Analysis of Heat Distributions on Riemannian Metric Measure Spaces with a Lower Ricci Curvature Bound

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Fractal Geometry and Related Areas (Mar. 21–22, 2016)

1. Introduction & overview

Analysis on metric measure spaces

$(X, d, \mathfrak{m})$: Polish metric measure space

$\mathfrak{m}$: σ-finite, loc. finite,

d: geodesic distance

$$\left[\begin{array}{l} {}^{\forall}x_0, x_1 \in X,\ {}^{\exists}\gamma : [0,1] \to X \text{ s.t.} \\ \gamma_i = x_i \ (i = 0, 1) \\ \& \ d(\gamma(s), \gamma(t)) = |s - t| d(x_0, x_1) \end{array}\right]$$

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$$\boxed{\text{gradients/energy functionals/diffusions/}\cdots},$$

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★ Optimal transport works as a fundamental tool

What's optimal transport?

- Bring a mass $\mu \in \mathcal{P}(X)$ to $\nu \in \mathcal{P}(X)$
- $c(x, y)$: cost to bring a unit mass from x to y

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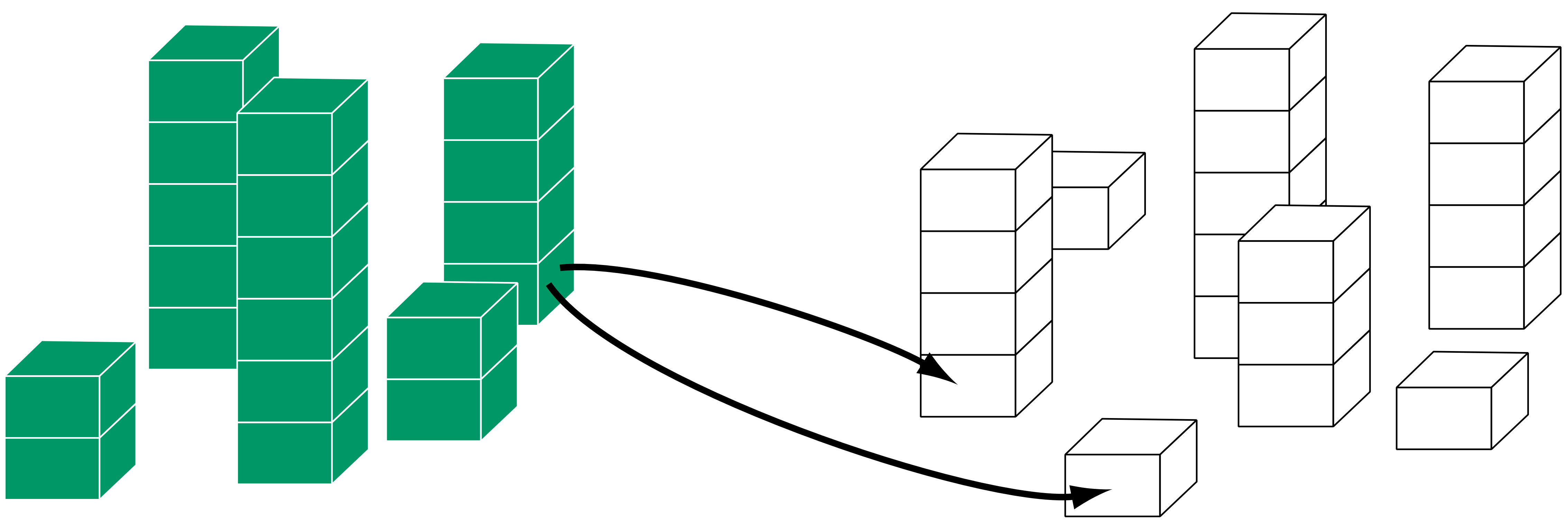

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Optimal transportation cost

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \int_{X \times X} c(x, y)\, \pi(dxdy)$$

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X^2) \,\middle|\, \begin{array}{l} \pi(A \times X) = \mu(A), \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

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- Fits well with gradient(s) in a Sobolev sense
- "$\mathrm{Ric} \geq K$" on $(X, d, \mathfrak{m})$ in terms of W_2
- Heat flow = a gradient flow in $(\mathcal{P}(X), W_2)$

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Q.

Modification/Generalization of these conditions?

Outline of the talk

1. **Introduction**

2. **Basics of optimal transport**

3. **Heat flow**

4. **Applications**

5. **Extensions to other framework**

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Wasserstein distance

L^p-Wasserstein distance ($p \in [1, \infty)$)

$$W_p(\mu, \nu) := \mathcal{T}_{d^p}(\mu, \nu)^{1/p} \left(= \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \right)$$

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- Dual representation (Kantorovich duality)
- Property of $(X, d) \Rightarrow$ the same for $(\mathcal{P}(X), W_p)$ (e.g. separability/completeness ($p < \infty$))

Superposition principle ($p < \infty$)

$$\mathcal{P}_p(X) := \{\mu \in \mathcal{P}(X) \mid W_p(\delta_o, \mu) < \infty\}$$

$\mathrm{Geo}(X)$: sp. of const. speed geod.'s on X

- ${}^\forall \mu_0, \mu_1 \in \mathcal{P}_p(X)$, ${}^\exists (\mu_r)_{r \in [0,1]}$: W_p-min. geod. (i.e. W_p is geodesic dist. on $\mathcal{P}_p(X)$)

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- ${}^{\exists}\Gamma \in \mathcal{P}(\mathrm{Geo}(X))$ s.t.

$$\mu_t(A) = \int_{\mathrm{Geo}(X)} 1_A(\gamma(t))\Gamma(d\gamma),$$

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Curvature-Dimension conditions

[von Renesse & Sturm '05]

For X: cpl Riem. mfds with $\partial X = \emptyset$,

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$\Rightarrow$ $\mathbf{CD}(K, \infty)$ cond. [Sturm '06 / Lott & Villani '09]

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$\underline{\mathrm{Ric} \geq K \ \& \ \dim \leq N}$ $(N < \infty)$

- **CD**(K, N): [Sturm '06] via Rényi entropy
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$$\text{“}\nabla^2 \mathrm{Ent} - \frac{1}{N} \nabla \mathrm{Ent}^{\otimes 2} \geq K\text{”}$$

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$$\bigstar \; |f(\gamma_1) - f(\gamma_0)| \le \int_0^1 |\nabla f|_w(\gamma_s)|\dot{\gamma}_s| \, ds$$

for a.e. trajectories $(\gamma_s)_{s\in[0,1]}$ of "nice" transports $\Gamma \in \mathcal{P}(C([0,1] \to X))$

[Ambrosio, Gigli & Savaré '13]

$\mathsf{RCD}^*(K, N)$ space

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(Basic) properties

- "$\partial_t(\mu P_t) = -\nabla \operatorname{Ent}(\mu P_t)$" on $(\mathcal{P}_2(X), W_2)$
- **Ch**: str. local quasi-reg. Dirichlet form admitting carré du champ ($\rightsquigarrow$ Brownian motion $(B(t), \mathbb{P}_x)$)
- Bakry-Émery's curv.-dim. cond. $\mathbf{BE}(K,N)$
  " $\frac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla \Delta f\rangle_w \geq K|\nabla f|_w^2 + \frac{1}{N}(\Delta f)^2$ "

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$\mathsf{RCD}^*(K,N) \Rightarrow \mathsf{BE}(K,N)$

$\underline{N = \infty}$ [AGS '13–'15 / AG, Mondino & Rajala '15]

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Notation

N: $\boldsymbol{N} < \infty$ is essential

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- (Sharp) Bonnet-Myers diameter bound
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  [K. & X.-D. Li, in progress]

1. Introduction
2. Basics of optimal transport
3. Heat flow
4. Applications
5. **Extensions to other framework**

Extensions to other class of sp.'s

- Alexandrov sp.'s $\subset$ **RCD** sp.'s [Petrunin '11]
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