On the speed in transportation costs of heat distributions

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1. Introduction

Speed in transportation cost

$\partial_t \mu_t = \Delta \mu_t$: heat distribution

$\Rightarrow (\mu_t)_{t \geq 0}$: curve in $\mathcal{P}(M)$

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Q.1 $\mathcal{T}_c(\mu_t, \mu_s) \approx ?\quad (s \to t)$

$$\mathcal{T}_c(\mu, \nu) := \inf \left\{ \int_{M \times M} c \, d\pi \;\middle|\; \begin{array}{c} \pi\text{: coupling of} \\ \mu \text{ and } \nu \end{array} \right\}$$

(Optimal transportation cost for a cost function c)

Q.2 Applications?

Background

On Q.1: Speed of gradient curve

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$\mu_t = \mathrm{e}^{t\Delta}\mu_0$: heat dist. on a met. meas. sp. (M, d, v)

$\Rightarrow$ "$\partial_t \mu_t = -\nabla \operatorname{Ent}_v(\mu_t)$" w.r.t. $W_2 = (\mathcal{T}_{d^2})^{1/2}$

[Jordan, Kinderlehrer & Otto '98]

[Ambrosio, Gigli & Savaré '05, ...]

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"$\partial_t \operatorname{Ent}_v(\mu_t) = \langle \nabla \operatorname{Ent}_v, \partial_t \mu_t \rangle = -|\partial_t \mu_t|^2$"

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$$\Downarrow$$

$$\overline{\lim_{s \downarrow t}} \left(\frac{W_2(\mu_s, \mu_t)}{s - t} \right)^2 = \int_M \frac{|\nabla \rho_t|^2}{\rho_t} dv =: I(\mu_t)$$

(Fisher information)

Background

On Q.2: Monotonicity of transportation costs

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"$\mathrm{Hess}\,\mathrm{Ent}_v \geq K$" ($\Leftrightarrow$ "$\mathrm{Ric} \geq K$") for $K \in \mathbb{R}$

$$\Downarrow$$

$e^{Kt} W_2(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$ in t

$$\Downarrow$$

$e^{Kt} W_2(\mu_t, \mu_{t+s}) \searrow$ in t

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$$\Downarrow$$

$$\boxed{I(\mu_t) \leq e^{-2Kt} I(\mu_0)}$$

($\Rightarrow$ log Sobolev ineq. (when $K > 0$))

Remark on backgrounds

★ $\left.\begin{array}{l}\text{Q1}\\ \text{Q2}\end{array}\right\} \leftrightsquigarrow$ grad. flow "$\dot{\mu}_t = -\nabla U(\mu_t)$" on $\mathcal{P}(M)$

[Ambrosio, Gigli & Savaré '05]

- Q1 $\leftrightsquigarrow$ Energy dissipation equality

$$-\frac{d}{dt}U(\mu_t) = \frac{1}{2}|\dot{\mu}_t|^2 + \frac{1}{2}|\nabla_- U|(\mu_t)^2$$

- Q2 $\leftrightsquigarrow$ (K-)Evolution variational ineqaulity

$$\frac{1}{2}\mathrm{e}^{-Kt}\frac{d}{dt}\left(\mathrm{e}^{Kt}W_2(\mu_t,\nu)^2\right) \le U(\nu) - U(\mu_t)$$

$$({}^\forall \nu \in \mathcal{P}_2(M))$$

Questions

- What happens for other trans. costs than $\mathcal{T}_{d^2}$?
- What happens when there is no gradient flow structure?

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$\rightsquigarrow$ Heat distributions on (backward) Ricci flow

1. Introduction
2. **Heat distributions on backward Ricci flow**
3. Coupling methods (Thm 1 & 2)
4. Idea of the proof of Thm 3 & 5
5. Further problems

Framework

- $(M, g(t))$: m-dim. cpl. Riem. mfds., $t \in [0, T]$

  $\partial_t g(t) = 2\,\mathrm{Ric}_t$ (backward Ricci flow)

- $((X(t))_{t \geq 0}, (\mathbb{P}_x)_{x \in M})$: $g(t)$-Brownian motion

  $\leftrightsquigarrow \Delta_{g(t)}$: generator

  $\mu_t = \mathbb{P}_{\mu_0} \circ X(t)^{-1}$: heat dist.

- v_t: $g(t)$-volume meas., $\mu_t = \rho_t v_t$

  ★ $\partial_t v_t = R_t v_t$ $\quad$ (R_t: $g(t)$-scalar curv.)

Ass. $\sup_t |\mathrm{Rm}_t|_{g(t)} < \infty$ (Rm_t: $g(t)$-curv. tensor)

$$\partial_t \mu_t \neq -\nabla \operatorname{Ent}_{v_t}(\mu_t)$$

$$\boxed{\operatorname{Ent}_{v_t}(\mu_t) := \int_M \rho_t \log \rho_t \, dv_t = \int_M \log \rho_t \, d\mu_t}$$

★ $\partial_t \mu_t = \Delta_t \mu_t$ (weakly)

$$\Rightarrow \partial_t \operatorname{Ent}_{v_t}(\mu_t) = -\int_M \left(\frac{|\nabla \rho_t|^2}{\rho_t^2} + R_t \right) d\mu_t$$

$$=: -\mathcal{F}(\mu_t) \qquad (\mathcal{F}\text{-functional})$$

$\Rightarrow$ No monotonicity of $\operatorname{Ent}_{v_t}(\mu_t)$!

Monotonicity of transportation costs

Observation

When $g(t) \equiv g_0$, $\partial_t g(t) = 2\,\mathrm{Ric}_t \Rightarrow \mathrm{Ric} \equiv 0$

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★ $\mathcal{T}_{d_t^2}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

- [McCann & Topping '10]: Opt. trans.
- [Arnaudon, Coulibaly & Thalmaier '09], [K. '12]: Stochastic analysis (coupling of BMs)

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$\not\Rightarrow$ $\mathcal{T}_{d_t^2}(\mu_t, \mu_{t+s}) \searrow$ (time-inhomogeneity)

Monotonicity of transportation costs

$$L_\alpha^{t,t'}(x,y) := \inf_{\substack{\gamma(t)=x,\\ \gamma(t')=y}} \left[\int_t^{t'} r^{\alpha/2} \left(|\dot\gamma(r)|_r^2 + R_r(\gamma(r)) \right) dr \right]$$

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Theorem 1 ([Lott '09], [Amaba & K.])

$\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s}) \searrow$ *in* t

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Theorem 2 ([Topping '09], [K. & Philipowski '11])

$$\Xi_{\tau_0,\tau_1}(t) := \left(\sqrt{\tau_1 t} - \sqrt{\tau_0 t}\right) \mathcal{T}_{L_1^{\tau_0 t, \tau_1 t}}(\mu_{\tau_0 t}, \mu_{\tau_1 t}) - m(\sqrt{\tau_1 t} - \sqrt{\tau_0 t})^2$$

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Monotonicity of transportation costs

Comparison of results

- [Lott '09], [Topping '09]:
  - Optimal transportation
  - Ass: M: cpt.
- [K. & Amaba], [K. & Philipowski '11]
  - Stochastic analysis
  - Ass: $\mathrm{Ric}_t \geq {}^{\exists}K g(t)$ $({}^{\forall}t)$

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Recall:

<u>Ass.</u> $\sup_t |\mathrm{Rm}_t|_{g(t)} < \infty$ (Rm_t: $g(t)$-curv. tensor)

Monotonicity of $\mathcal{F}$

Theorem 3

Suppose $\mathrm{Ent}_{v_0}(\mu_0) < \infty$ *and* $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_0^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \mathcal{F}(\mu_t) \text{ a.e. } t \in [0, T]$$

Corollary 4

$\mathcal{F}(\mu_t) \searrow$

- Rem: $g(t) \equiv g$, $\mathrm{Ric} \geq 0 \Rightarrow I(\mu_t) \searrow$
- [Lott '09] when M: cpt.
  by Eulerian calculus (requires smoothness)

Monotonicity of $\mathcal{W}$-entropy

Theorem 5

Suppose $\mathrm{Ent}_{v_0}(\mu_0) < \infty$ *and* $\mathcal{F}(\mu_0) < \infty$

$$\Rightarrow \lim_{s \downarrow 0} \frac{\mathcal{T}_{L_1^{t,t+s}}(\mu_t, \mu_{t+s})}{s} = \sqrt{t}\mathcal{F}(\mu_t) \text{ a.e. } t \in (0, T]$$

Corollary 6

$t^2\mathcal{F}(\mu_t) - \frac{mt}{2} \searrow$. *In particular,* $\mathcal{W}(\mu_t) \searrow$

$$\mathcal{W}(t) := t\mathcal{F}(\mu_t) - \mathrm{Ent}(\mu_t) - \frac{m \log t}{2} + \text{const.}$$

- [Topping '09] when M: cpt.

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$(\boldsymbol{X}_0(\boldsymbol{t}), \boldsymbol{X}_1(\boldsymbol{t}))$: coupling of BMs moving parallely

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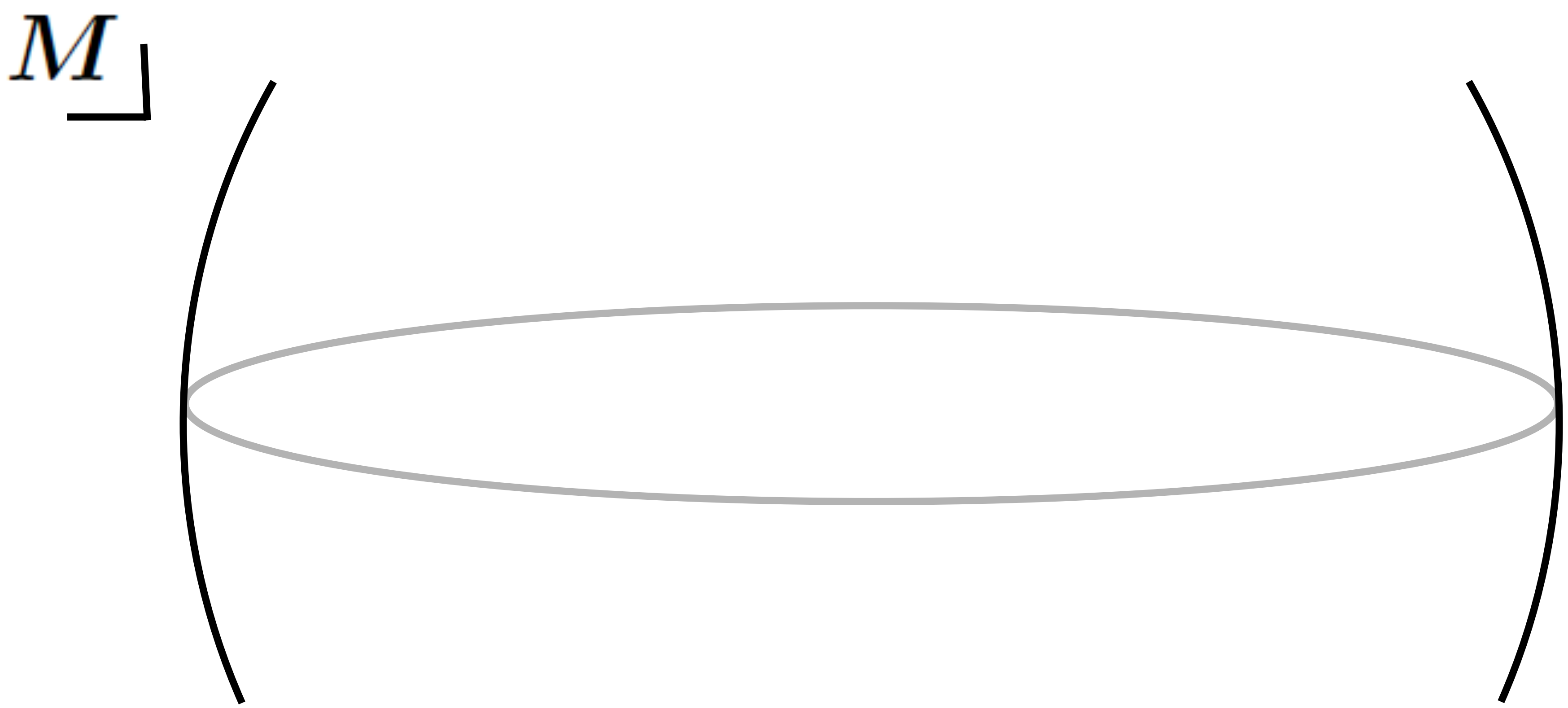
$(\boldsymbol{X}_0(t), \boldsymbol{X}_1(t))$: coupling of BMs moving parallelly



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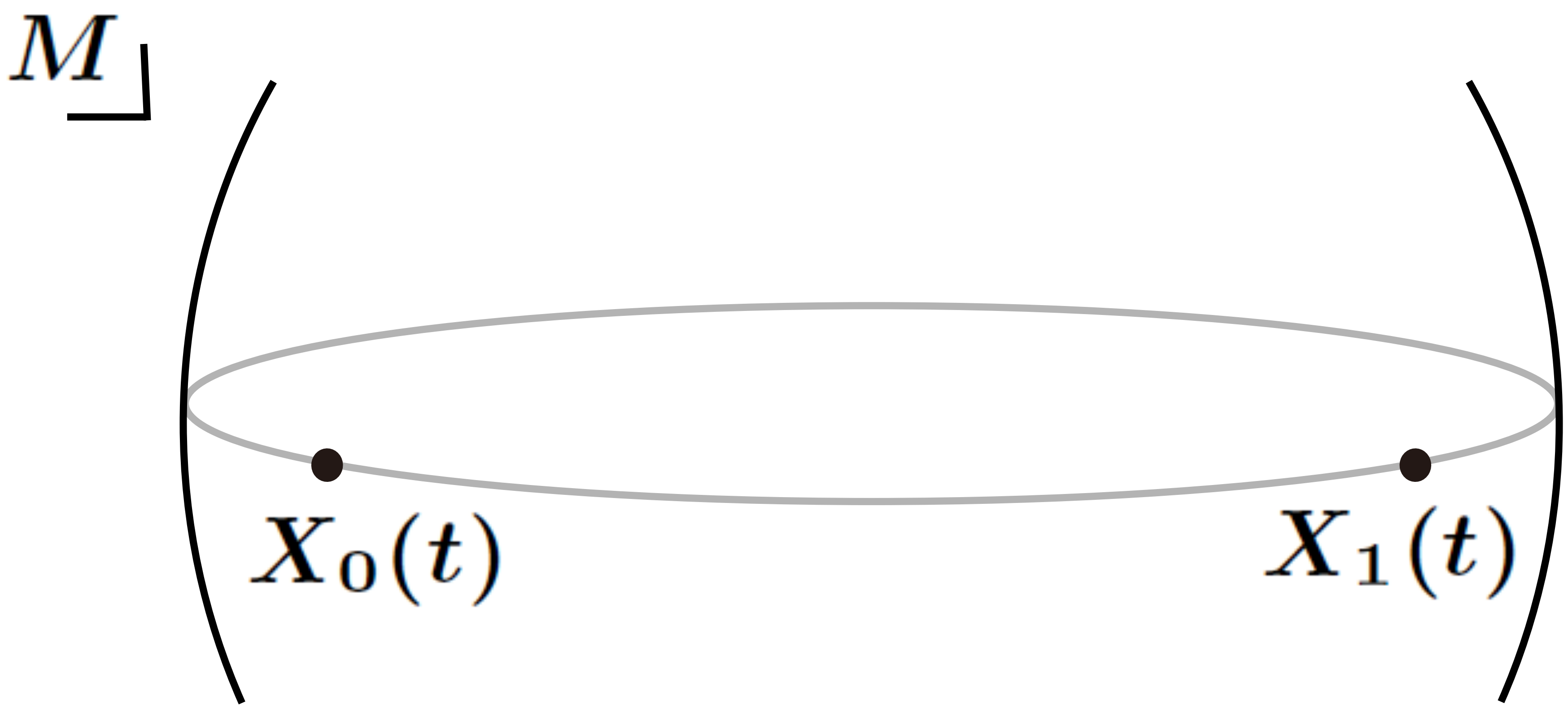
$(X_0(t), X_1(t))$: coupling of BMs moving parallelely



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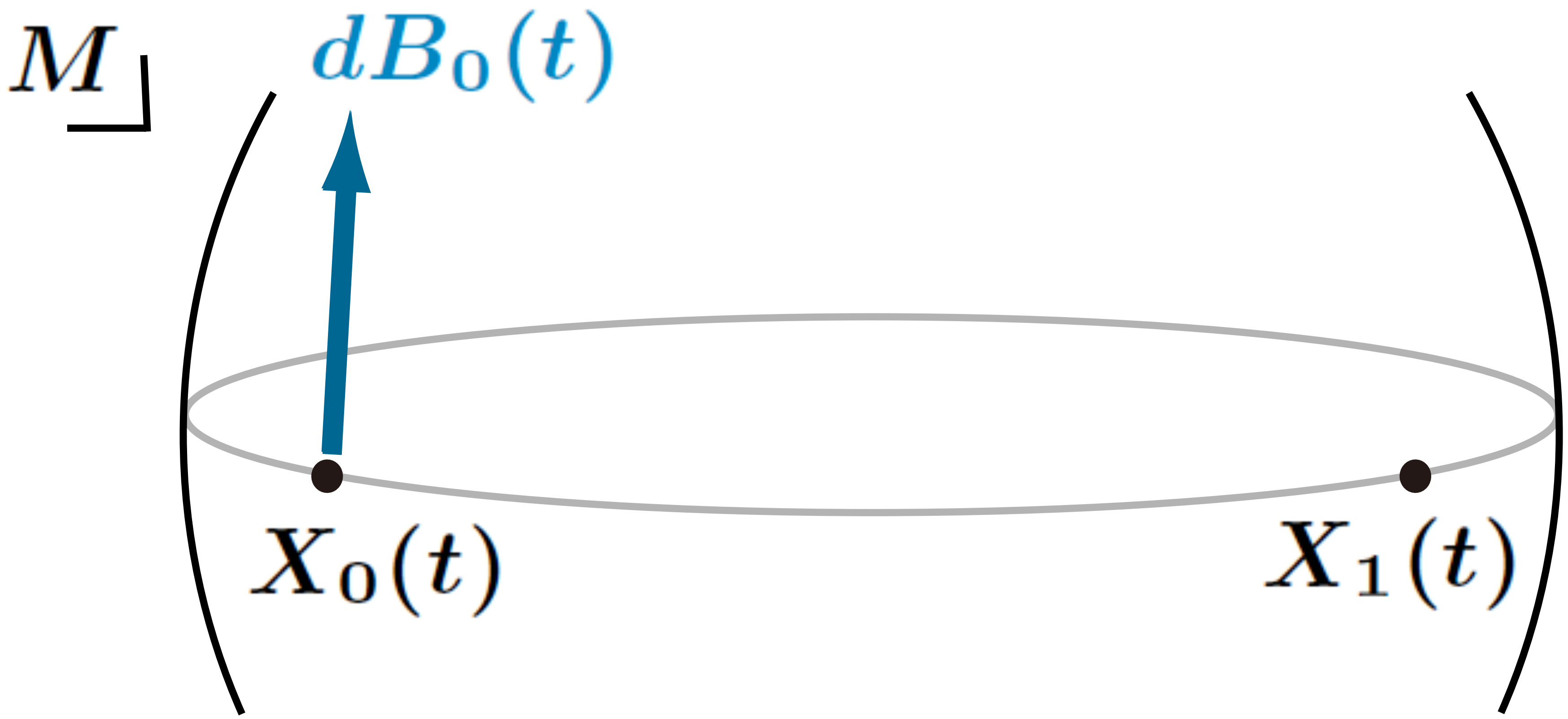

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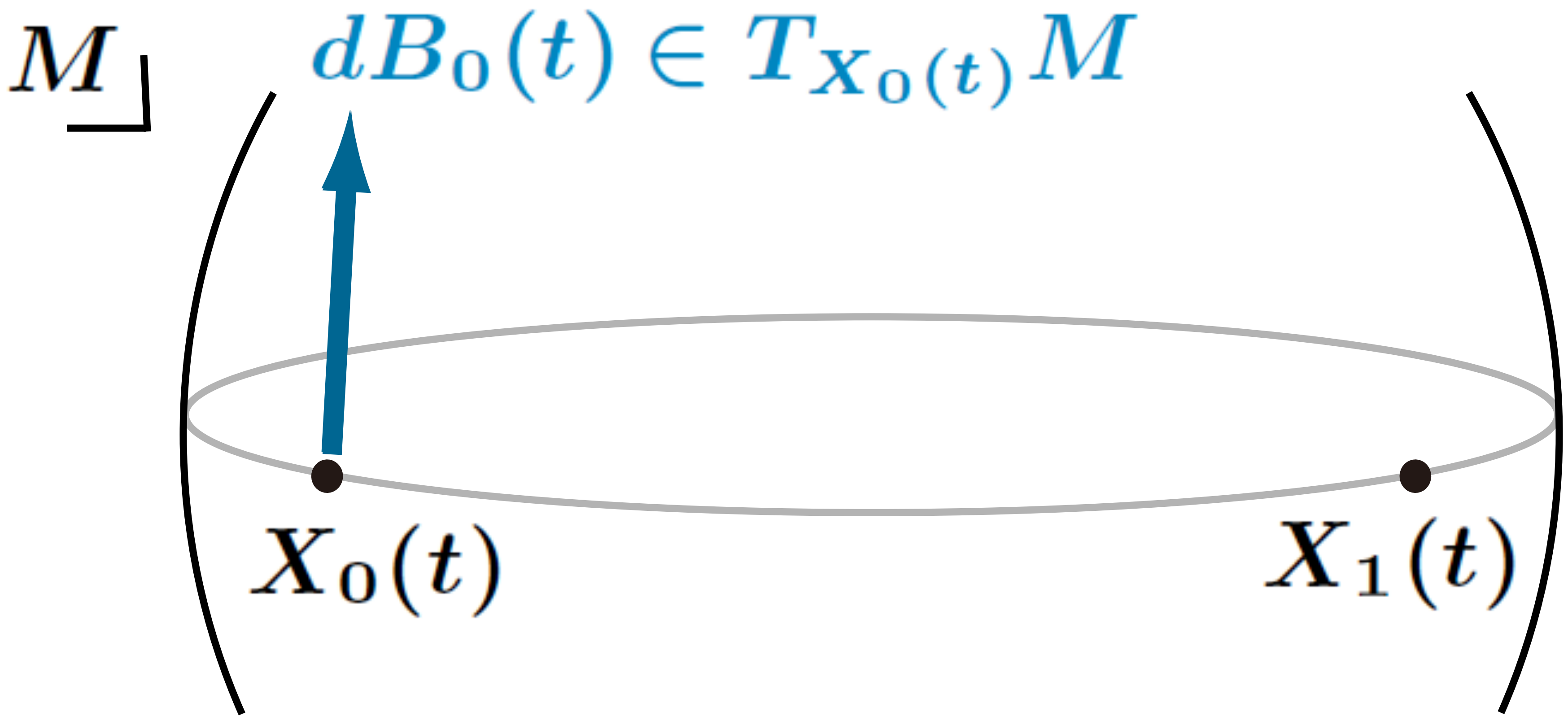

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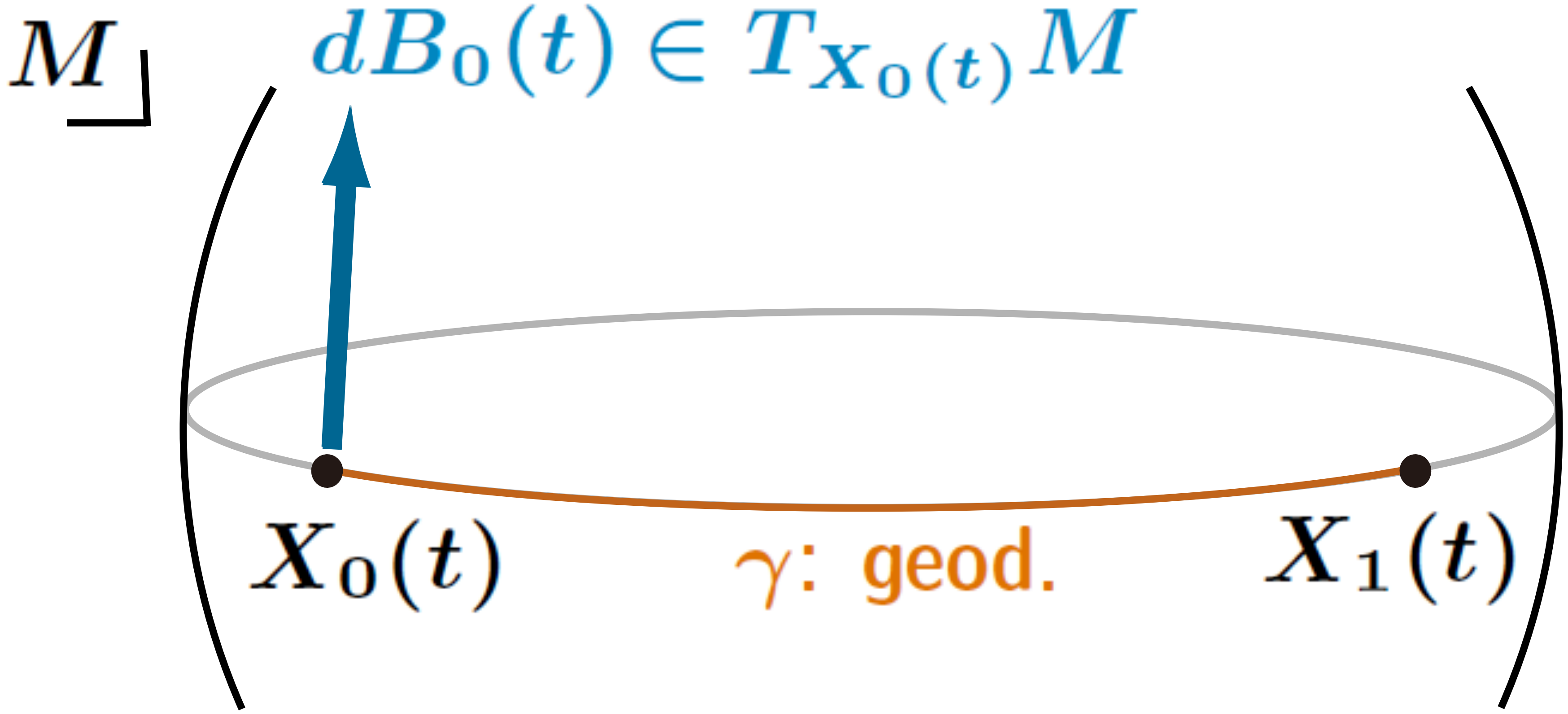

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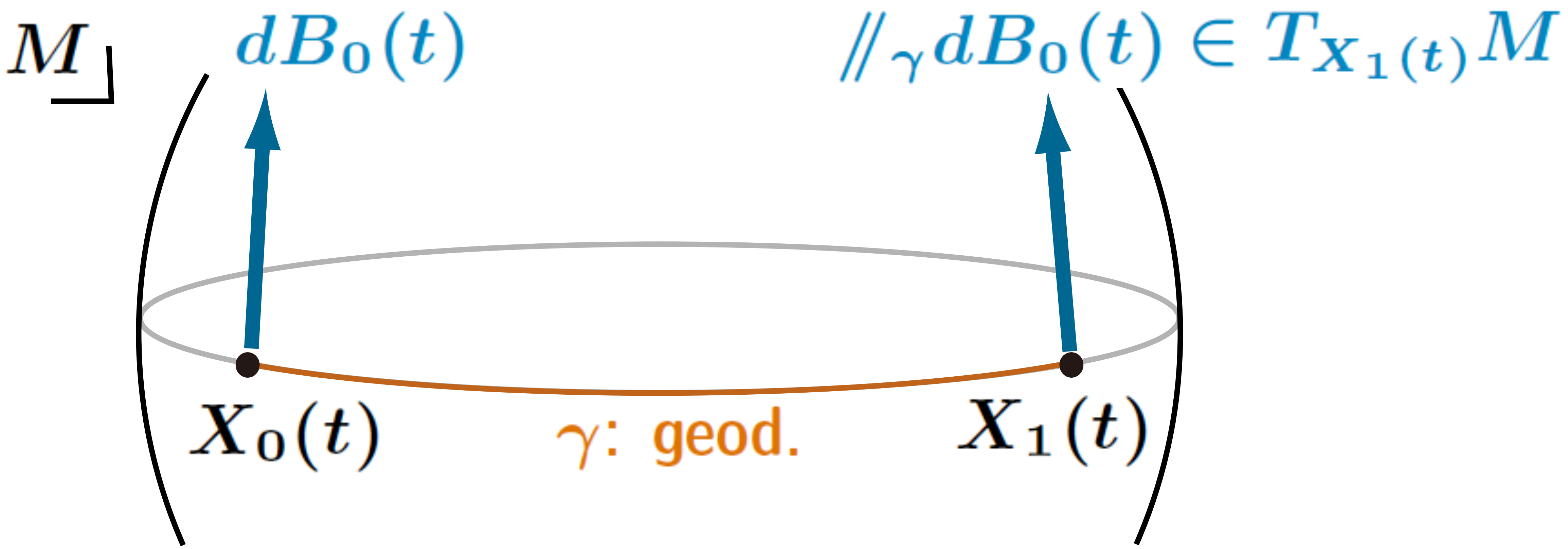

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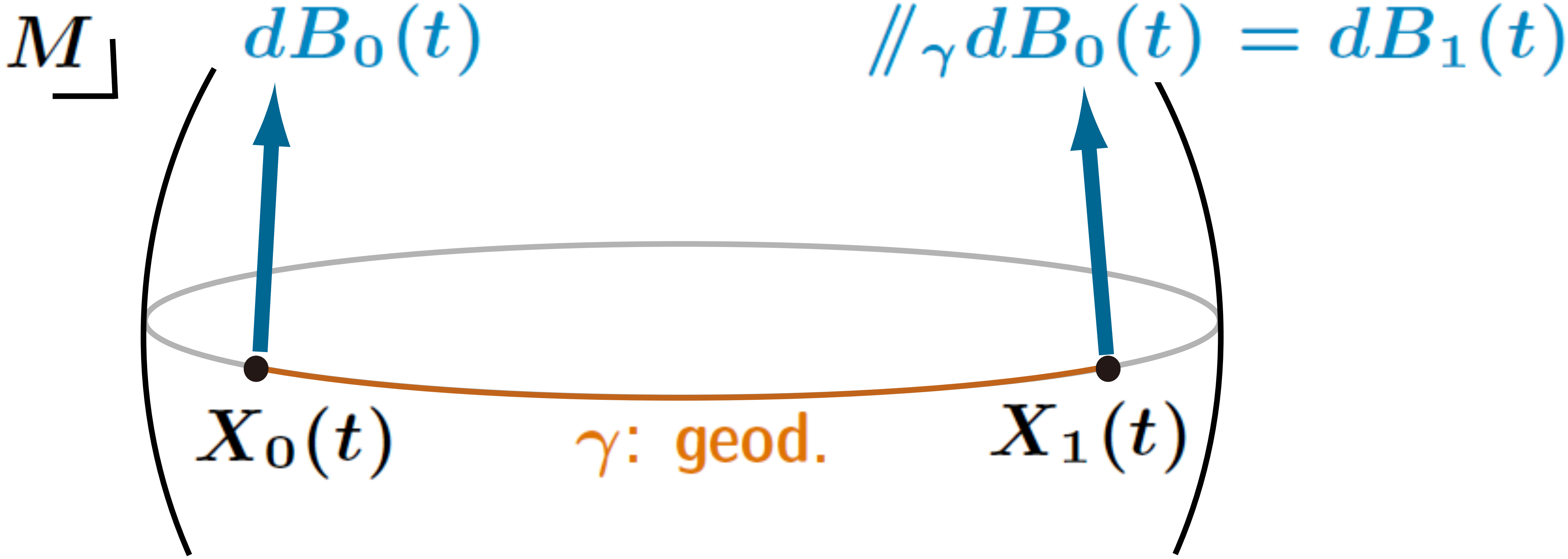

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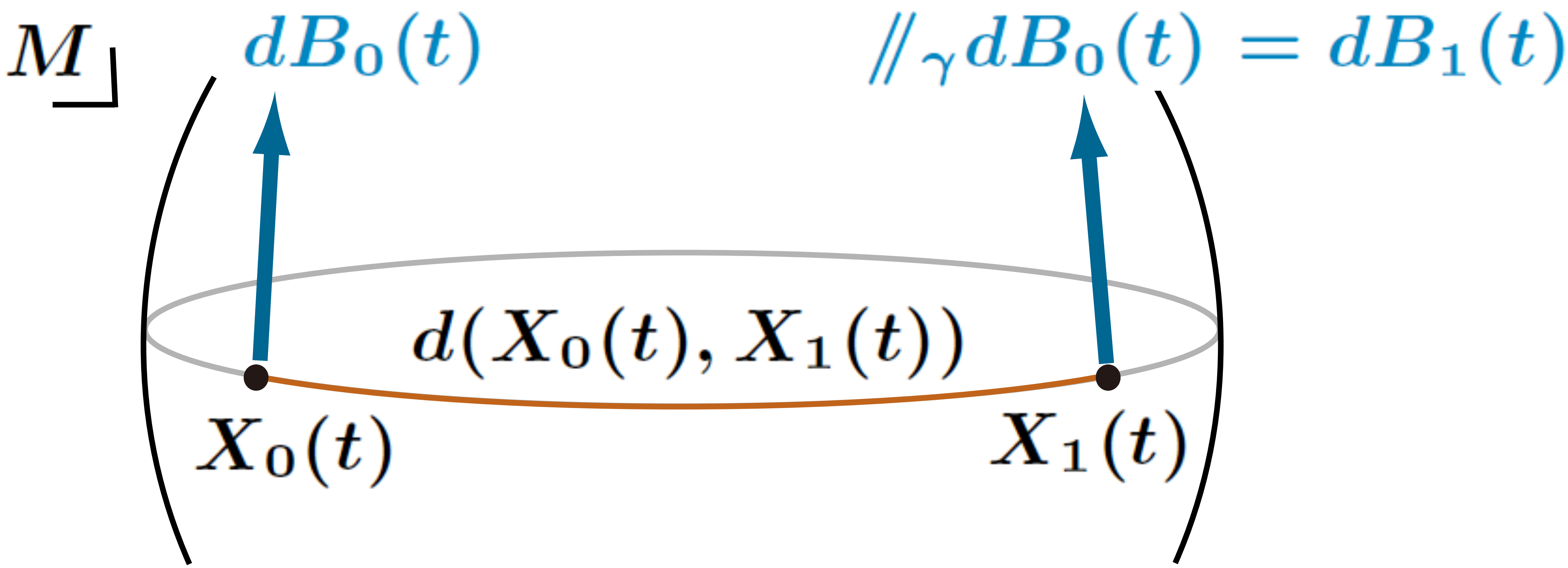

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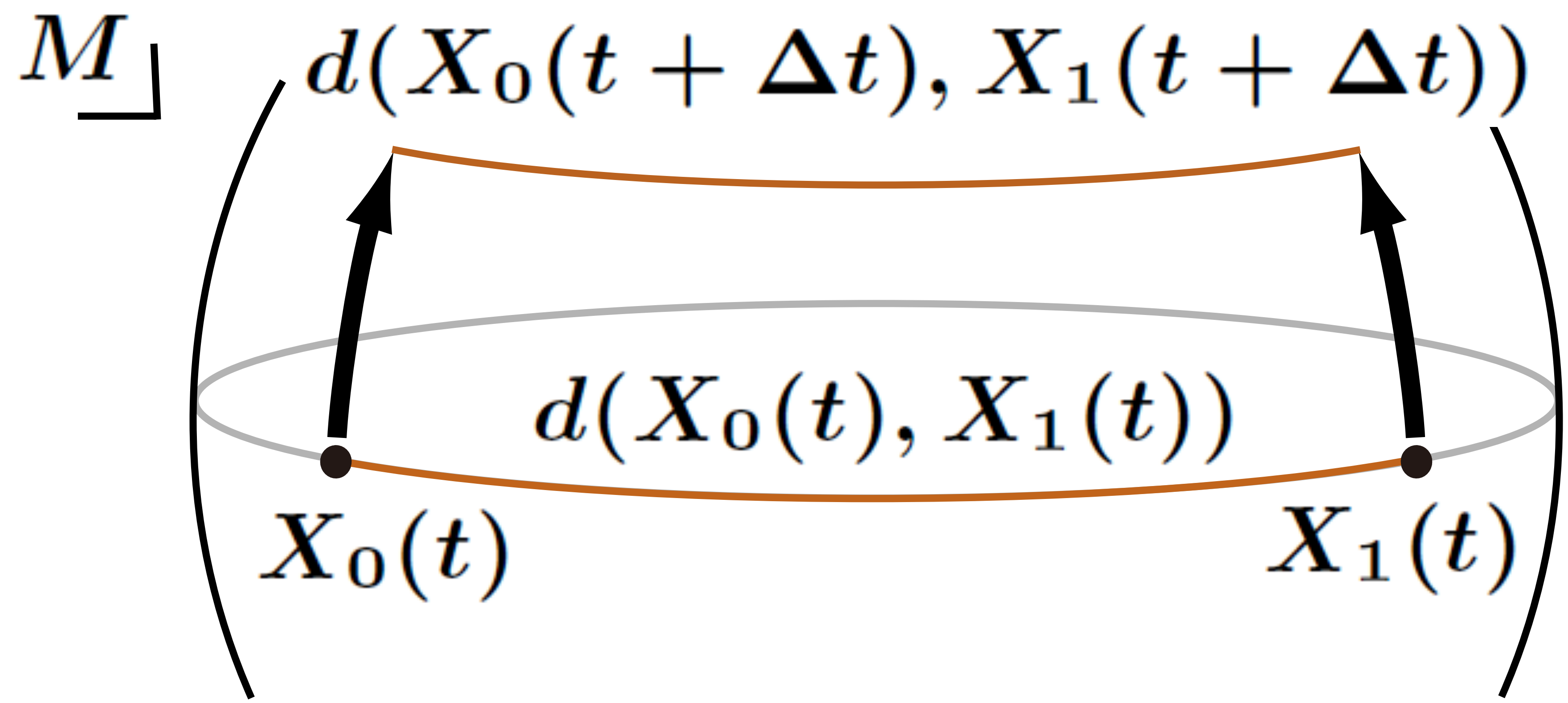

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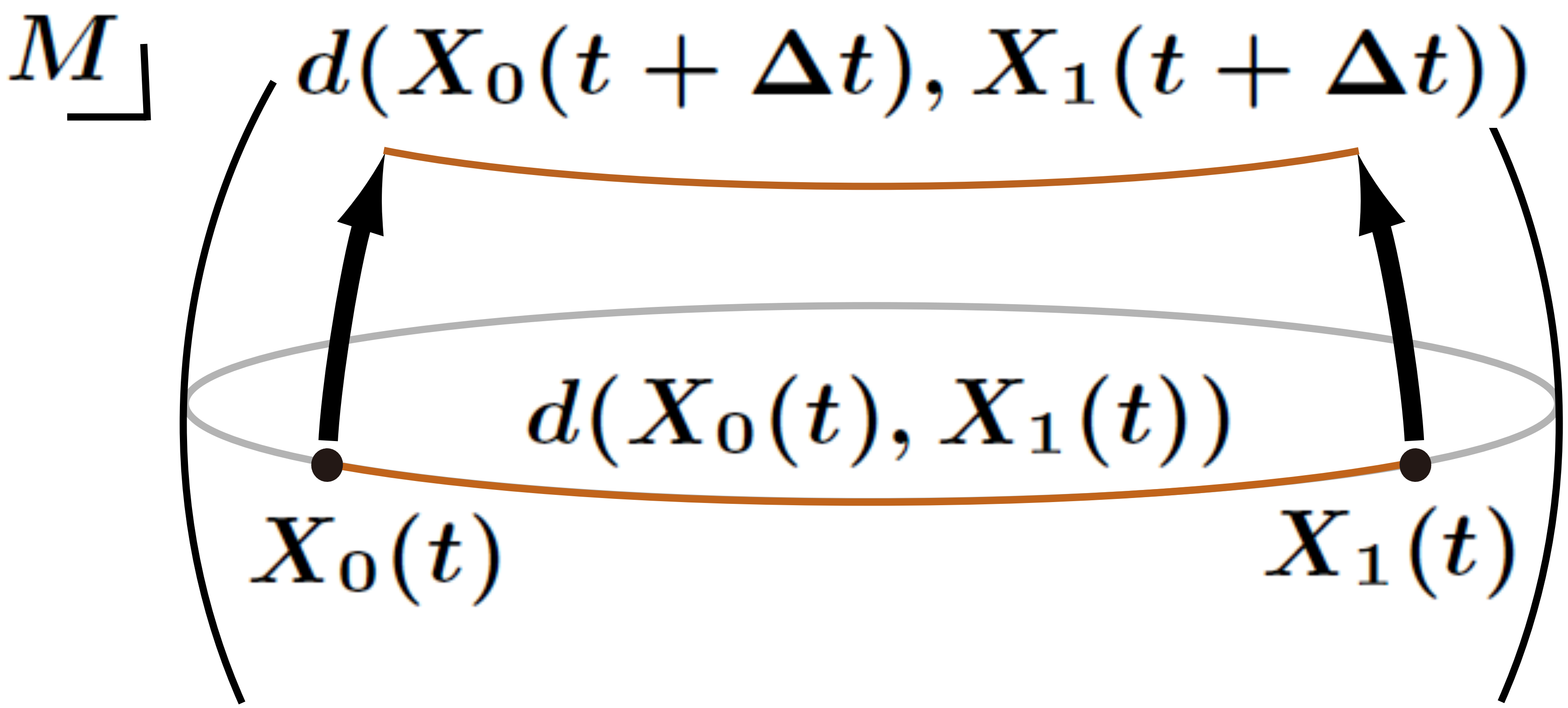

- (mart. part of $d(X_0(t), X_1(t))) = 0$

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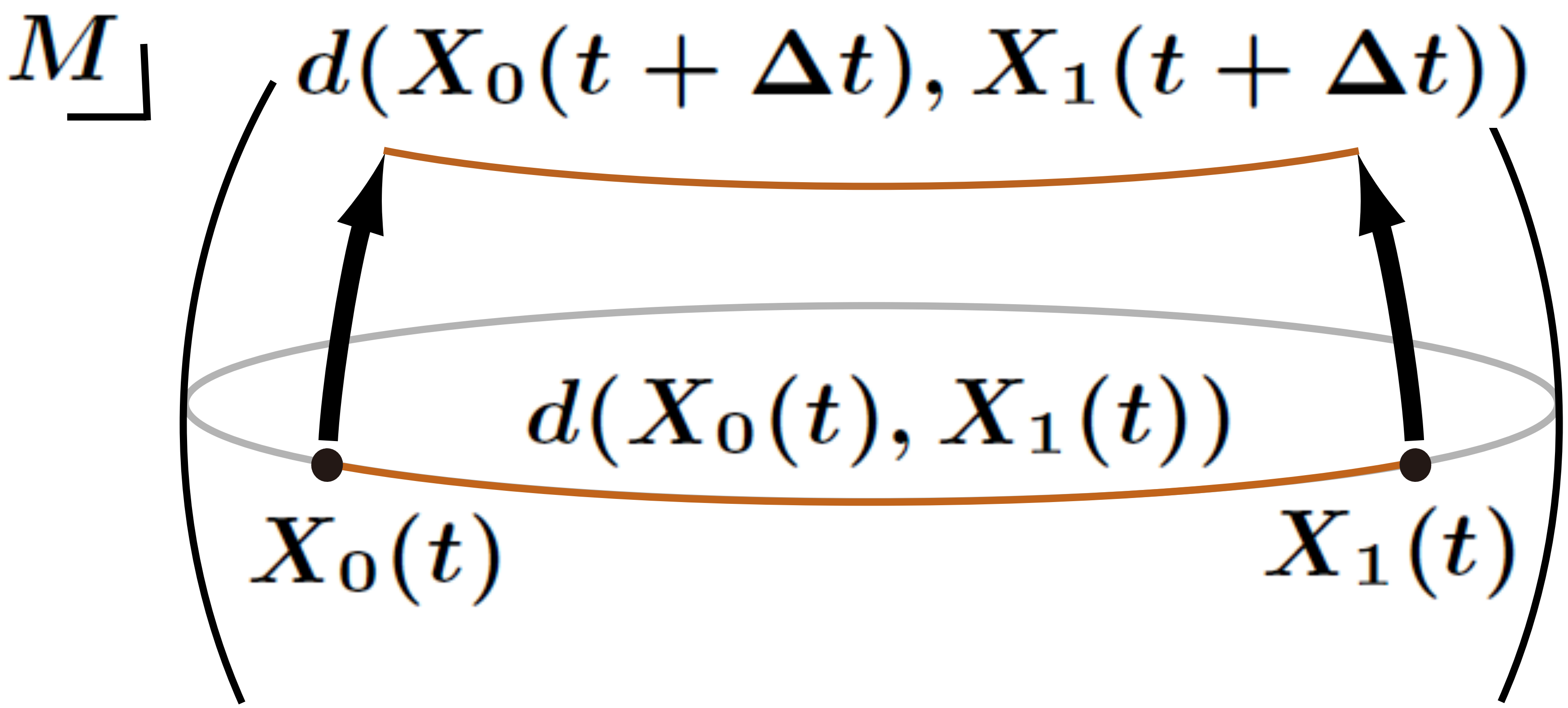

- (mart. part of $d(X_0(t), X_1(t))$) $= 0$
- ("bdd. var." part): Controlled by $\mathrm{Ric} \geq K$

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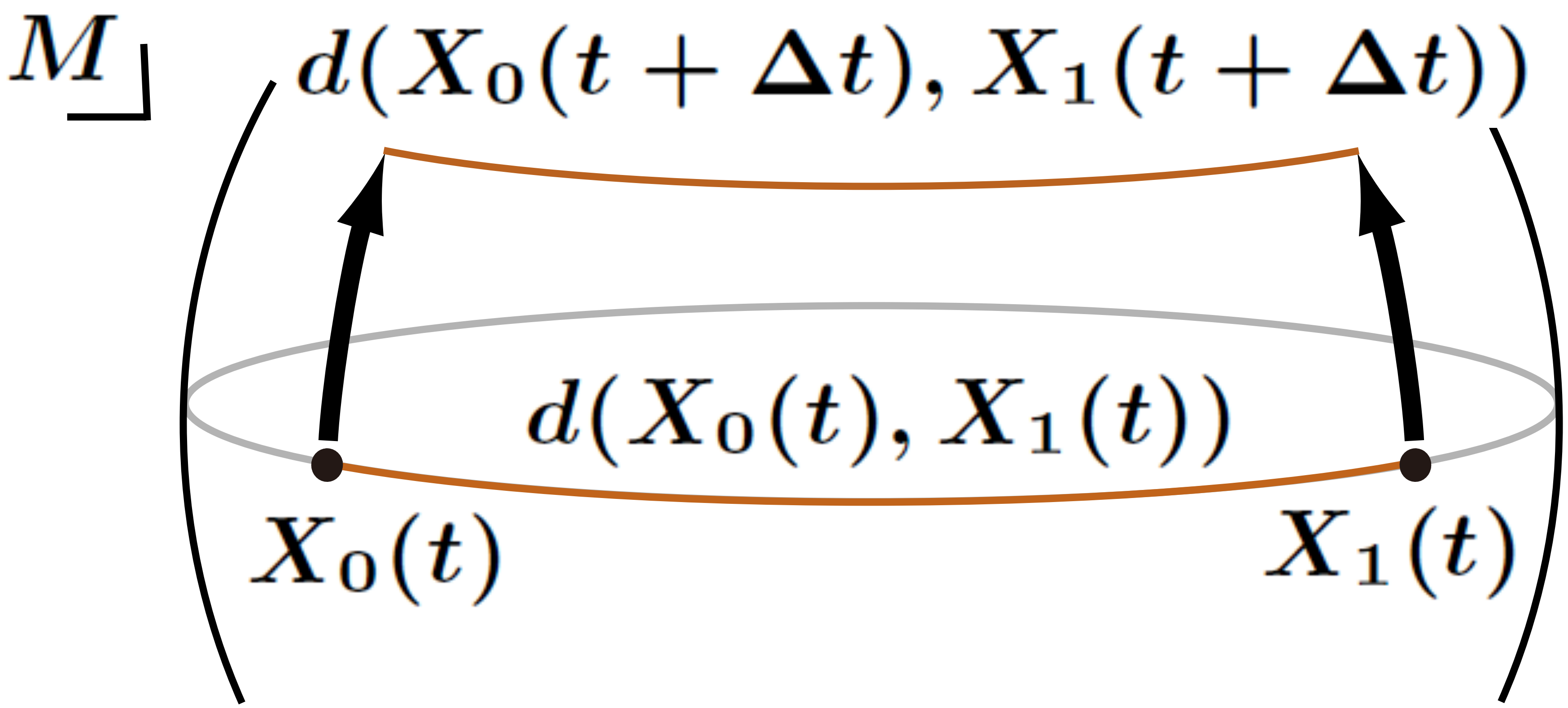

- (mart. part of $d(X_0(t), X_1(t))) = 0$
- ("bdd. var." part): Controlled by $\mathrm{Ric} \geq K$

$$\Downarrow$$

$$\text{“}\frac{\partial}{\partial t} d(X_0(t), X_1(t)) \leq -K d(X_0(t), X_1(t))\text{”}$$

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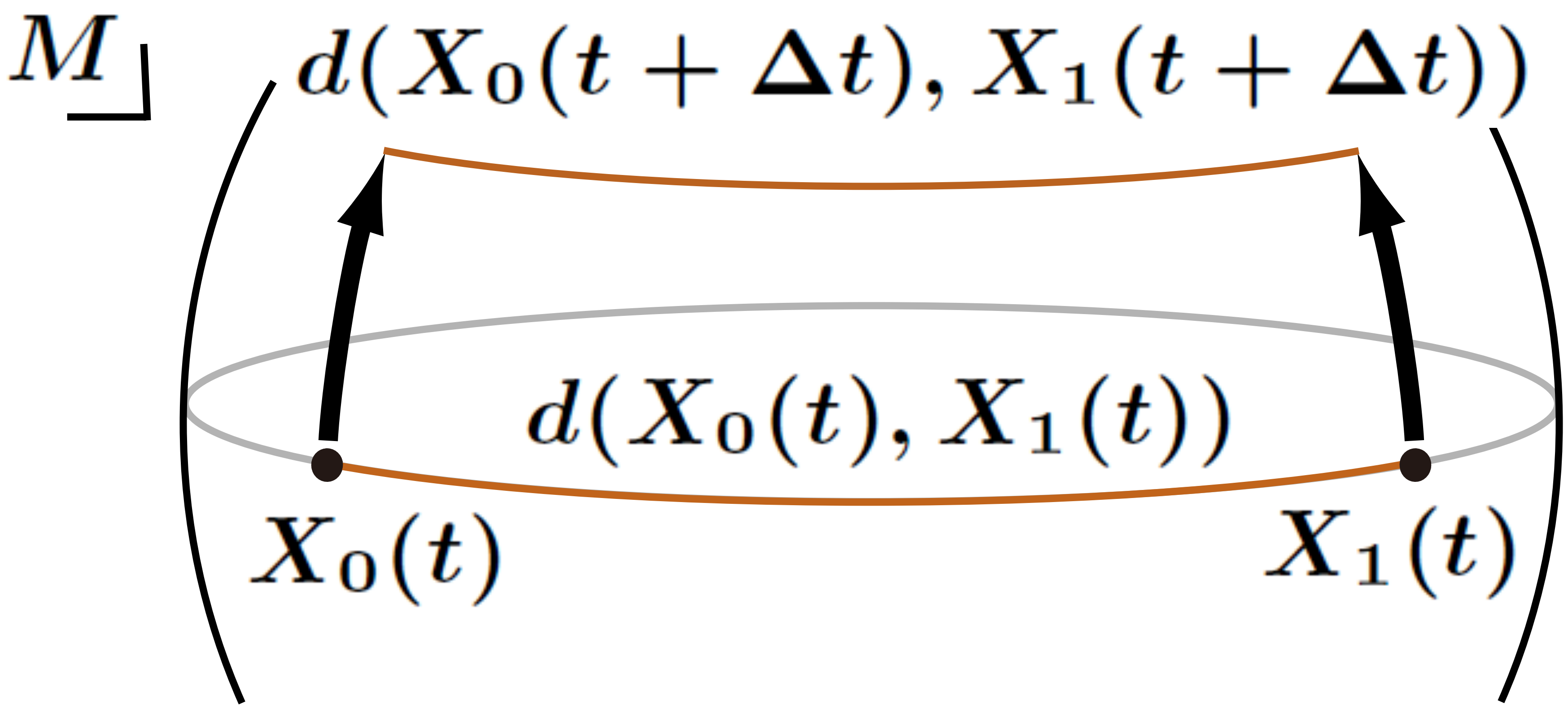

$\therefore \mathrm{Ric} \geq K$

$$\Rightarrow \mathrm{e}^{pKt}\mathcal{T}_{d^p}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow \text{ in } t \quad (1 \leq {}^\forall p < \infty)$$

Ricci flow case (for L_0/L_1)

- Properties of L_0

  being analogous to the Riem. dist.

  $\left(\begin{array}{l}\text{geodesic (minizing curve), 1st \& 2nd variation,} \\ \text{index lemma, cut locus, } \ldots\end{array}\right)$

- Coupling of $dX_0(t)$ and $dX_1(t+s)$

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- Coupling of $dX_0(t)$ and $dX_1(t+s)$

  by space-time parallel transport

For $\gamma : [s, t] \to M$ & V: vector field along γ,

$$\nabla^{g(u)}_{\dot\gamma(u)} V(u) = -\frac{1}{2}\partial_u g(u)^{\#} V(u)$$

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- Coupling of $dX_0(t)$ and $dX_1(t+s)$

  by space-time parallel transport along L_0-geodesic

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Ricci flow case (for L_0/L_1)

- Properties of $\boldsymbol{L_1}$
  being analogous to the Riem. dist.
  $\left(\begin{array}{l}\text{geodesic (minizing curve), 1st \& 2nd variation,}\\ \text{index lemma, cut locus, } \ldots\end{array}\right)$
- Coupling of $\boldsymbol{dX_0(\tau_0 t)}$ and $\boldsymbol{dX_1(\tau_1 t)}$
  by space-time parallel transport along $\boldsymbol{L_1}$-geodesic
  & scaling

For $\gamma : [s, t] \to M$ & V: vector field along γ,

$$\nabla^{g(u)}_{\dot{\gamma}(u)} V(u) = -\frac{1}{2}\partial_u g(u)^{\#} V(u)$$

Ricci flow case (for L_0/L_1)

<u>Technicalities</u>

- Non-smootheness of L_0/L_1 at their cut loci
  $\leftsquigarrow$ Approximation by coupling of random walks
  (Differential ineq. $\rightsquigarrow$ Difference ineq.)
- Lack of a (global) upper bound of Ric
  $\leftsquigarrow$ Localization by stopping times

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Remark

- Many other approaches in time-homogeneous case
- A method in [Arnaudon, Coulibaly & Thalmaier '09] does not seem to work

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Kantorovich duality

Observation: $\mathcal{T}_{d^2}$ under $g(t) \equiv g$

$$\frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{2s} = \sup_{\varphi \in C_b} \left[\int_M Q_s\varphi \, d\mu_{t+s} - \int_M \varphi \, d\mu_t \right]$$

$$Q_s\varphi(x) := \inf_{y \in M} \left(\varphi(y) + \frac{d(y,x)^2}{2s} \right)$$

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$$= \inf_{\substack{\gamma:[t,t+s]\to M \\ \gamma(t+s)=x}} \left(\varphi(\gamma(t)) + \frac{1}{2} \int_t^{t+s} |\dot{\gamma}(r)|^2 \, dr \right)$$

(Hopf-Lax semigroup)

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$$= \inf_{\substack{\gamma:[t,t+s]\to M \\ \gamma(t+s)=x}} \left(\varphi(\gamma(t)) + \frac{1}{2} \int_t^{t+s} |\dot{\gamma}(r)|^2 \, dr \right)$$

(Hopf-Lax semigroup)

$$\bigstar \ \partial_s Q_s\varphi + \frac{1}{2} |\nabla Q_s\varphi|^2 = 0 \quad \text{(Hamilton-Jacobi eq.)}$$

Upper bound

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$$(\partial_r \cdots) = \int \left(-\langle \nabla Q_r\varphi, \frac{\nabla \rho_{t+r}}{\rho_{t+r}} \rangle - \frac{1}{2} |\nabla Q_r\varphi|^2 \right) d\mu_{t+r}$$

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Outline of the proof of Thm 3

- Kantorovich duality / Hopf-Lax semigr. for $L_0^{t,t+s}$
- Difficulty: Dependency on t of geometry

  No a priori integrability of $\nabla \rho_t$ & $\Delta_t \rho_t$

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1. Introduction
2. Heat distributions on backward Ricci flow
3. Coupling methods (Thm 1 & 2)
4. Idea of the proof of Thm 3 & 5
5. **Further problems**

A remark on lower bound

Alternative proof for $\mathcal{T}_{d^2}$ in the time-homogeneous case:

$\mathbf{Ric} \geq \boldsymbol{K} \Rightarrow$ Wang's Harnack inequality

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($\because$ Kantorovich duality)

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$$g_1 := c^{-1}\alpha \log P_s \rho_t, \quad g_0 := c^{-1} \log P_s(\rho_t^\alpha)$$

$$\begin{aligned}
\varliminf_{s \downarrow 0} \frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{s^2} \\
&\geq \varliminf_{s \downarrow 0} \frac{1}{s^2} \left(\int g_1 \, d\mu_{t+s} - \int g_0 \, d\mu_t \right) \\
&= \cdots = 4(\alpha - 1)(2 - \alpha) I(\mu_t)
\end{aligned}$$

$\Downarrow \alpha = 3/2$

$$\varliminf_{s \downarrow 0} \frac{\mathcal{T}_{d^2}(\mu_t, \mu_{t+s})}{s^2} \geq I(\mu_t) \qquad \square$$

$\mathcal{T}_{d^p}$ (or W_p)

- For $\mathcal{T}_{d^p}$ ($p \in [1, \infty)$) in time-homogeneous case?

† $\mathrm{Ric} \geq K \Rightarrow \mathrm{e}^{pKt} \mathcal{T}_{d^p}(\mu_t^{(0)}, \mu_t^{(1)}) \searrow$

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$\mathrm{Ric} \geq K > 0$, $\mu_t \to v \in \mathcal{P}(M)$

$$\Rightarrow \boxed{\mathcal{T}_{d^p}(\mu, v) \leq \frac{1}{K^p} \int_M \frac{|\nabla \rho|^p}{\rho^{p-1}} dv}$$

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- $u_t \in L^1_+(v)$: The solution to the p_*-heat eq.

$$\partial_t u = \mathrm{div}(|\nabla u|^{p_*-2}\nabla u)$$

(or gradient flow of $\int |\nabla f|^{p_*} dv$)

$$\Rightarrow \overline{\lim_{s\downarrow 0}} \frac{W_p(u_t v, u_{t+s} v)^p}{s^p} = \int \frac{|\nabla u_t|^{p_*}}{u_t^{p-1}} dv$$

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Q. Monotonicity of W_p on "Riemannian" spaces?

(False on Finsler mfds with $p = 2$)

$\mathcal{W}$-entropy

Facts

- Monotonicity of $\mathcal{W}$-entropy

$$\mathcal{W}(t) := tI(\mu_t) - \mathrm{Ent}(\mu_t) - \frac{N}{2}\log t + \text{(const.)}$$

on N-dim. Riem. mfds with $\mathrm{Ric} \geq 0$

- Rigidity: $\mathcal{W}$-entropy is constant iff $M \simeq \mathbb{R}^n$

[L. Ni '04], [X.-D. Li '11], ...

Q. (X.-D. Li) The same for **RCD**$^*(0, N)$ spaces?
(The proof on smooth spaces relies on differential calc.)

★ Monotonicity holds on **RCD**$^*(0, N)$ met. meas. sp.