(\gamma_r) - P_{\alpha(r)}(|\nabla Q_r f|^2)(\gamma_r) \end{aligned}$$

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&\le 2\sqrt{P_t(|\nabla Q_r f|^2)\left(2N(\sqrt{t}-\sqrt{s})^2 + |\dot\gamma_r|^2\right)}\\
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&\quad \le 2N(\sqrt{t}-\sqrt{s})^2 + d(x,y)^2
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cf. Another proof when $K = 0$ or $N = \infty$

[Bakry, Gentil & Ledoux '15]

Outline of pf: $\widehat{\mathbf{W}}_2'(0, N) \Rightarrow \mathbf{BE}(0, N)$

$\widehat{\mathbf{W}}_2'(0, N)$:

$$\partial_t W_2(\mu P_t, \nu P_t)^2 \le -\frac{2}{N}(\mathrm{Ent}(\mu P_t) - \mathrm{Ent}(\nu P_t))^2$$

$\mathbf{BE}(0, N)$: $\frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle \ge \frac{1}{N}(\mathcal{L} f)^2$

Remarks

- No use of $\mathbf{G}_2(K, N)$:

  $$\widehat{\mathbf{W}}_2'(K, N) \Rightarrow \mathbf{BE}(K, N) \text{ directly}$$

- On $\mathbb{R}^m$,

  $$\widehat{\mathbf{W}}_2'(K, N) \Leftrightarrow \mathbf{W}_2(K, \infty)$$

  for Dirac meas.'s

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$$\boxed{\boldsymbol{\rho}_r = \boldsymbol{\rho}(1 - r\boldsymbol{\mathcal{L}}^{\rho} \boldsymbol{f}) + \boldsymbol{o}(\boldsymbol{r})} \qquad (\boldsymbol{r} \to 0),$$

$$\boldsymbol{\mathcal{L}}^{\rho} := \boldsymbol{\mathcal{L}} + \nabla \log \boldsymbol{\rho} \cdot \nabla \qquad (\text{symm. for } \boldsymbol{\mu})$$

Outline of pf: $\widehat{\mathsf{W}}'_2(0, N) \Rightarrow \mathsf{BE}(0, N)$

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$$\boxed{\rho_r = \rho(1 - r\mathcal{L}^\rho f) + o(r)} \qquad (r \to 0),$$

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Otto calc.

$$\boxed{\text{“}\dot{\nu}_0 = \nabla f \text{ in } T_{\nu_0}\mathcal{P}(X)\text{”}} \text{ for } (\nu_r)_r \subset \mathcal{P}(X)$$

$$\overset{\text{def}}{\Leftrightarrow} \quad \forall\varphi, \boxed{\partial_r \int \varphi \, d\nu_r \bigg|_{r=0} = \int \langle \nabla f, \nabla\varphi \rangle d\nu_0}$$

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$\Rightarrow \dot{\mu}_0 = \nabla f$

Outline of pf: $\widehat{\mathsf{W}}'_2(0, N) \Rightarrow \mathsf{BE}(0, N)$

$$\mu_r = \rho_r \mathfrak{m},\ \rho_r = \rho(1 - r\mathcal{L}^\rho f) + o(r)$$

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$\mu_r = \rho_r \mathfrak{m}$, $\rho_r = \rho(1 - r\mathcal{L}^\rho f) + o(r)$

$$\begin{aligned} W_2(\mu_0 P_t, \mu_r P_t)^2 &\le W_2(\mu_0, \mu_r)^2 \\ &\quad - \frac{2}{N} \int_0^t (\mathrm{Ent}(\mu_0 P_s) - \mathrm{Ent}(\mu_r P_s))^2 \quad ds \end{aligned}$$

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$$\frac{(1)}{r^2} \ge \int \left(-P_t(|\nabla f|^2) + 2\langle \nabla f, \nabla P_t f\rangle\right) \rho \, d\mathfrak{m} + o(1)$$

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$$\frac{(1)}{r^2} \ge \int \left(-P_t(|\nabla f|^2) + 2\langle \nabla f, \nabla P_t f\rangle\right) \rho\, d\mathfrak{m} + o(1)$$

$$\frac{(2)}{r^2} \le \int |\nabla f|^2 \rho\, d\mathfrak{m} + o(1)$$

$$\frac{(3)}{r^2} \ge \left(\int P_s(\rho \mathcal{L}^\rho f) \log P_s \rho\, d\mathfrak{m}\right)^2 + o(1)$$

Outline of pf: $\widehat{\mathsf{W}'_2}(0,N) \Rightarrow \mathsf{BE}(0,N)$

$$\underbrace{W_2(\mu_0 P_t, \mu_r P_t)^2}_{(1)} \le \underbrace{W_2(\mu_0, \mu_r)^2}_{(2)} - \frac{2}{N}\int_0^t \underbrace{(\mathrm{Ent}(\mu_0 P_s) - \mathrm{Ent}(\mu_r P_s))^2}_{(3)} ds$$

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- $r \to 0$, $\times \dfrac{1}{t}$ & $t \to 0$

Outline of pf: $\widehat{\mathsf{W}}_2'(0, N) \Rightarrow \mathsf{BE}(0, N)$

- $r \to 0$, $\times \frac{1}{t}$ & $t \to 0$

$$\int \left(-P_t(|\nabla f|^2) + 2\langle \nabla f, \nabla P_t f\rangle - |\nabla f|^2 \right) \rho \, d\mathfrak{m}$$

$$\leq -\frac{2}{N} \int_0^t \left(\int P_s(\rho \mathcal{L}^\rho f) \log P_s \rho \, d\mathfrak{m} \right)^2 ds$$

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$$\Downarrow$$

$$-2\int \Gamma_2(f)\rho\, d\mathfrak{m} \leq -\frac{2}{N}\left(\int \log \rho \mathcal{L}^\rho f\, \rho\, d\mathfrak{m}\right)^2$$

Outline of pf: $\widehat{\mathsf{W}}_2'(0,N) \Rightarrow \mathsf{BE}(0,N)$

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\end{aligned}
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$\therefore\ \rho\mathfrak{m} \to \delta_x \Rightarrow \mathsf{BE}(0,N)$

Summary

- RCD$^*(K,N)$: $\nabla^2 \operatorname{Ent} - \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \geq K$
  & P_t: linear ($\leftarrow$ Cheeger energy)
- RCD$^*(K,N) \Leftrightarrow {}^{\exists}$sol. to EVI$(K,N)$
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★ $\mathbf{G}_1(K,\infty)$: $|\nabla P_t f|_w \le \mathrm{e}^{-Kt} P_t(|\nabla f|_w)$

$\Rightarrow$ (P_t-)(reversed) log Sobolev ineq.,

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Coupling by parallel transport [Sturm '15]

$^\forall x_0, x_1 \in X$ $^\exists (Z_t^{(0)}, Z_t^{(1)})$: coupling of BMs on X s.t.

- $(Z_0^{(0)}, Z_0^{(1)}) = (x_0, x_1)$
- $\mathrm{e}^{Kt} d(Z_t^{(0)}, Z_t^{(1)}) \searrow$ a.s.

Total variation bound

Theorem 2 ([K.] cf. [K. & Sturm '13])

Suppose **RCD**(K, ∞)*. Then*

$$\frac{1}{2}\|\mu P_t - \nu P_t\|_{var} \le \mathcal{T}_{\varphi_t(d)}(\mu, \nu),$$

where $\varphi_t(r) := 2\Phi\left(\frac{r}{2\sqrt{\sigma(t)}}\right) - 1,$

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$\because$ Use P_t-reversed Gaussian isoperimetric ineq.

$$I(P_t f)^2 - P_t(I(f))^2 \ge \sigma(t) |\nabla P_t f|^2 \ (I := \Phi' \circ \Phi^{-1})$$

L^p/L^q-extension when $N < \infty$

Theorem 3 (Dimensional self-improvement [K.])

Suppose **RCD**$^*(K, N)$. *For* $p \geq 2$ & $q = p/(p-1)$,

$$\Gamma_2(f,f) - K|\nabla f|_w^2 - \frac{1}{N+p-2}(\mathcal{L}f)^2 \geq \frac{2-q}{4}\frac{|\nabla|\nabla f|_w^2|_w^2}{|\nabla f|_w^2}$$

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L^p/L^q-extension when $N < \infty$

Theorem 4 (A reverse isop. ineq. [K.])

Suppose **RCD***(K, N), $K > 0$. *Then*

$$
\begin{aligned}
(\mathrm{e}^{2Kt} - 1)|\nabla P_t f|^2 & \\
\le I_N(P_t f)^2 &- P_t({I_N}^{N/(N-1)}(f))^{2(N-1)/N} \\
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\end{aligned}
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$I_N := \Phi_N' \circ \Phi_N^{-1}$, $\Phi_N(r) := \overline{\mathrm{vol}}_{K,N}(B_r(o))$,
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Application of $\mathsf{W}_2(0, N)$: $\mathcal{W}$-entropy

- $\boxed{\mathsf{W}_2(0, N) \Rightarrow \mathcal{W}(t) \searrow \text{[K.]}}$ (cf. [Topping '09])

$$\boxed{\mathcal{W}(t) := tI(\mu P_t) - \mathrm{Ent}(\mu P_t) - \frac{N}{2}\log t + c}$$

($\mathcal{W}$-entropy)

$$\left(I(\rho \mathfrak{m}) := \int \frac{|\nabla \rho|_w^2}{\rho}\, d\mathfrak{m}\text{: Fisher information} \right)$$

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[. . . / X.-D. Li '14]

Theorem 5 ([K. & X.-D. Li]; in progress)

On $\mathsf{RCD}^*(0,N)$ *sp.,* $\mathcal{W}'(t) = 0$

$\Rightarrow X \simeq (0,N)$*-cone over* $\mathsf{RCD}^*(0,N-1)$ *sp.*

Coupling by parallel transport

On n-dim. Riem. mfd M,

$$\delta_x P_t \leftrightarrow \text{sol. to SDE: } \begin{cases} dZ_t = dB_t - \nabla V(Z_t)dt, \\ Z_0 = x \end{cases}$$

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Extensions

- The last argument $\Rightarrow \mathrm{e}^{Kt} d({}^{\exists}Z_t^{(0)}, {}^{\exists}Z_t^{(1)}) \searrow$
- A similar argument works also on Riem. mfds (2nd deriv. of $d^2 \leftrightarrow \mathrm{Ric}$ / non-symm. $\mathcal{L}$)

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On a weighted Riem. mfd, for $p \geq 2$,

$$\mathcal{T}_{\mathfrak{s}^p_{K^*}(d/2)}(\mu P_t, \nu P_s)^{2/p} \leq \mathrm{e}^{-\theta}\mathcal{T}_{\mathfrak{s}^p_{K^*}(d/2)}(\mu, \nu)^{2/p} + \frac{\hat{N}(1-\mathrm{e}^{\theta})}{2\theta}(\sqrt{t}-\sqrt{s})^2,$$

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2. Wasserstein distance

3. Convexities of Ent

4. Bakry-Émery theory

5. Self-improvements and their applications

6. Stochastic analysis

7. **Problems**

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- Semigroup approach to Lévy-Gromov isop. ineq.
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