Monotonicity and rigidity of the $\mathcal{W}$-entropy on RCD$^*(0, N)$ spaces

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Ongoing joint work with X.-D. Li

Geometric Analysis on Riemannian and Metric spaces
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1. Introduction

Perelman's $\mathcal{W}$-entropy

(M, g): m-dim. cpt. Riem. mfd, $\tau > 0$,

$f \in C^\infty(M)$, $\displaystyle\int_M \frac{e^{-f}}{(4\pi\tau)^{m/2}}\, d\,\mathrm{vol} = 1$

$$\mathcal{W}(g, f, \tau)$$

$$:= \int_M \left[\tau(R + |\nabla f|^2) + f - m\right] \frac{e^{-f}}{(4\pi\tau)^{m/2}}\, d\,\mathrm{vol}$$

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$\partial_t g = -2\,\mathrm{Ric}$, $\partial_t f = -\Delta f + |\nabla f|^2 - R + \dfrac{m}{2\tau}$

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$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g, f, \tau) \geq 0$$

Entropy formula

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 2\int_M \tau \left| \mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}}\,\mathrm{d}\,\mathrm{vol}$$

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- $\partial_\tau \mathrm{vol} = R\,\mathrm{vol} \Rightarrow \partial_\tau (u\,\mathrm{vol}) = \Delta(u\,\mathrm{vol})$

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[L. Ni '04]

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⇝ Extension to weighted Riem. mfds [X.-D. Li '12]

Purpose

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Can one extend the monotonicity/rigidity of $\boldsymbol{\mathcal{W}}$ on metric measure spaces with "$\mathbf{Ric} \geq \mathbf{0}$ & $\mathbf{dim} \leq \boldsymbol{N}$" ($\mathbf{RCD}^e(0, N)$ spaces)?

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  $\leftsquigarrow$ optimal transport approach
- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof

4.1 Monotonicity

4.2 Rigidity

4.3 Additional remarks

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($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

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Definition 1

$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form ($\Leftrightarrow$ P_t: linear)

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$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form ($\Leftrightarrow P_t$: linear)

$\Rightarrow {}^{\exists}\langle \nabla \cdot, \nabla \cdot \rangle_w$ bilinear s.t. $\langle \nabla f, \nabla f \rangle_w = |\nabla f|_w^2$

Entropic curvature-dimension cond.

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) \;\middle|\; \int_X d(x_0, \cdot)^2 \, \mathrm{d}\mu < \infty \right\}$$

$$W_2(\mu, \nu) := \inf_\pi \left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l} \pi(A \times X) = \mu(A) \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

★ $(\mathcal{P}_2(X), W_2)$: Polish geod. met. meas. sp.

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$^\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, $^\exists(\mu_t)_{t\in[0,1]}$: W_2-min. geod. s.t.

$$\mathrm{Ent}(\mu_t) \le (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1)$$

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★ **RCD**$^e(K,N) \overset{\text{def}}{\Leftrightarrow}$ **CD**$^e(K,N)$ & infin. Hilb.

Examples

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$$\mathsf{RCD}^e(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

- (Pointed) measured GH lim. of $\mathsf{RCD}^e(K, N)$ sp.'s
  [Gigli, Mondino & Savaré '15]
- m-dim. Alexandrov sp. of curv. $\geq k$
  $\Rightarrow \mathsf{RCD}^e((m-1)k, m)$ sp.
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Heat flow

Properties of the heat semigr. P_t under $\mathbf{RCD}^e(K, N)$

- $\|P_t f\|_{L^1(\mathfrak{m})} = \|f\|_{L^1(\mathfrak{m})}$ for $f \geq 0$
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  in the (K, N)-evolution variational inequality sense
- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
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  $$\text{“}\dot{\mu}_t = -\nabla \operatorname{Ent}(\mu_t)\text{”} \text{ on } (\mathcal{P}_2(X), W_2)$$

  in the (K, N)-evolution variational inequality sense
- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
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Heat flow

Properties of the heat semigr. P_t under $\mathbf{RCD}^e(K,N)$

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Characterizations of RCD cond.

On $\mathbf{RCD}^e(0,N)$ sp. ($K = 0$ for simplicity),

- ${}^\forall\mu_0$, ${}^\exists(\mu_t)_{t\ge 0}$ sol. to $(0,N)$-EVI
- $W_2(P_s\mu, P_t\nu)^2 \le W_2(\mu,\nu)^2 + 2N(\sqrt{t}-\sqrt{s})^2$

  (Space-time W_2-control)
- $|\nabla P_t f|_w^2 + \dfrac{2t}{N}|\Delta P_t f|^2 \le P_t(|\nabla f|^2)$

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- "$\dfrac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle_w \ge \dfrac{1}{N}|\Delta f|^2$"

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Each of them $\Leftrightarrow$ $\mathbf{RCD}^e(0,N)$ under "regularity ass."

History

- **CD** conditions ($N = \infty$ / $N < \infty$ by Rényi ent.) on met. meas. sp.'s [Sturm '06 / Lott & Villani '09]
- Reduced **CD** cond. [Bacher & Sturm '10]
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Under infin. Hilb., $\mathbf{CD}^e \Leftrightarrow$ reduced **CD** [EKS]

Gradient flow of Ent

Otto calculus (When X: Riem. mfd)

$$T_\mu \mathcal{P}_2(X) \text{``}=\text{''} \overline{\{\nabla\varphi \mid \varphi \in C_0^\infty(X)\}}^{L^2(\mu)},$$

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$$\dot{\mu}_t = \nabla\varphi \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \int_X f \, \mathrm{d}\mu_t = \int_X \langle \nabla f, \nabla\varphi \rangle \, \mathrm{d}\mu_t$$

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(Fisher information)

Geometric properties of $\mathsf{RCD}^e(0,N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $\Delta d(x_0,\cdot)^2 \le 2N$ in the distributional sense

- Splitting thm [Gigli]: $N > 1$,

  $\mathbb{R} \hookrightarrow X$ isometry

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- Volume rigidity [Gigli & De Philippis]: $N > 2$,

$$^\exists x_0,\ ^\forall r, R > 0,\ \mathfrak{m}(B_R(x_0)) = \left(\frac{R}{r}\right)^N \mathfrak{m}(B_r(x_0))$$

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$$-\Delta \log P_t f = \frac{|\nabla P_t f|^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \leq \frac{N}{2t}$$

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  $^\forall \varepsilon > 0$, $^\exists C_\varepsilon > 0$ s.t.

$$\frac{C_\varepsilon^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x,y)^2}{(4-\varepsilon)t}\right) \leq p_t(x,y) \leq \frac{C_\varepsilon}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x,y)^2}{(4+\varepsilon)t}\right)$$

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$\Rightarrow$ $\boxed{\lim_{t\downarrow 0} 4t \log p_t(x,y) = -d(x,y)^2}$ cpt. unif in y

1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof
4.1 Monotonicity
4.2 Rigidity
4.3 Additional remarks

$\mathcal{W}$-entropy

$$\mu = \rho\,\mathfrak{m} \in \mathcal{P}(X),\ \rho =: \frac{e^{-f}}{(4\pi t)^{N/2}} \qquad (\tau \rightsquigarrow t)$$

$$\mathcal{W}(\mu, t) := \int_X \left[t|\nabla f|_w^2 + f - N\right] \rho\, \mathrm{d}\mathfrak{m}$$

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Main thm

Theorem 3 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: $\mathsf{RCD}^e(0, N)$, $N \geq 2$, $\mu_t := P_t\mu$

(1) $\mathcal{W}(\mu_t, t) \searrow$ *in* $t \in (0, \infty)$

(2) *Suppose* $^\exists t_* > 0$ *s.t.*

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Definition 4 ($(0, N)$-cone)

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$\overset{\text{def}}{\Longleftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
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- Theorem 1 (1) is known when X: cpt.
  [Jiang & Zhang '16]
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- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
- $(0, N)$-cone of Y is a (smooth) Riem. mfd
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1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof

4.1 Monotonicity

4.2 Rigidity

4.3 Additional remarks

4.1. Monotonicity

Optimal transport approach on Ricci flow

$\partial_\tau g_\tau = 2\,\mathrm{Ric}$, μ_τ: $\partial_\tau \mu_\tau = \Delta_\tau \mu_\tau$

$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r} \left(|\dot{\gamma}_r|_r^2 + R(\gamma_r) \right) \mathrm{d}r \right]$$

$\mathcal{T}_{L_s^t}(\mu,\nu) := \inf_\pi \int_{X \times X} L_s^t \, \mathrm{d}\pi$: L-opt. trans. cost

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

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Toward the time-inhomogeneous case

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$$\Rightarrow 2(\sqrt{t} - \sqrt{s}) \int_s^t \sqrt{r}\, |\dot{\gamma}_r|^2 \, \mathrm{d}r = \int_0^1 |\dot{\gamma}_u^*|^2 \, \mathrm{d}u$$

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$$\Downarrow$$

$$\Xi_1^{\tau}(t) \searrow$$

$$\Updownarrow$$

$$\begin{aligned}W_2(\mu_t,\mu_{\tau t})^2 \le W_2(\mu_s,\mu_{\tau s})^2 \\ +2N(\sqrt{\tau(t-s)}-\sqrt{t-s})^2\end{aligned}$$

Derivation from RCD$^e(0, N)$

$$W_2(P_{t'}\mu', P_{s'}\nu')^2 \leq W_2(\mu', \nu')^2 + 2N(\sqrt{t'} - \sqrt{s'})^2$$

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$\mu' = \mu_s$, $\nu' = \mu_{s+s^\alpha\delta}$

$t' = t - s$, $s' = (t^\alpha - s^\alpha)\delta + t - s$

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4.2. Rigidity

Equality in Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ $(\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m})$

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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$\Rightarrow$ Volume rigidity [Gigli & De Philippis] is applicable $\square$

4.3. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0, \cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0, x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0, x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0, x)^2}{4t}\right)$$

$\because$ Compute I for "Gaussian kernel" in two ways & RHS enjoys the energy dissipation identity for Ent □

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Proposition 1

Suppose $\Delta d(x_0,\cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0,x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0,x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0,x)^2}{4t}\right)$$

In particular, X *is non-compact*

$\because$ Compute I for "Gaussian kernel" in two ways &
RHS enjoys the energy dissipation identity for Ent □

Initial data

Lemma 2

Suppose $I(\mu_t) = \frac{N}{2t}$. *Then* $I(p_t^x \mathfrak{m}) = \frac{N}{2t}$ μ*-a.e.* x.

$\because \mu \mapsto I(\mu)$ convex

$$\Rightarrow \frac{N}{2t} = I(\mu_t) \text{``}\leq\text{''} \int_X I(p_t^x)\mu(\mathrm{d}x) \overset{\text{Li-Yau}}{\leq} \frac{N}{2t} \qquad \square$$

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Lemma 3

Suppose $I(\mu_t) = \dfrac{N}{2t}$*. Then* μ *is Dirac.*

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$$\because\ \frac{|\nabla p_t^x|}{p_t^x} = \frac{|\nabla p_t^y|}{p_t^y} \Rightarrow d(x,\cdot) = d(y,\cdot) \qquad \square$$

Heat flow is a W_2-geodesic

Proposition 4

Suppose $\Delta d(x_0,\cdot)^2 = 2N$ *and* $\mu_t = P_t\delta_{x_0}$.
$\Rightarrow (\mu_{t^2/(2N)})_{t\ge 0}$: W_2*-min. geod.*

$\because$

- $\dfrac{N}{2t} = I(\mu_t) = \dfrac{1}{4t^2}\displaystyle\int_X d(x_0,x)^2\,\mu_t(\mathrm{d}x)$
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$\Rightarrow \mu_t^* := \mu_{t^2/(2N)}$ satisfies
$W_2(\mu_0^*, \mu_t^*) = t$ & $\|\dot{\mu}_t^*\| = 1$ $\square$