Optimal transport, heat flow and coupling of Brownian motions

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偏微分方程式に付随する確率論的問題
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1. Introduction

Heat equation $\leftrightarrow$ Brownian motion

Heat equation

$$\partial_t u = \Delta u$$

$\longleftrightarrow$

Brownian motion

$$(B_t)_{t \geq 0}$$

Heat equation ↔ Brownian motion

$$u(t,x) = \mathbb{E}[u(0,B_t) \mid B_0 = x]$$

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Optimal transport

Geometry of the underlying space X

What's optimal transport?

- Bring a mass $\mu_0 \in \mathcal{P}(X)$ to $\mu_1 \in \mathcal{P}(X)$
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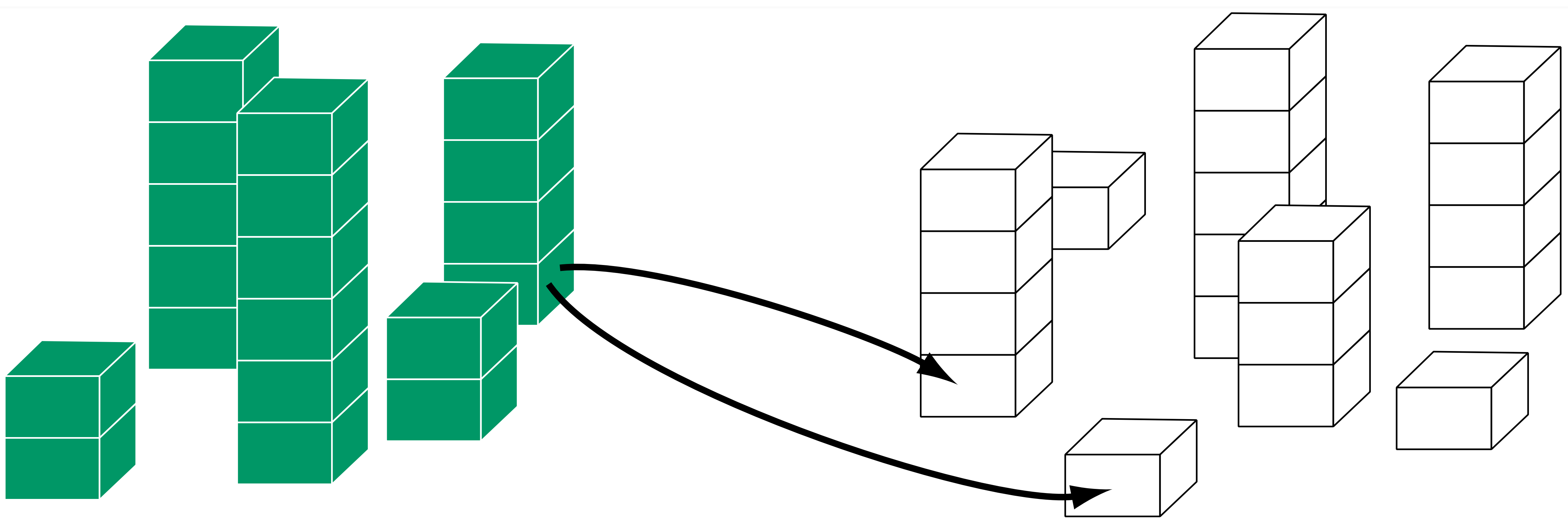

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- How do we minimize the total cost?
- Properties of the optimal cost (as a fn. of μ_0, μ_1)?
  - ↳ "difference" of μ_0 and μ_1

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> Characterizations of curv.-dim. cond'n by $(\nu_t)_{t\geq 0}$

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  How about BMs?

Outline of the talk

1. Introduction

2. Basics of optimal transport

3. Framework

4. Some couplings of Brownian motions
4.1 Coupling by parallel transport
4.2 Coupling by reflection

5. Concluding remarks

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(X, d): Polish metric space

$c : X \times X \to [0, \infty)$: symm., conti.

Opt. trans. cost

$$\mathcal{T}_c(\mu, \nu) := \inf_{\pi \in \Pi(\mu,\nu)} \left[\int_{X \times X} c \, d\pi \right]$$

$$\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X^2) \,\middle|\, \begin{array}{l} \pi(A \times X) = \mu(A) \\ \pi(X \times A) = \nu(A) \end{array} \right\}$$

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Basic problems in Opt. Trans.

- Characterizations of minimizer(s)

  (e.g. $\rightsquigarrow$ Monge-Ampère eq.)
- Trends to equilibrium in opt. trans. cost
- Functional inequalities

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Other connections
with optimal control/Hamilton-Jacobi eq., Euler eq., ...

Kantorovich duality

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⇝ Connection with

Hopf-Lax formulae & associated H.-J. eq.'s

Wasserstein distance

L^p-Wasserstein distance ($p \in [1, \infty)$)

$$W_p(\mu, \nu) := \mathcal{T}_{d^p}(\mu, \nu)^{1/p} \left(= \inf_{\pi \in \Pi(\mu,\nu)} \|d\|_{L^p(\pi)} \right)$$

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Other gradient flows on $(\mathcal{P}(X), W_2)$:
Porous medium eq./McKean-Vlasov eq./p-heat eq./$\cdots$

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,

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$$\bigstar \; |f(\gamma_1) - f(\gamma_0)| \le \int_0^1 |\nabla f|_w(\gamma_s)|\dot{\gamma}_s| \, ds$$

for a.e. trajectories $(\gamma_s)_{s \in [0,1]}$ of "nice" transports

$\mathsf{RCD}(K,\infty)$ space

$$\mathrm{Ent}(\rho\mathfrak{m}) := \int_X \rho \log \rho \, d\mathfrak{m}$$

Definition 1

$(X,d,\mathfrak{m})$: Riemannian $\mathbf{CD}(K,\infty)$ sp. $(K \in \mathbb{R})$

$\overset{\text{def}}{\Leftrightarrow}$ "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}(X), W_2)$

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Examples

- (M, g): cpl. Riem. mfd, $\partial M = \emptyset$, V: fn. on M
  $\rightsquigarrow (M, d_g, \mathfrak{m})$ (d_g: Riem. dist., $\mathfrak{m} = \mathrm{e}^{-V} \mathrm{vol}_g$)

  $$\rightsquigarrow \quad \mathsf{RCD}(K, \infty) \Leftrightarrow \mathrm{Ric} + \mathrm{Hess}\, V \geq K$$

- $M = \mathbb{R}^d$, $V(x) = \dfrac{K|x|^2}{2}$
  $(\leftrightsquigarrow \partial_t u = \Delta u - Kx \cdot \nabla u)$
- Meas. Gromov-Hausdorff lim. of $\mathsf{RCD}(K, \infty)$ sp.'s
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[Ambrosio, Gigli & Savaré '15]

Analytic properties of RCD(K,∞) sp.

- (Quantitative) Lipschitz regularization of P_t:
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- Functional inequalities (mainly when $K > 0$):
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- A moment bound of $\mathfrak{m}$
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Q.

- Something more for sample paths of BMs?
- In particular, coupling of BMs?

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(M, g): cpl. Riem. mfd, $\mathrm{Ric} \geq K$

$(B_t^{(0)}, B_t^{(1)})$: coupling by parallel transport of BMs

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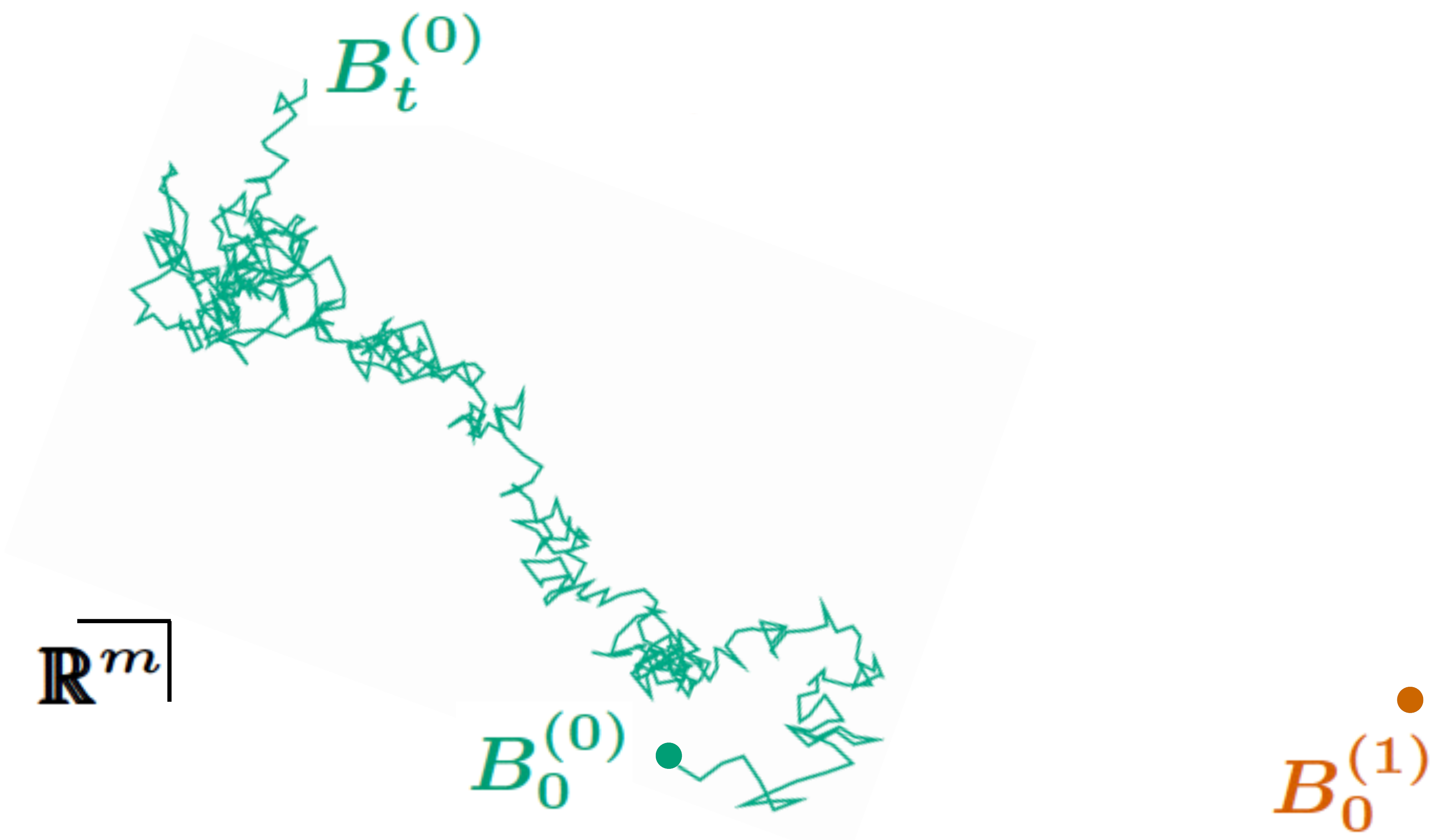

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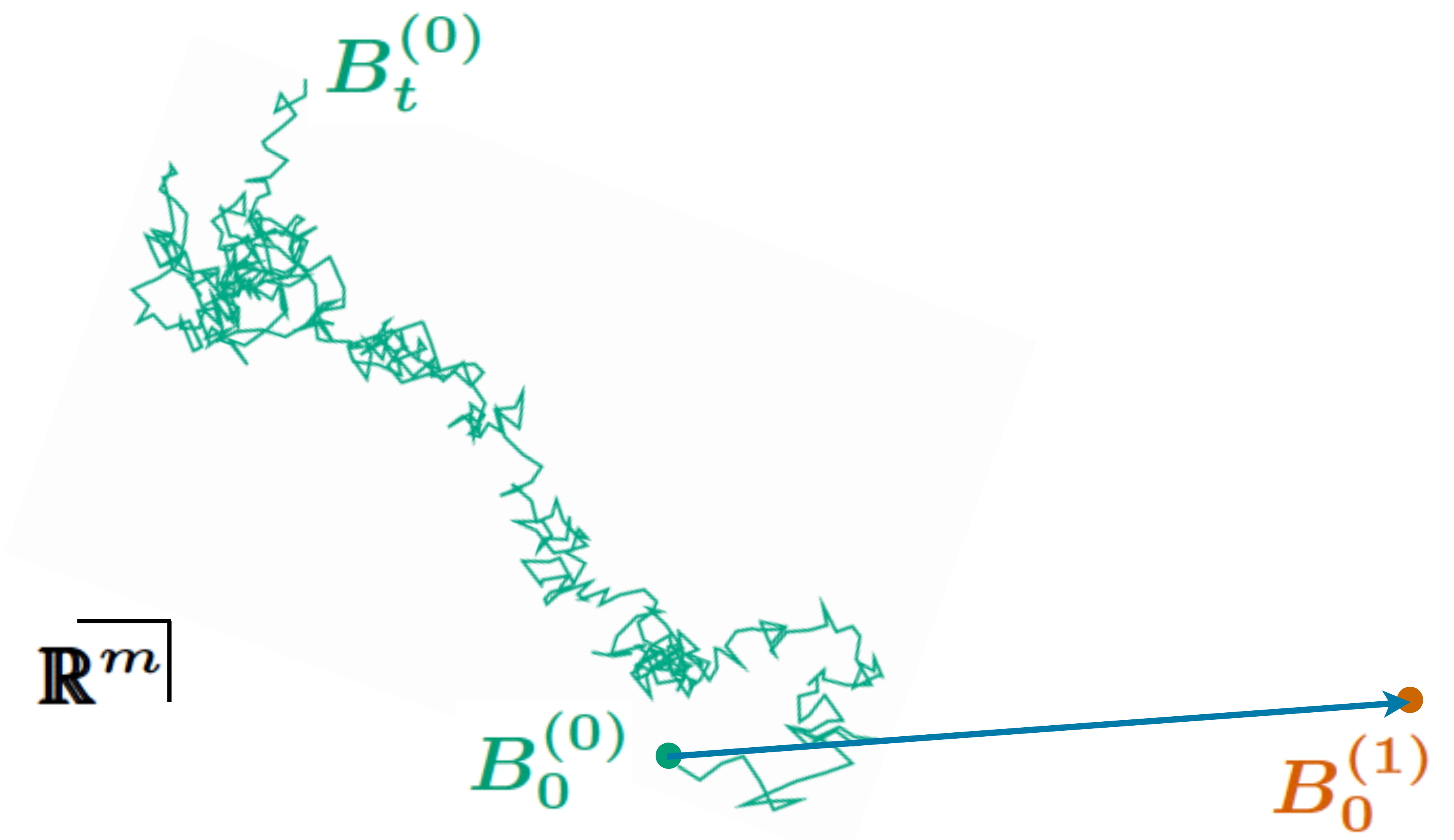

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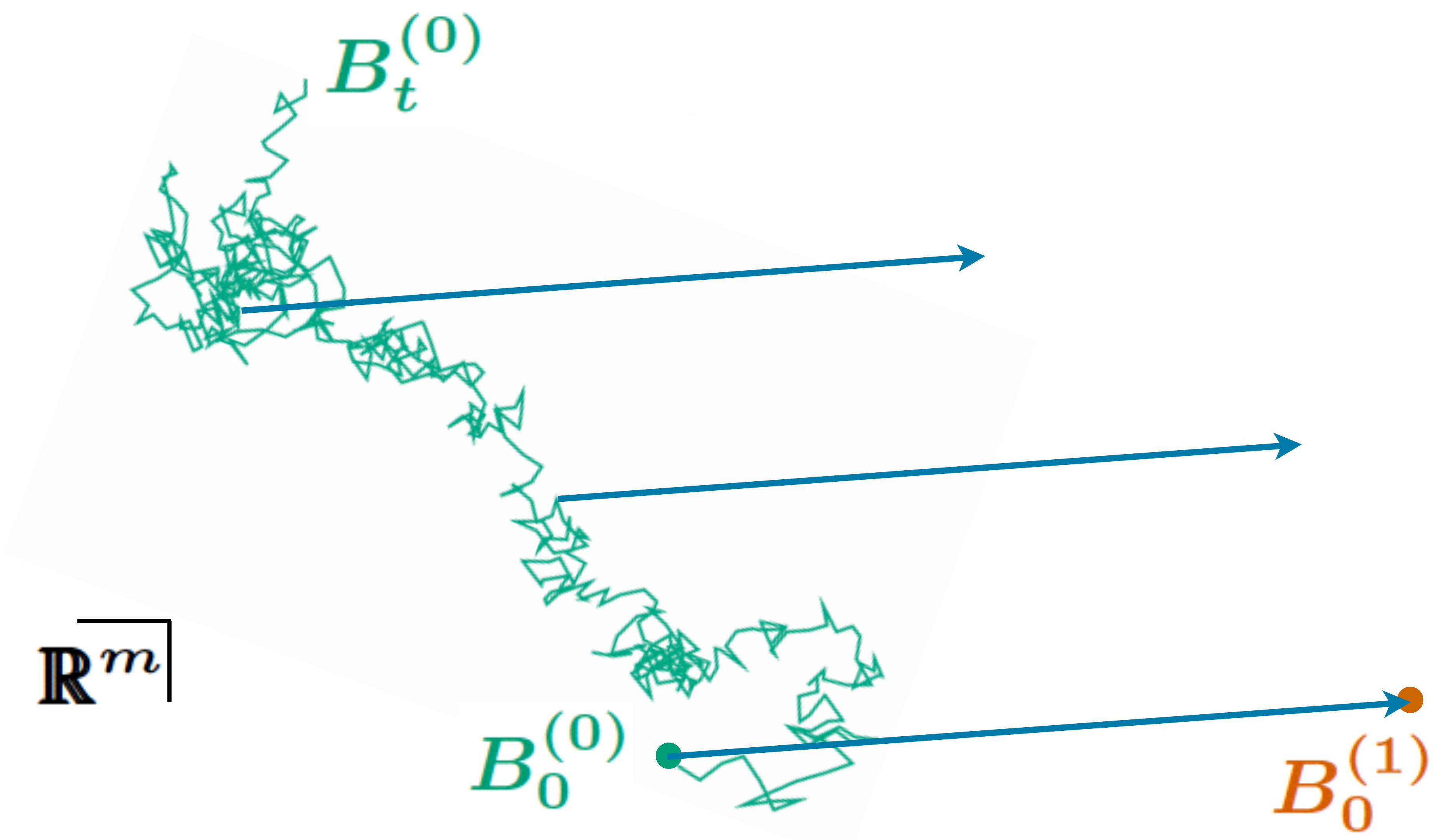

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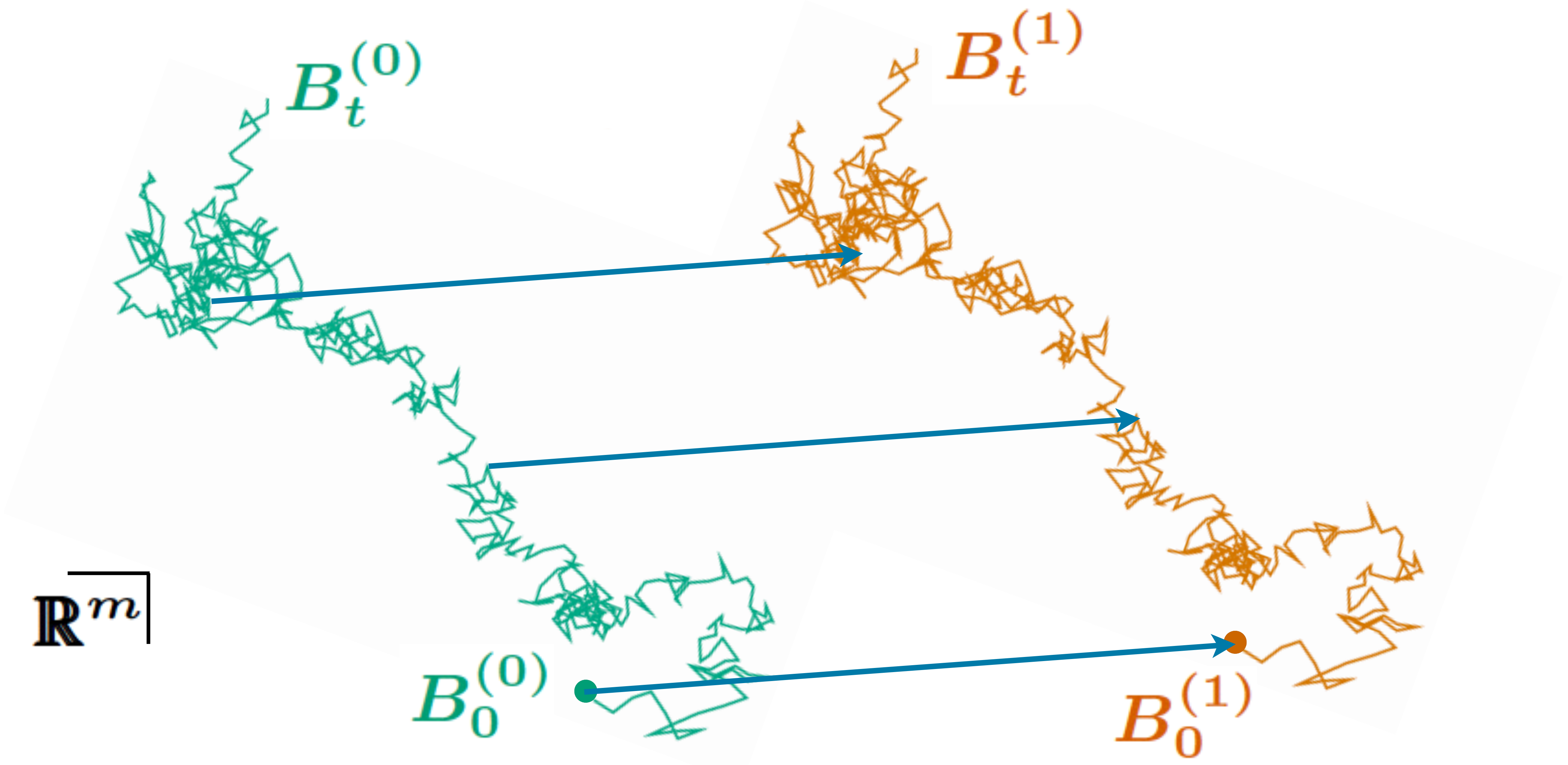

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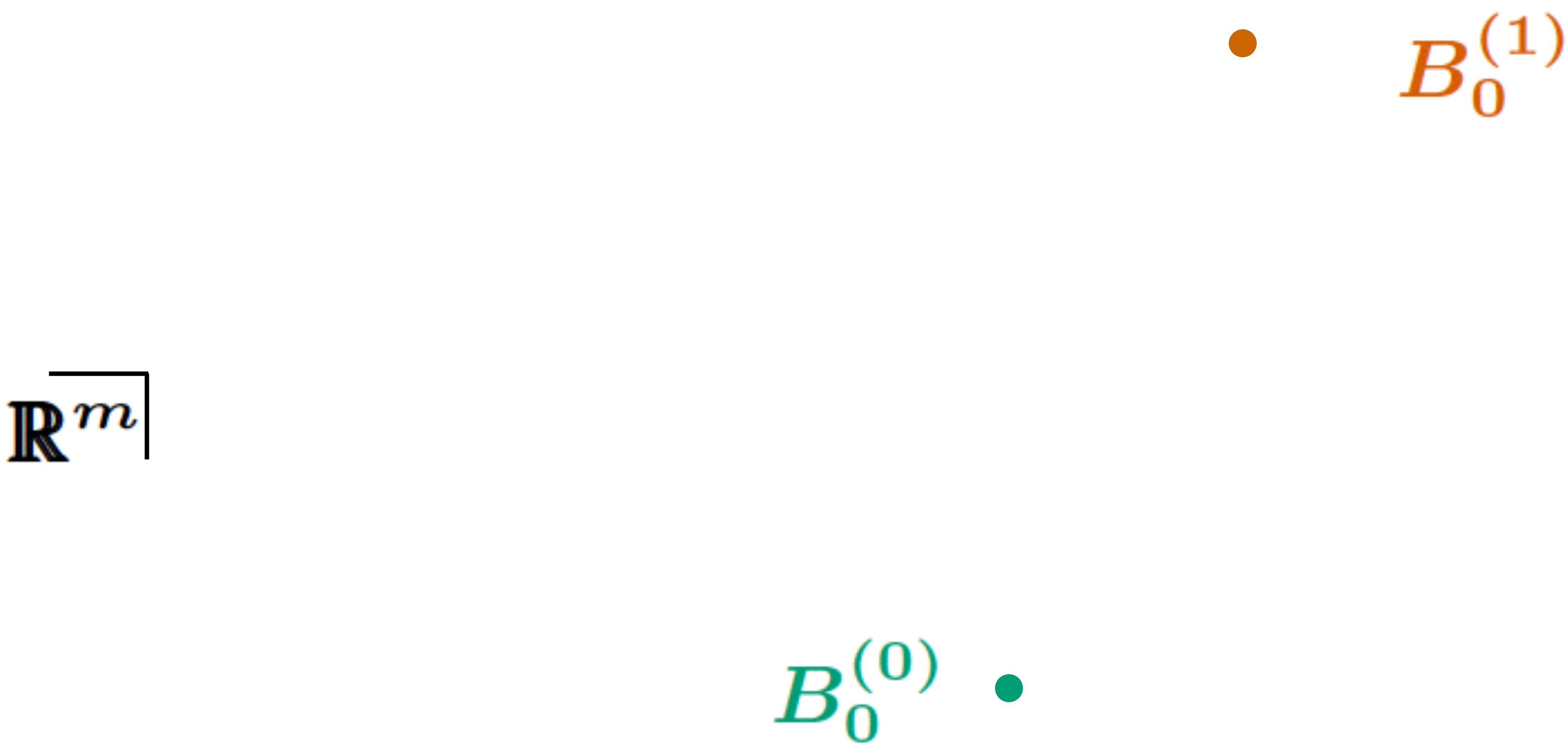

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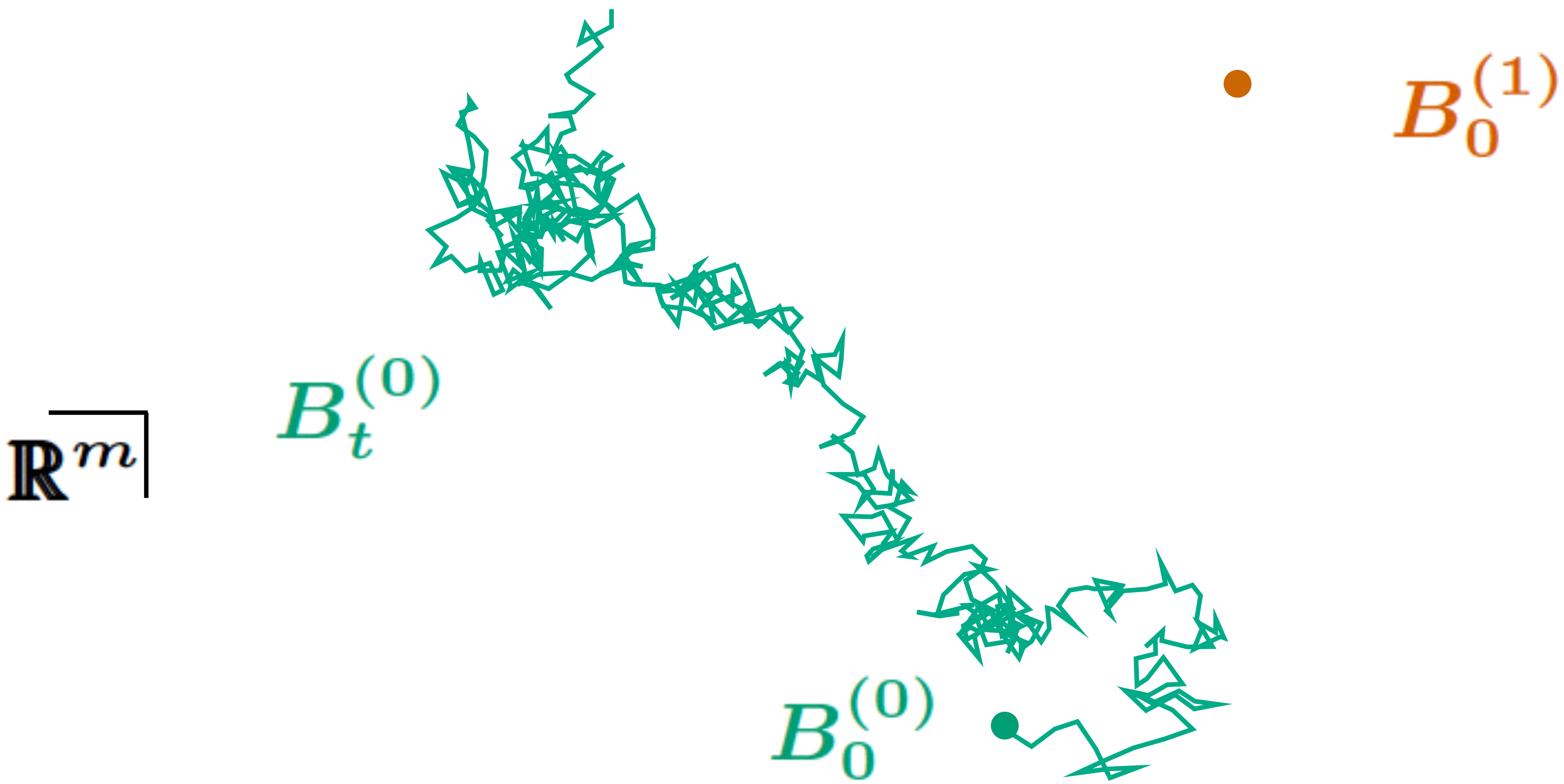

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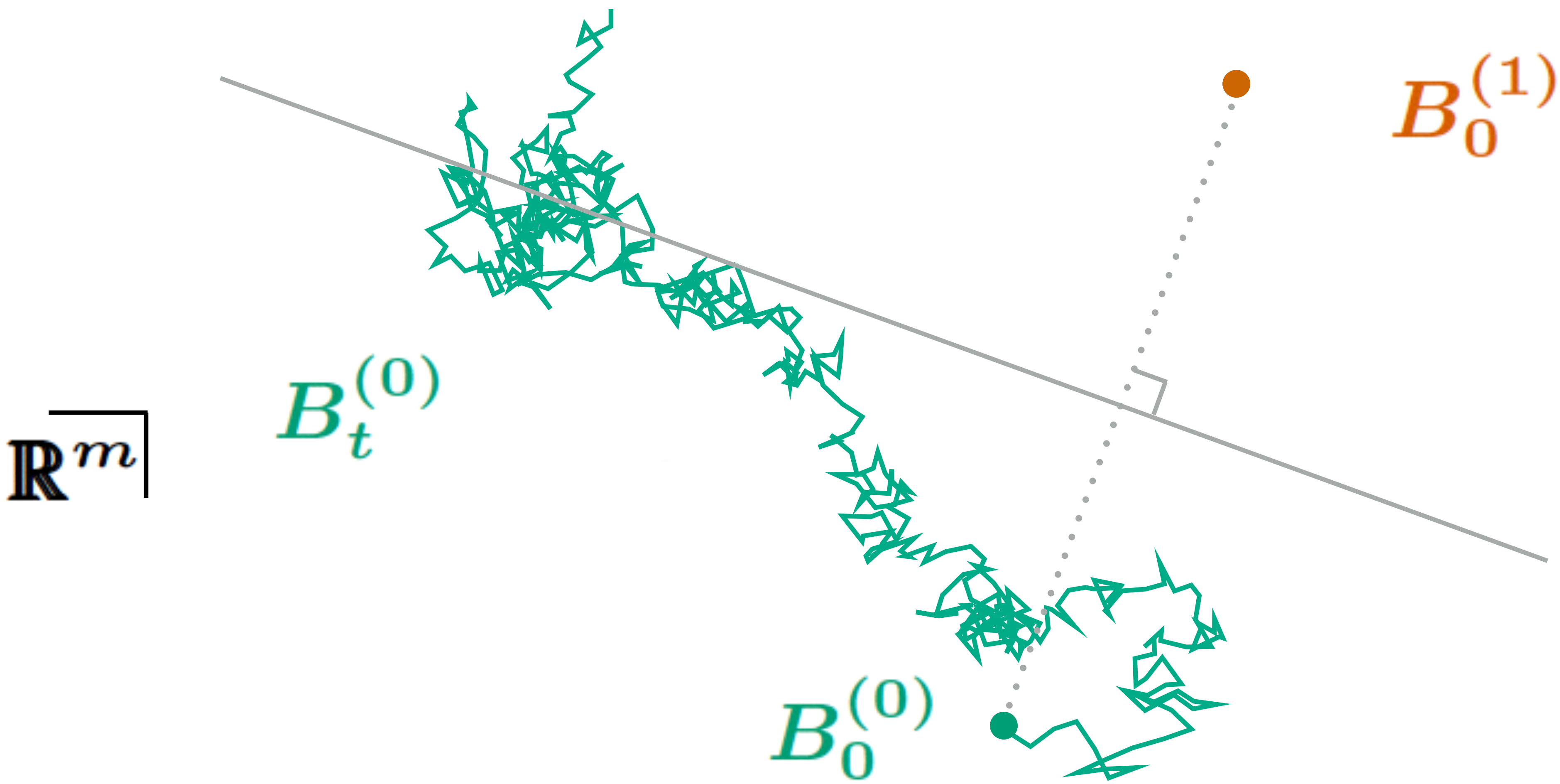

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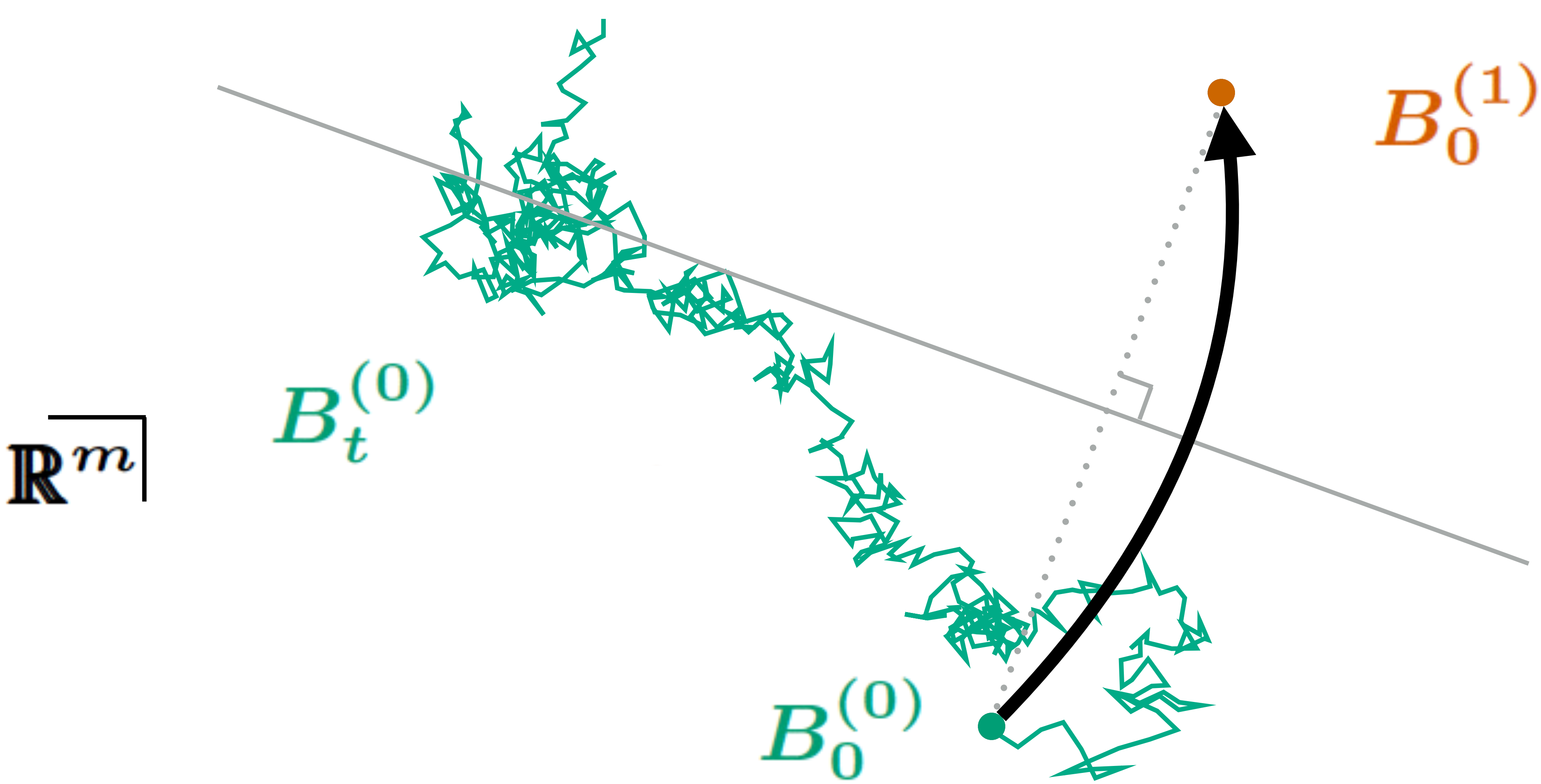

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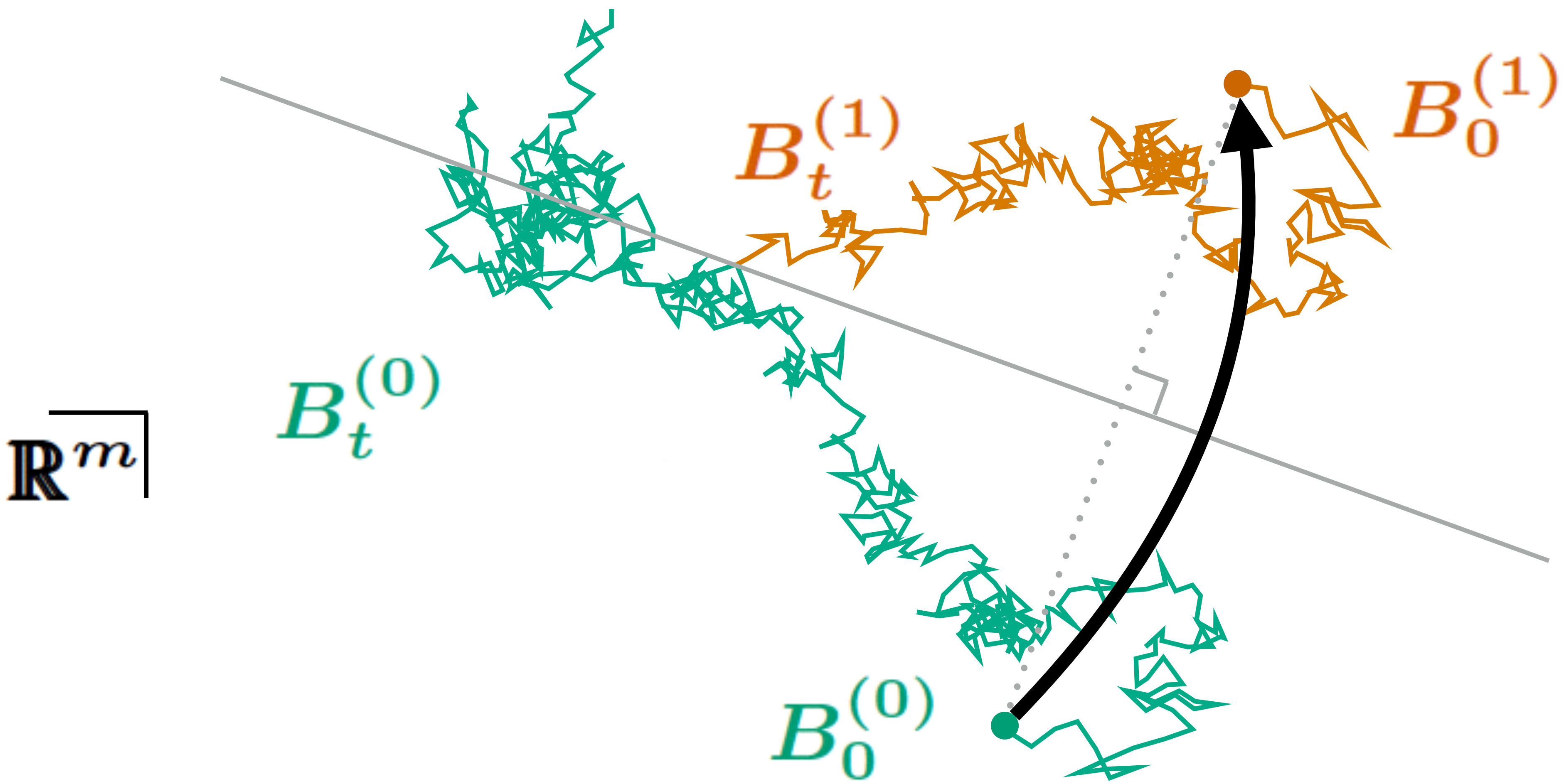

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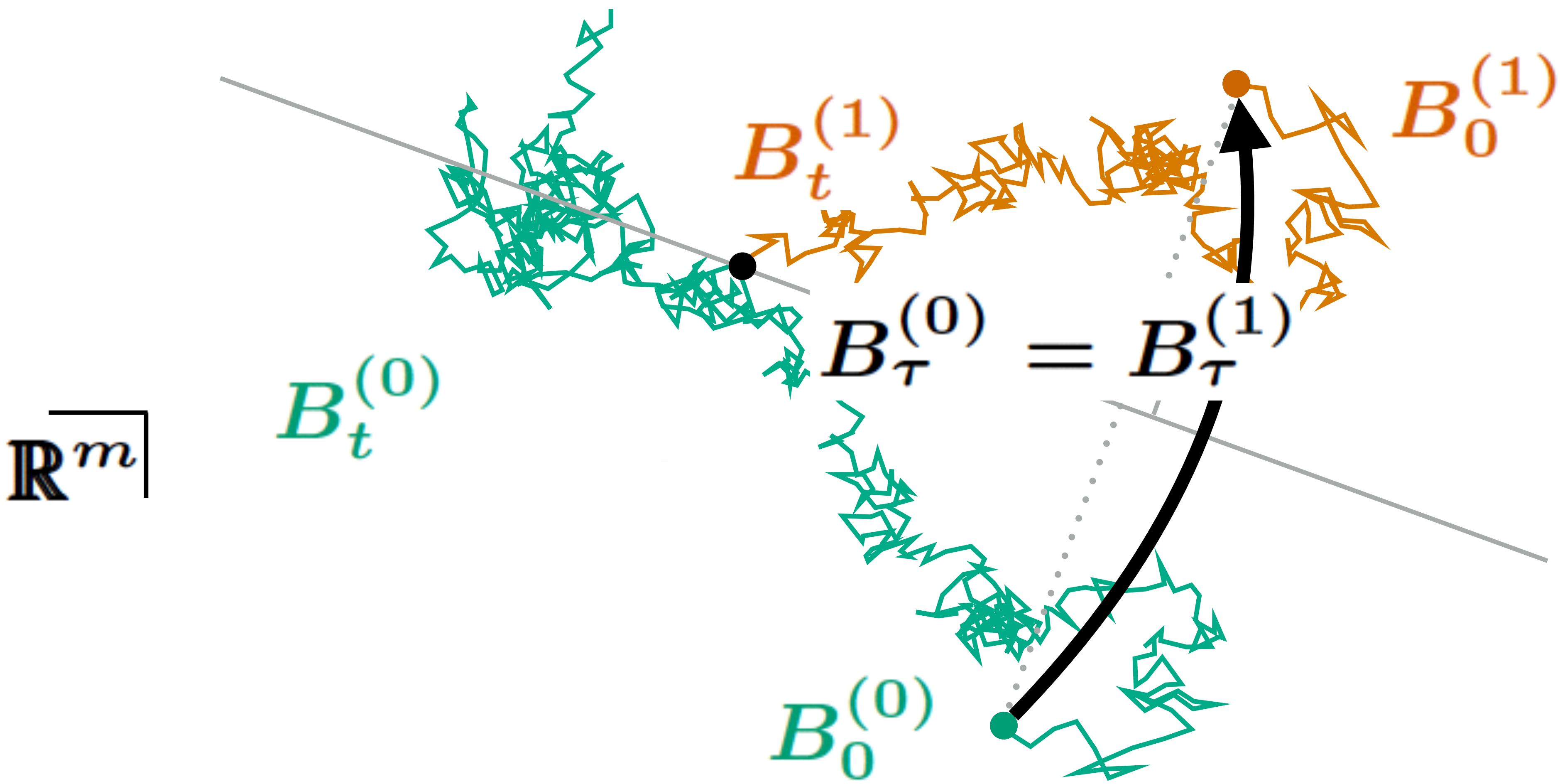

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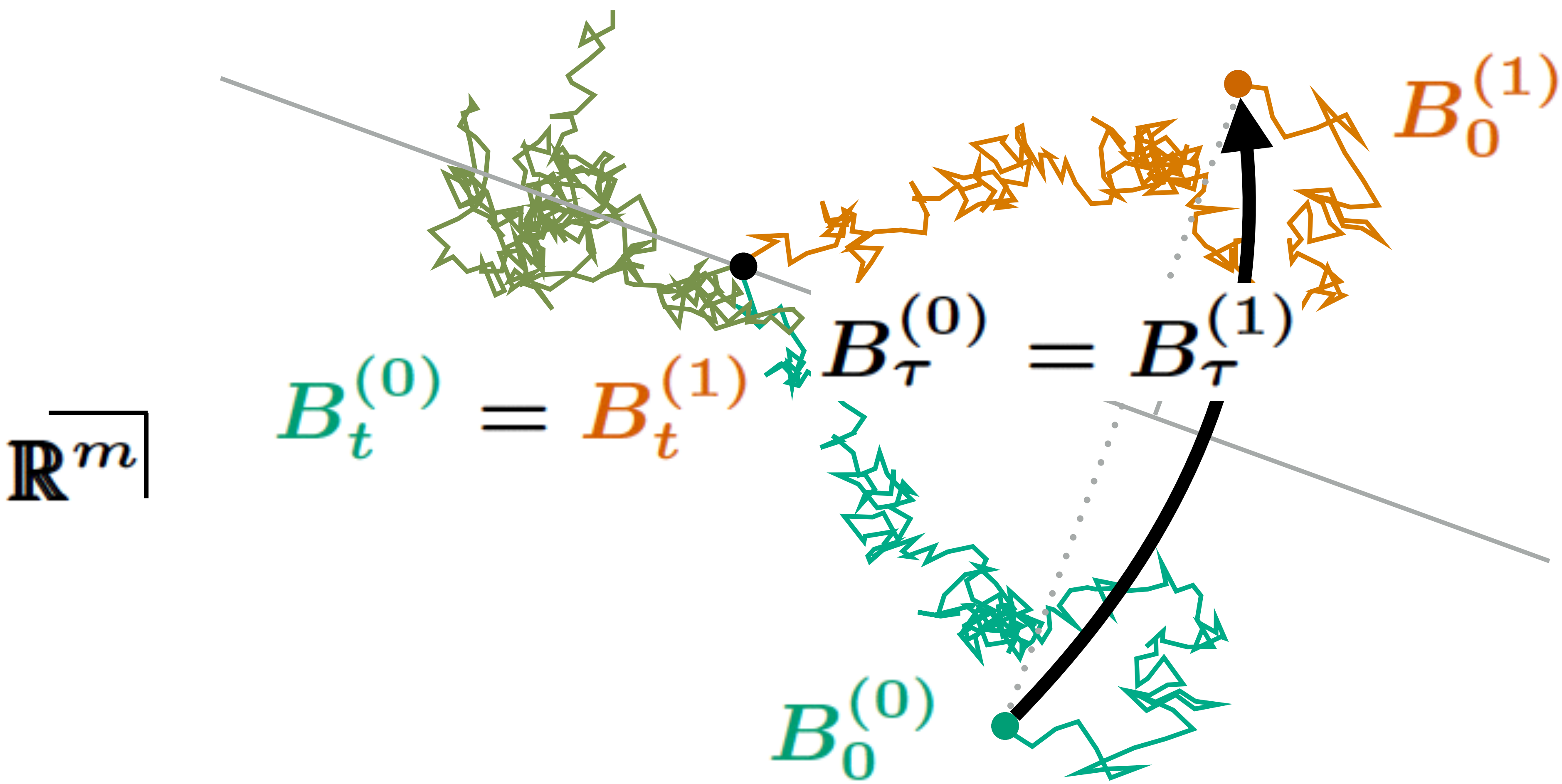

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[Kendall '86 / Cranston '91 / F.-Y. Wang '94,'05 / E.-P. Hsu '03 / von Renesse '04 / K. '10,'12 / Arnaudon, Coulibaly & Thalmaier '09 / Neel & Popescu '15+ / $\cdots$]

How do we extend?

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$\Downarrow$

Change the definition:

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⇝

Change the definition:

From the structure we used for construction
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Sturm's coupling by parallel transport

$^\forall x_0, x_1 \in X$ $^\exists(B_t^{(0)}, B_t^{(1)})$: coupling of BMs on X s.t.

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- Self-improvement of $\mathbf{BE}(K, \infty)$

[Bakry & Émery '85/Savaré '14]

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[Bakry & Émery '85/Savaré '14]

$\Rightarrow W_{\infty}(\delta_x P_t, \delta_y P_t) \le e^{-Kt} d(x, y)$

$\Rightarrow$ ${}^{\exists}$a "nice" coupling of trans. prob.

Self-improvement

$$\mathsf{W}_p(K,\infty):\ W_p(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_p(\mu,\nu)$$

$$\mathsf{G}_q(K,\infty):\ |\nabla P_t f|_w \le \mathrm{e}^{-Kt} P_t(|\nabla f|_w^q)^{1/q}$$

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  Monotonicity of a transportation cost

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Observe it on Riem. mfd.

Coupling by refl. and opt. trans.

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where $\begin{cases} d\rho_t^r = 2\sqrt{2}dW_t - K\rho_t^r dt, \\ \rho_0^r = r \end{cases}$

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Corollary 4 (cf. [K. & Sturm '13])

$$\frac{1}{2}\|\delta_{x_0}P_T - \delta_{x_1}P_T\|_{var} \leq \varphi_T(d(x_0,x_1))$$

(Comparison theorem for total variations)

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★ "Thm. 3 ⇒ Thm. 5" is simliar to the corresponding argument in coupling by parallel transport

1. Introduction

2. Basics of optimal transport

3. Framework

4. Some couplings of Brownian motions

4.1 Coupling by parallel transport

4.2 Coupling by reflection

5. Concluding remarks

Summary

- “$\mathbf{Ric} \geq \boldsymbol{K}$” on “Riemannian” met. meas. sp
  - $\rightarrow$ Brownian motion is defined
  - $\rightarrow$ Bakry-Émery theory is available
- “Coupling by parallel transport/reflection” of BMs
  - $\leftarrow$ Define them by characteristic properties
  - $\leftarrow$ Use of f’nal ineq’s & Bakry-Émery theory
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