Monotonicity and rigidity of the $\mathcal{W}$-entropy on Riemannian metric measure spaces with nonnegative Ricci curvature

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1. Introduction

Perelman's $\mathcal{W}$-entropy

(M, g): m-dim. cpt. Riem. mfd, $\tau > 0$,

$f \in C^\infty(M)$, $\displaystyle\int_M \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}} \,\mathrm{d}\,\mathrm{vol} = 1$

$$\mathcal{W}(g, f, \tau)$$
$$:= \int_M \left[\tau(R + |\nabla f|^2) + f - m\right] \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}} \,\mathrm{d}\,\mathrm{vol}$$

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★ $(g(t), f(t), \tau(t))$: $\partial_t \tau = -1$,

$\partial_t g = -2\,\mathrm{Ric}$, $\partial_t f = -\Delta f + |\nabla f|^2 - R + \dfrac{m}{2\tau}$

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$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g, f, \tau) \geq 0$$

Entropy formula

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$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 2\int_M \tau \left| \mathrm{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 \frac{\mathrm{e}^{-f}}{(4\pi\tau)^{m/2}} \, \mathrm{d}\,\mathrm{vol}$$

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Entropy formula and rigidity

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[L. Ni '04]

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$\rightsquigarrow$ Extension to weighted Riem. mfds [X.-D. Li '12]

Purpose

Q.

Can one extend the monotonicity/rigidity of $\boldsymbol{\mathcal{W}}$ on metric measure spaces with "$\mathbf{Ric} \geq \mathbf{0}$ & $\mathbf{dim} \leq \boldsymbol{N}$" ($\mathbf{RCD}(0, N)$ spaces)?

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- Singular sp.'s other than $\mathbb{R}^m$ appear in rigidity

Outline of the talk

1. Introduction

2. Framework: RCD spaces

3. Main results

4. Proof
4.1 Monotonicity
4.2 Rigidity
4.3 Additional remarks

Met. meas. sp. & heat flow on it

$(X, d, \mathfrak{m})$: Polish geod. met. meas. sp.

($\mathfrak{m}$: loc.-finite, $\operatorname{supp} \mathfrak{m} = X$)

$P_t = \mathrm{e}^{t\Delta} \leftrightarrow$ Cheeger's L^2-energy

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$(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form ($\Leftrightarrow P_t$: linear $\Leftrightarrow \Delta$: linear)

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$\overset{\text{def}}{\Leftrightarrow}$ **Ch**: quadratic form ($\Leftrightarrow P_t$: linear $\Leftrightarrow \Delta$: linear)

$\Rightarrow {}^{\exists}\langle D\cdot, D\cdot\rangle_w$ bilinear s.t. $\langle Df, Df\rangle_w = |Df|_w^2$

RCD spaces

Definition 2 (RCD$(0, N)$ $(N \in (0, \infty))$)

- $(X, d, \mathfrak{m})$: infin. Hilb.
- $\displaystyle\int_X \exp\left(-{}^{\exists}c\, d({}^{\exists}x_0, x)^2\right) \mathfrak{m}(\mathrm{d}x) < \infty$
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★ **RCD**$^*(K, N)$ $(K \neq 0)$ can be defined similarly

Examples

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$$\textbf{RCD}^*(K, N) \Leftrightarrow \mathrm{Ric} + \nabla^2 V - \frac{\nabla V^{\otimes 2}}{N - m} \geq K$$

- (Pointed) measured GH lim. of $\textbf{RCD}^*(K, N)$ sp.'s
  [Gigli, Mondino & Savaré '15]
- m-dim. Alexandrov sp. of curv. $\geq k$
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Characterizations of RCD cond.

RCD$(0, N)$: Some regularity ass'ns &

$$W_2(P_s f\,\mathfrak{m}, P_t g\,\mathfrak{m})^2 \le W_2(f\,\mathfrak{m}, g\,\mathfrak{m})^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

★ Equiv. cond'ns to **RCD**$(0, N)$ (up to reg. assn's)

- "$\frac{1}{2}\Delta|Df|_w^2 - \langle Df, D\Delta f\rangle_w \ge \frac{1}{N}|\Delta f|^2$"

  (Bakry-Émery's curv.-dim. cond.)

- On $(\mathcal{P}_2(X), W_2)$, $^\forall\mu_0$, $^\exists(\mu_t)_{t\ge 0}$ sol. to $(0, N)$-evolution variational inequality of Ent (a (metric) formulation of "$\dot{\mu}_t = -\nabla\,\mathrm{Ent}(\mu_t)$")

[Erbar, K. & Sturm '15]

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Heat flow

Properties of the heat semigr. P_t under **RCD**$^*(K, N)$

- $P_t : L^2(\mathfrak{m}) \to L^2(\mathfrak{m})$ can be extended to $P_t : \mathcal{P}_2(X) \to \mathcal{P}_2(X)$
- P_t admits a continuous kernel (heat kernel) p_t
- $\mu_t = P_t \mu (= \rho_t \mathfrak{m}) \in \mathcal{P}(X)$ satisfies

$$\|\dot{\mu}_t\|^2 := \varlimsup_{\delta \downarrow 0} \frac{W_2(\mu_t, \mu_{t+\delta})^2}{\delta^2}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ent}(\mu_t) = \int \frac{|D\rho_t|_w^2}{\rho_t} \mathrm{d}\mathfrak{m} =: I(\mu_t)$$

(Fisher information)

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(Fisher information)

Heat flow

Properties of the heat semigr. P_t under **RCD**$^*(K, N)$

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Geometric properties of $\mathsf{RCD}(0, N)$ sp.

- Laplacian comparison thm [Gigli '15]:

  $\Delta d(x_0, \cdot)^2 \le 2N$ in the distributional sense

- Splitting thm [Gigli]: $N > 1$,

  $\mathbb{R} \hookrightarrow X$ isometry

  $\Rightarrow X \simeq \mathbb{R} \times$ (an **RCD**$(0, N-1)$ sp.)

- Volume rigidity [Gigli & De Philippis]: $N \ge 2$,

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$$-\Delta \log P_t f = \frac{|DP_t f|_w^2}{(P_t f)^2} - \frac{\Delta P_t f}{P_t f} \le \frac{N}{2t}$$

- Sharp heat kernel estimate [Jiang, Li & Zhang '16]:
$^\forall \varepsilon > 0$, $^\exists C_\varepsilon > 0$ s.t.

$$\frac{C_\varepsilon^{-1}}{\mathfrak{m}(B_{\sqrt{t}}(x))} \exp\left(-\frac{d(x,y)^2}{(4-\varepsilon)t}\right) \le p_t(x,y)$$

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$\Rightarrow$ $\lim_{t \downarrow 0} 4t \log p_t(x,y) = -d(x,y)^2$ cpt. unif. in y

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$$\mu = \rho\,\mathfrak{m} \in \mathcal{P}(X),\ \rho =: \frac{e^{-f}}{(4\pi t)^{N/2}} \qquad (\tau \rightsquigarrow t)$$

$$\mathcal{W}(\mu, t) := \int_X \left[t|Df|_w^2 + f - N\right] \rho\, d\mathfrak{m}$$

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Main thm

Theorem 3 ([X.-D. Li & K.])

$(X, d, \mathfrak{m})$: $\mathsf{RCD}(0, N)$, $N \geq 2$, $\mu_t := P_t\mu$

(1) $\mathcal{W}(\mu_t, t) \searrow$ *in* $t \in (0, \infty)$

(2) *Suppose* $^\exists t_* > 0$ *s.t.*

$$\varlimsup_{t \downarrow t_*} \frac{\mathcal{W}(\mu_t, t) - \mathcal{W}(\mu_{t_*}, t_*)}{t - t_*} = 0$$

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Cone

Definition 4 ($(0, N)$-cone)

$(X, d, \mathfrak{m})$: $(0, N)$-cone of $(Y, d_Y, \mathfrak{m}_Y)$

$\overset{\text{def}}{\Longleftrightarrow}$

- $X = [0, \infty) \times Y / \{0\} \times Y$,
- $d((r, x), (s, y))^2 := r^2 + s^2 - 2rs \cos(d_Y(x, y) \wedge \pi)$
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Remarks

- Theorem 1 (1) is known when X: cpt.
  [Jiang & Zhang '16]
- In previous results, $\mu = \delta_{x_0}$ (intial data) is assumed
- Considering the right upper derivative of $\mathcal{W}(\mu_t, t)$
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4.1. Monotonicity

Optimal transport approach on Ricci flow

$\partial_\tau g_\tau = 2\,\mathrm{Ric}$, μ_τ: $\partial_\tau \mu_\tau = \Delta_\tau \mu_\tau$

$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r}\left(|\dot{\gamma}_r|_r^2 + R(\gamma_r)\right) \mathrm{d}r \right]$$

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$(\tau_0 < \tau_1)$

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

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$\Rightarrow \Xi_{\tau_0}^{\tau_1}(t) \searrow$ in t [Topping '09 / K. & Philipowski '11]

$$\Rightarrow \lim_{\tau \downarrow 1} \frac{\Xi_1^\tau(t)}{(\tau-1)^2} \searrow$$

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Toward the time-inhomogeneous case

$$L_s^t(x,y) := \inf_{\gamma_s = x,\, \gamma_t = y} \left[\int_s^t \sqrt{r}\left(|\dot{\gamma}_r|_r^2 + R(\gamma_r)\right) \mathrm{d}r \right]$$

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$$\Rightarrow 2(\sqrt{t}-\sqrt{s})\int_s^t \sqrt{r}|\dot\gamma_r|^2\,\mathrm{d}r = \int_0^1 |\dot\gamma_u^*|^2\,\mathrm{d}u$$

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$$\Rightarrow 2(\sqrt{t} - \sqrt{s}) L_s^t(x,y) = d(x,y)^2,$$

$$2(\sqrt{t} - \sqrt{s}) \mathcal{T}_{L_s^t}(\mu,\nu) = W_2(\mu,\nu)^2$$

Toward the time-inhomogeneous case

$$2(\sqrt{t} - \sqrt{s})\mathcal{T}_{L_s^t}(\mu, \nu) = W_2(\mu, \nu)^2$$

$$\Xi_{\tau_0}^{\tau_1}(t) := 2(\sqrt{\tau_1 t} - \sqrt{\tau_0 t})\mathcal{T}_{L_{\tau_0 t}^{\tau_1 t}}(\mu_{\tau_0 t}, \mu_{\tau_1 t}) - 2N(\sqrt{\tau_1 t} - \sqrt{\tau_0 t})^2$$

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$$\Downarrow$$

$$\Xi_1^{\tau}(t) \searrow$$

$$\Updownarrow$$

$$\begin{aligned}W_2(\mu_t, \mu_{\tau t})^2 \le W_2(\mu_s, \mu_{\tau s})^2 \\ +2N(\sqrt{\tau(t-s)} - \sqrt{t-s})^2\end{aligned}$$

Derivation from RCD$(0, N)$

$$W_2(P_{t'}\mu', P_{s'}\nu')^2 \leq W_2(\mu', \nu')^2 + 2N(\sqrt{t'} - \sqrt{s'})^2$$

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$\mu' = \mu_s$, $\nu' = \mu_{s+s^\alpha\delta}$

$t' = t - s$, $s' = (t^\alpha - s^\alpha)\delta + t - s$

& "$\overline{\lim_{\delta\downarrow 0}} \frac{1}{\delta^2}$" with $\|\dot{\mu}_t\|^2 = I(\mu_t)$

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Derivation from $\mathsf{RCD}(0,N)$

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$$\Downarrow \quad \begin{array}{l} \mu' = \mu_s,\ \nu' = \mu_{s+s^\alpha\delta} \\ t' = t-s,\ s' = (t^\alpha - s^\alpha)\delta + t - s \\ \&\ \text{“}\overline{\lim_{\delta\downarrow 0}}\frac{1}{\delta^2}\text{”}\ \text{with}\ \|\dot{\mu}_t\|^2 = I(\mu_t) \end{array}$$

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★ "$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(\mu_t, t) = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}(t\,h(t))$" $\Rightarrow \mathcal{W}(\mu_t, t) \searrow$ □

4.2. Rigidity

Identification of Fisher info.

For simplicity, suppose $\mu = \delta_{x_0}$ ($\Rightarrow \mu_t = p_t^{x_0}\,\mathfrak{m}$)

$$h(t) = tI(\mu_t) - \frac{N}{2}, \quad t\,h(t) \searrow$$

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$$\Rightarrow h(t) = 0, \text{ i.e. } I(\mu_t) = \frac{N}{2t}\ (t \in (0, t_*])$$

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- Li-Yau: $\dfrac{|Dp_t^{x_0}|^2}{(p_t^{x_0})^2} - \dfrac{\Delta p_t^{x_0}}{p_t^{x_0}} \le \dfrac{N}{2t}$

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- $(F(\alpha))$ & "$\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = 0$" $\overset{\alpha\uparrow 1}{\Rightarrow} h(t_*) = 0$

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Equality in Laplacian comparison

$$I(\mu_t) = \frac{N}{2t} \ (t \in (0, t_*])$$

Li-Yau: $-\Delta \log p_t^{x_0} = \frac{|Dp_t^{x_0}|^2}{(p_t^{x_0})^2} - \frac{\Delta p_t^{x_0}}{p_t^{x_0}} \leq \frac{N}{2t}$

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$$\Downarrow \quad \lim_{t \downarrow 0} 4t \log p_t^{x_0}(x) = -d(x_0, x)^2$$

$$\Delta d(x_0, \cdot)^2 = 2N$$

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$$\Delta d(x_0, \cdot)^2 = 2N$$

$\Rightarrow$ Volume rigidity [Gigli & De Philippis] is applicable □

4.3. Additional remarks

Heat kernel

Proposition 1

Suppose $\Delta d(x_0, \cdot)^2 = 2N$. *Then* $^\exists C, C' > 0$ *s.t.*

$$p_t(x_0, x) = \frac{C}{t^{N/2}} \exp\left(-\frac{d(x_0, x)^2}{4t}\right) = \frac{C'}{\mathfrak{m}(B_{\sqrt{t}}(x_0))} \exp\left(-\frac{d(x_0, x)^2}{4t}\right)$$

$\because$ Compute I for "Gaussian kernel" in two ways &
RHS enjoys the energy dissipation identity for Ent $\square$

Heat kernel

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In particular, X *is non-compact*

$\because$ Compute I for "Gaussian kernel" in two ways &

RHS enjoys the energy dissipation identity for Ent $\square$

Initial data

Lemma 2

Suppose $\boldsymbol{I(\mu_t) = \frac{N}{2t}}$. *Then* $\boldsymbol{I(p_t^{x}\,\mathfrak{m}) = \frac{N}{2t}}$ $\boldsymbol{\mu}$*-a.e.* $\boldsymbol{x}$.

$\because \mu \mapsto I(\mu)$ convex

$$\Rightarrow \frac{N}{2t} = I(\mu_t) \text{“}\leq\text{”} \int_X I(p_t^x)\mu(\mathrm{d}x) \overset{\text{Li-Yau}}{\leq} \frac{N}{2t} \qquad \square$$

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Lemma 3

Suppose $\boldsymbol{I}(\boldsymbol{\mu}_t) = \dfrac{\boldsymbol{N}}{\boldsymbol{2t}}$. *Then* $\boldsymbol{\mu}$ *is Dirac.*

$\because$ Reduce to $\boldsymbol{\mu} = \dfrac{\boldsymbol{\delta}_{\boldsymbol{x}} + \boldsymbol{\delta}_{\boldsymbol{y}}}{\boldsymbol{2}} \Rightarrow \dfrac{|\boldsymbol{D}\boldsymbol{p}_t^{\boldsymbol{x}}|}{\boldsymbol{p}_t^{\boldsymbol{x}}} = \dfrac{|\boldsymbol{D}\boldsymbol{p}_t^{\boldsymbol{y}}|}{\boldsymbol{p}_t^{\boldsymbol{y}}}$

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Suppose $I(\mu_t) = \frac{N}{2t}$*. Then* $I(p_t^x\,\mathfrak{m}) = \frac{N}{2t}$ μ*-a.e.* x*.*

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Heat flow is a W_2-geodesic

Proposition 4

Suppose $\Delta d(x_0, \cdot)^2 = 2N$ *and* $\mu_t = P_t \delta_{x_0}$.
$\Rightarrow (\mu_{t^2/(2N)})_{t \geq 0}$: W_2*-min. geod.*

$\because$

- $\dfrac{N}{2t} = I(\mu_t) = \dfrac{1}{4t^2} \displaystyle\int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$
- $W_2(\mu_0, \mu_t)^2 = \displaystyle\int_X d(x_0, x)^2 \, \mu_t(\mathrm{d}x)$
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- $$\|\dot{\mu}_t\|^2 = I(\mu_t)$$

$\Rightarrow \mu_t^* := \mu_{t^2/(2N)}$ satisfies
$W_2(\mu_0^*, \mu_t^*) = t$ & $\|\dot{\mu}_t^*\| = 1$ $\square$