Coupling by reflection of Brownian motions on metric measure spaces with a lower Ricci curvature bound

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1. Introduction

Brownian motion $\leftrightarrow$ curvature

M: manifold

- Riem. met. $g \leftrightarrow$ Laplace-Beltrami op. Δ
  $\leftrightarrow$ Heat semigroup $P_t = \mathrm{e}^{t\Delta}$
  $\leftrightarrow$ Brownian motion B_t on M
- Shape of $(M, g) \leftrightsquigarrow$ "curvature" of (M, g)
- $\Delta = \mathrm{tr}\,\mathrm{Hess}$: "averaged (in direction)" 2nd deriv.

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  "$\leftrightsquigarrow$" averaged (in direction) curvature
  (Ricci curvature)

Recent developments on “$\mathrm{Ric} \geq K$”

- Characterizations of “$\mathrm{Ric} \geq K$” on met. meas. sp.
- Equivalence of characterizations:

  [Ambrosio, Gigli & Savaré ’13–’15]

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Couplings of B_t

- Coupling by parallel transport [Sturm]
- Coupling by reflection

Review: Couplings of BMs

(M, g): Riem. mfd, $\mathrm{Ric} \geq K$

$(B_t^{(0)}, B_t^{(1)})$: coupling by parallel transport of BMs

$$\Rightarrow d(B_t^{(0)}, B_t^{(1)}) \leq \mathrm{e}^{-Kt} d(B_0^{(0)}, B_0^{(1)})$$

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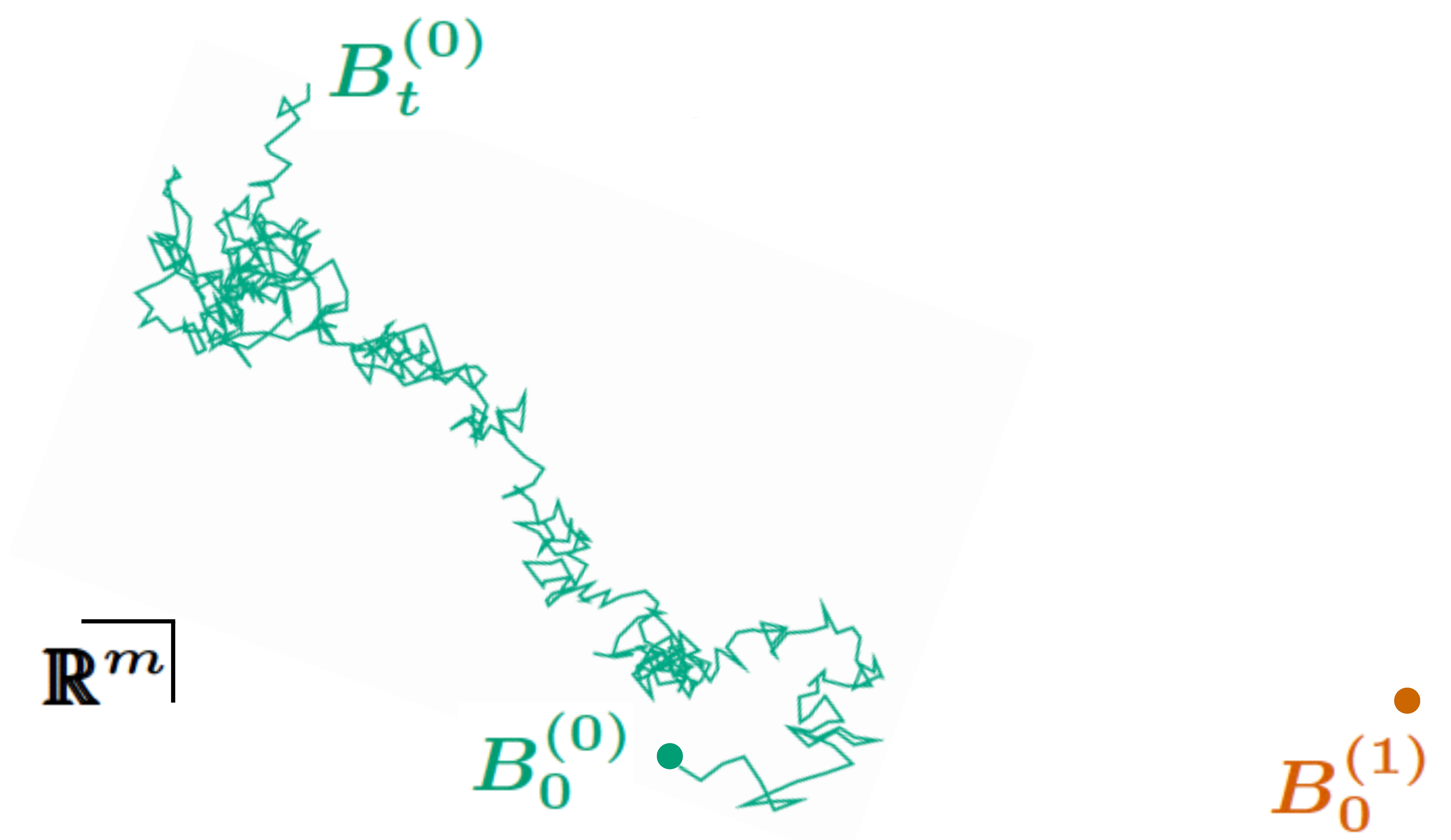

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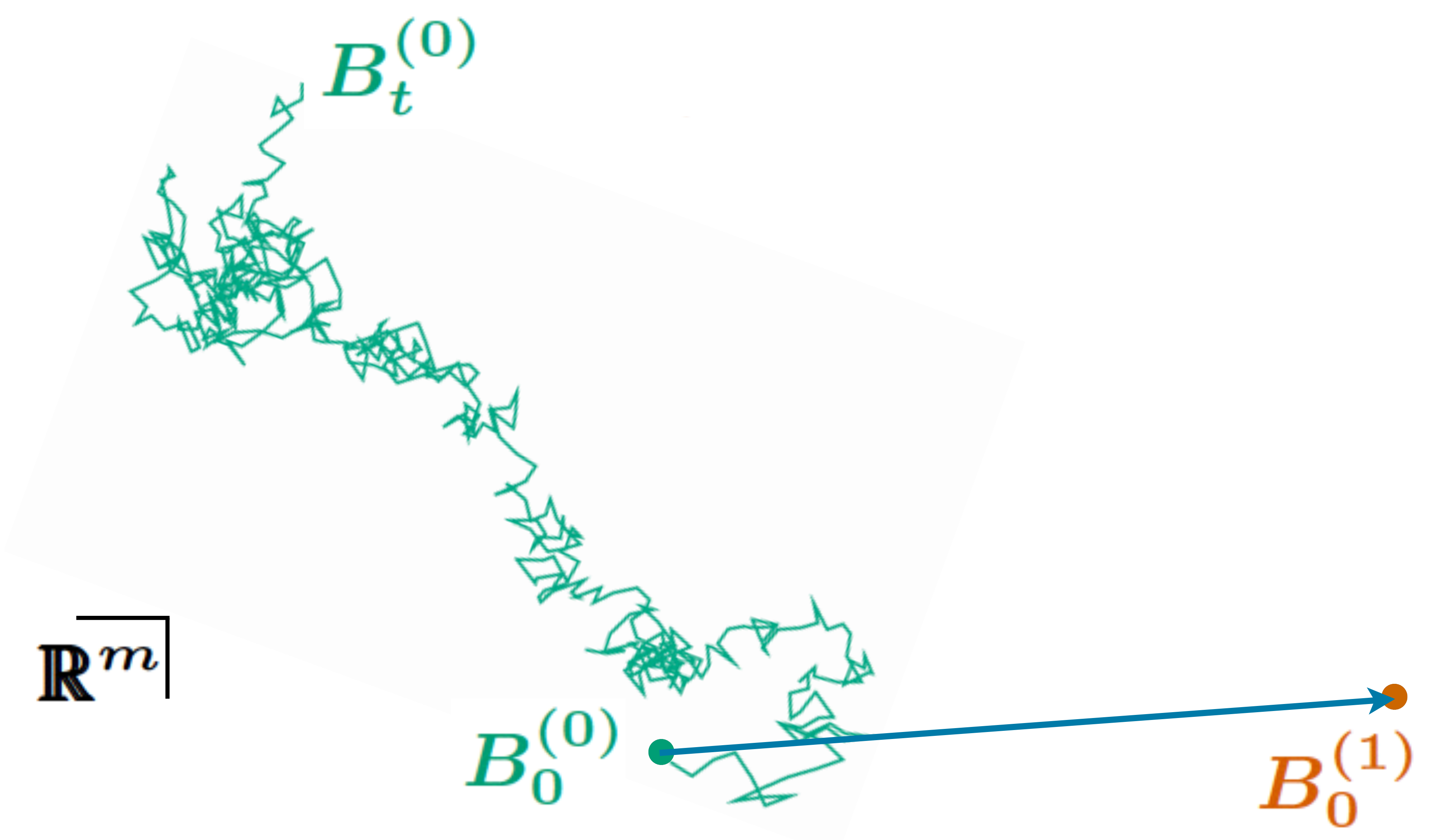

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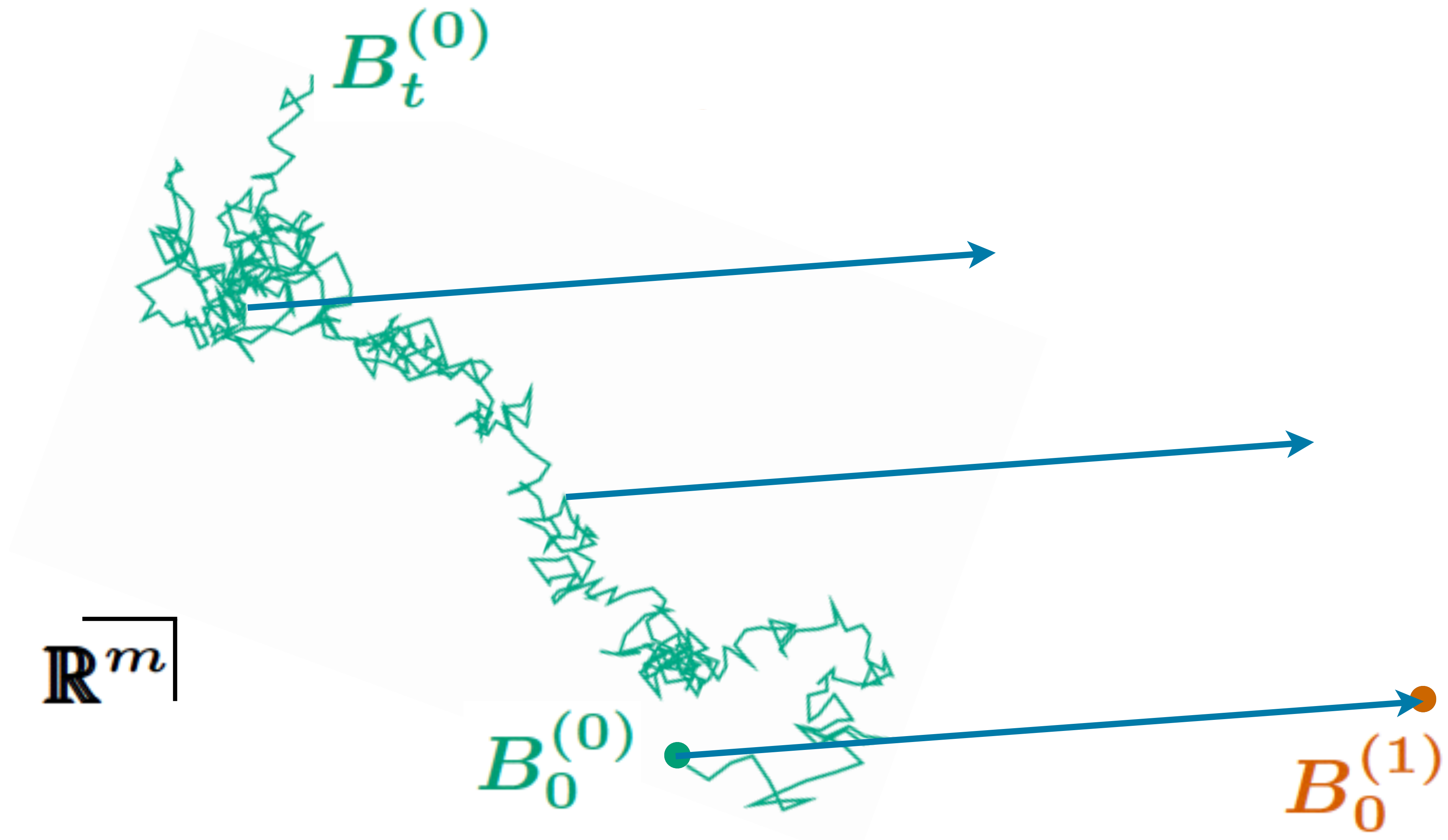

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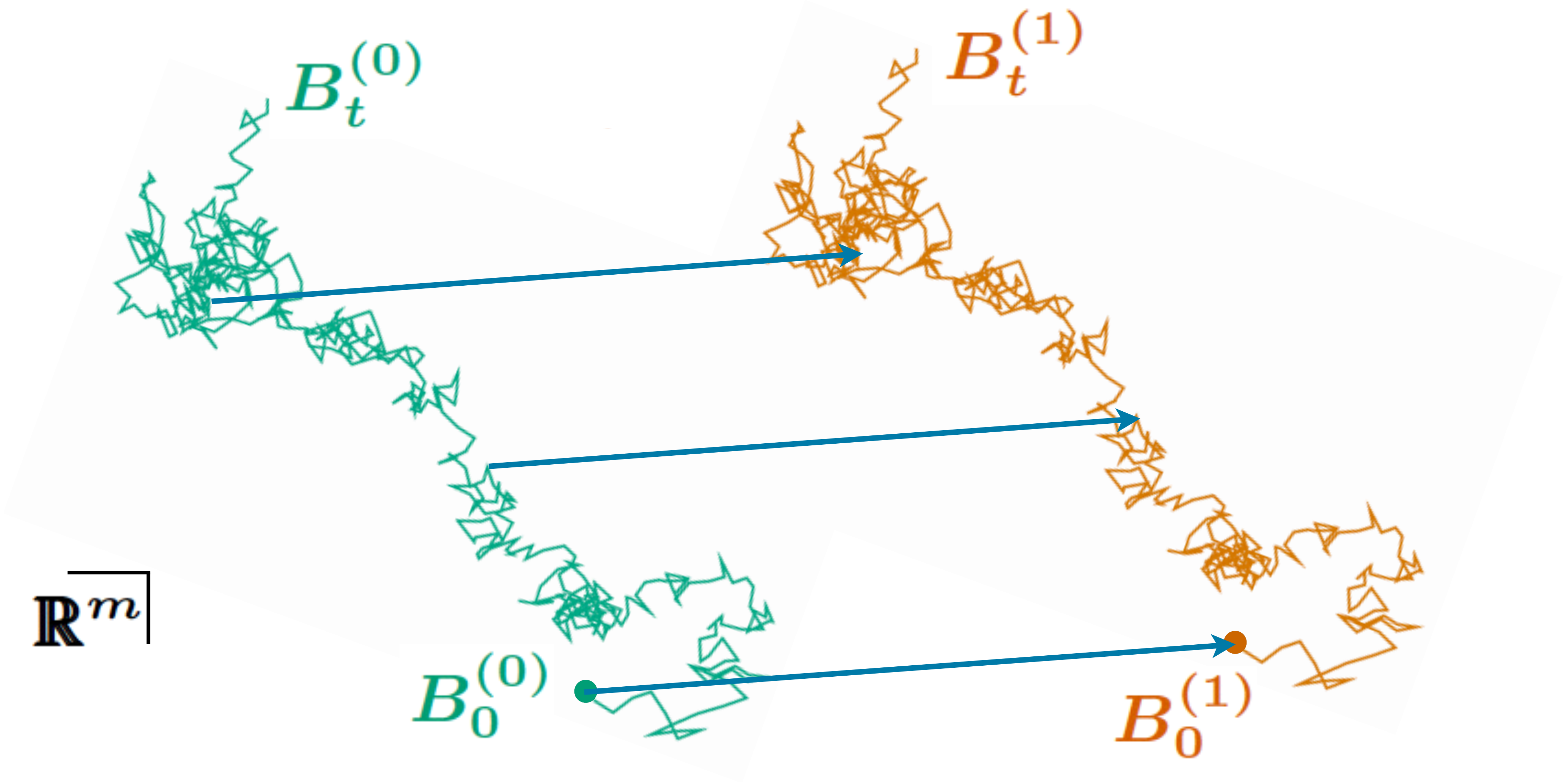

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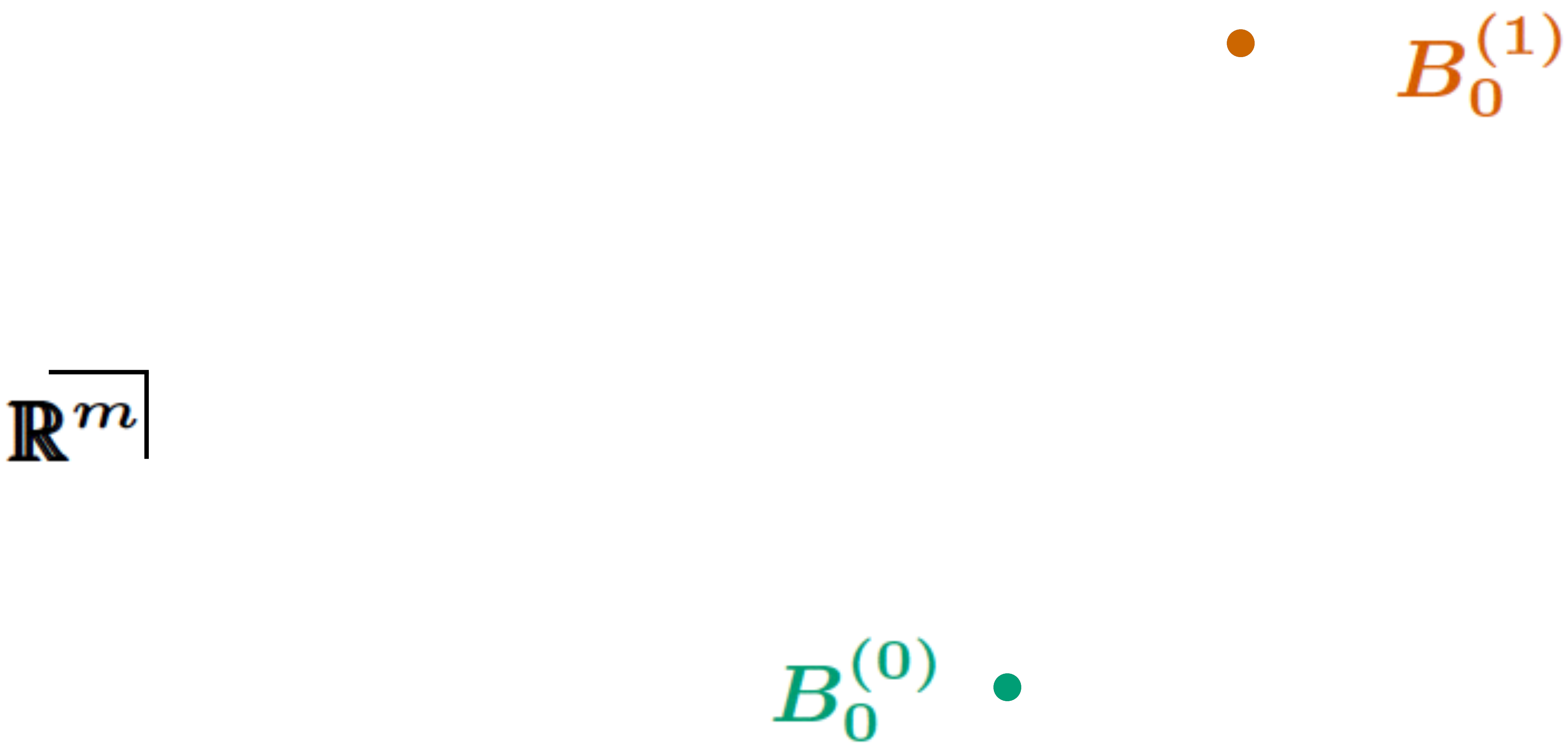

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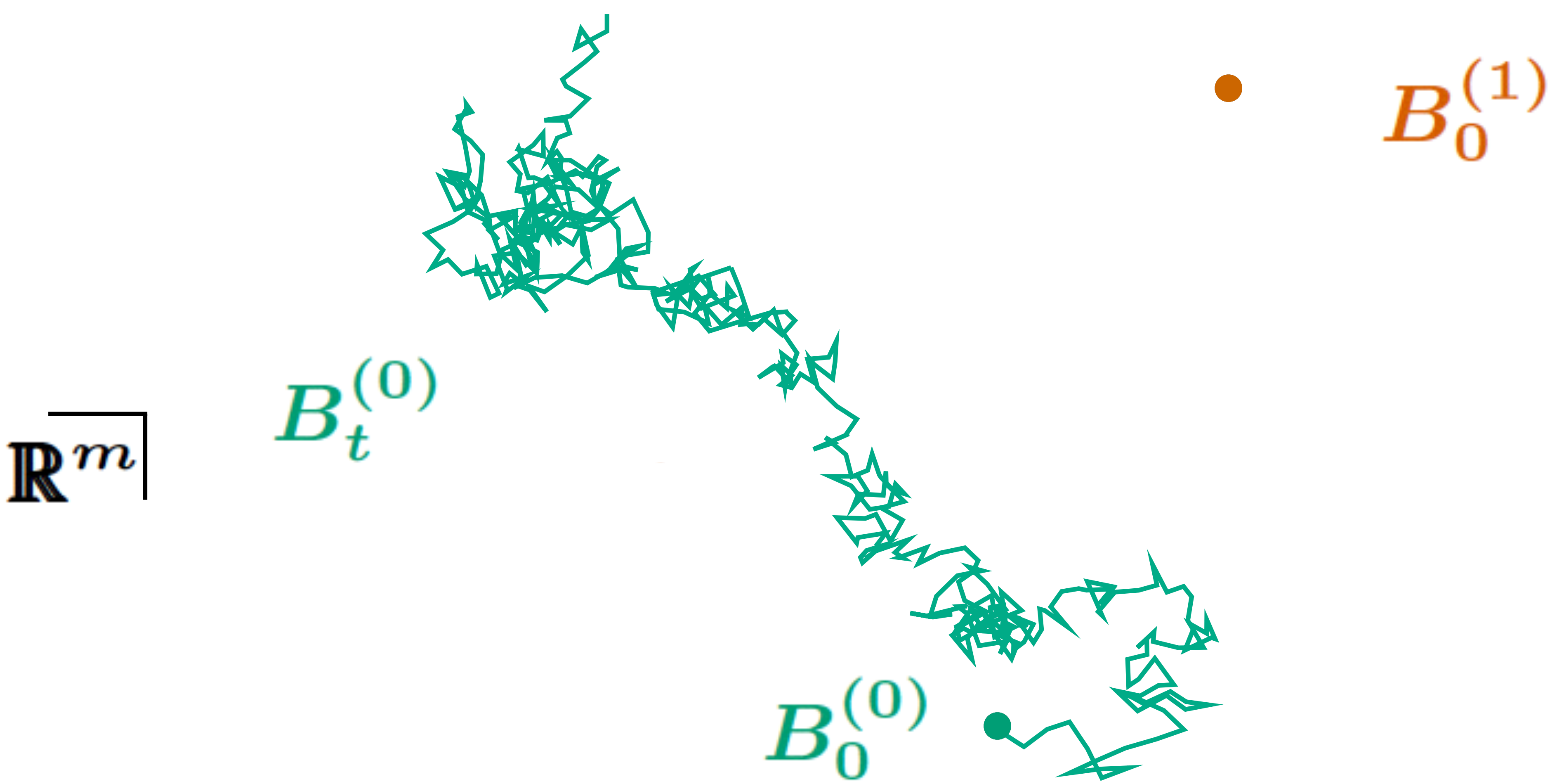

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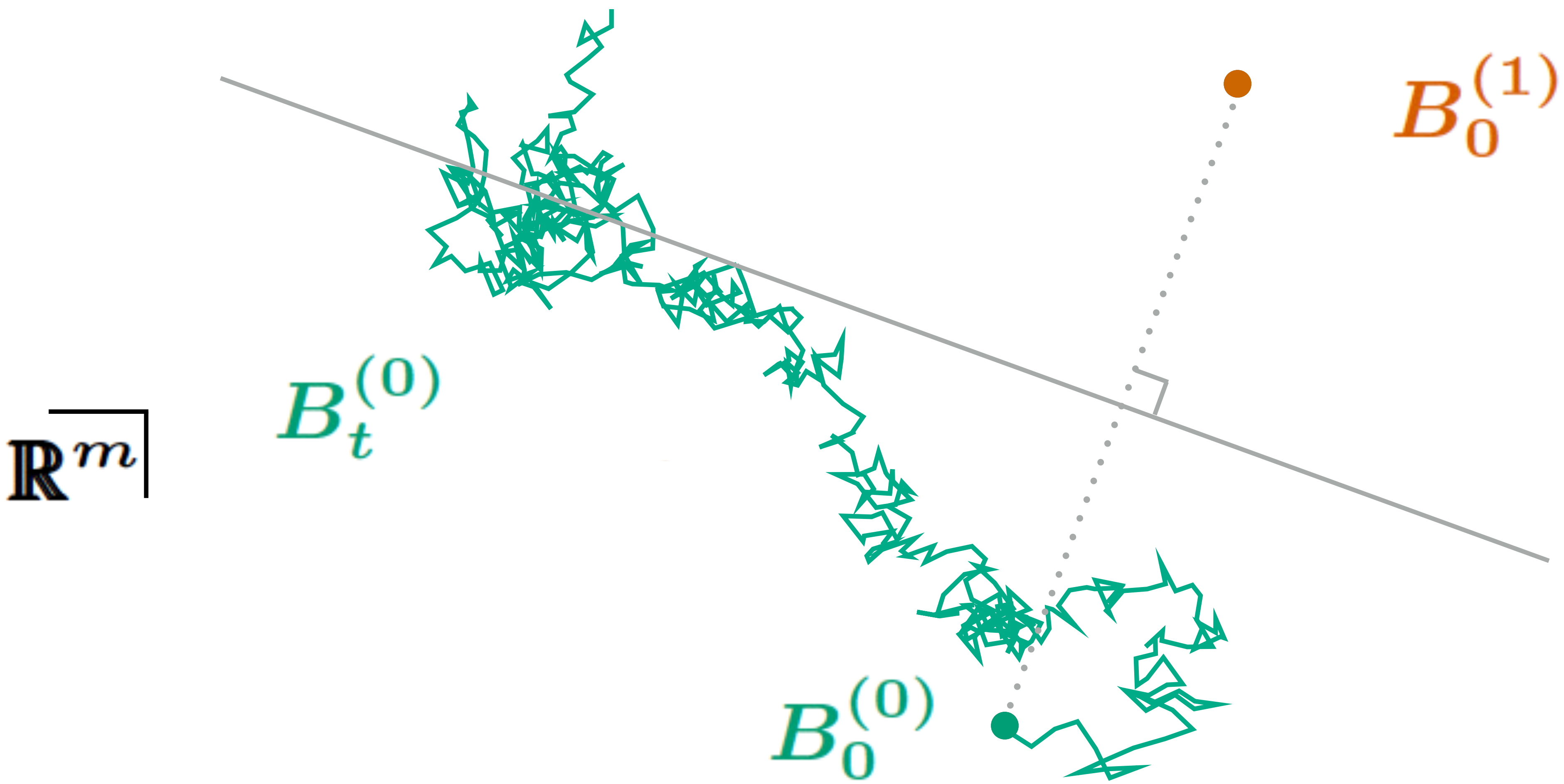

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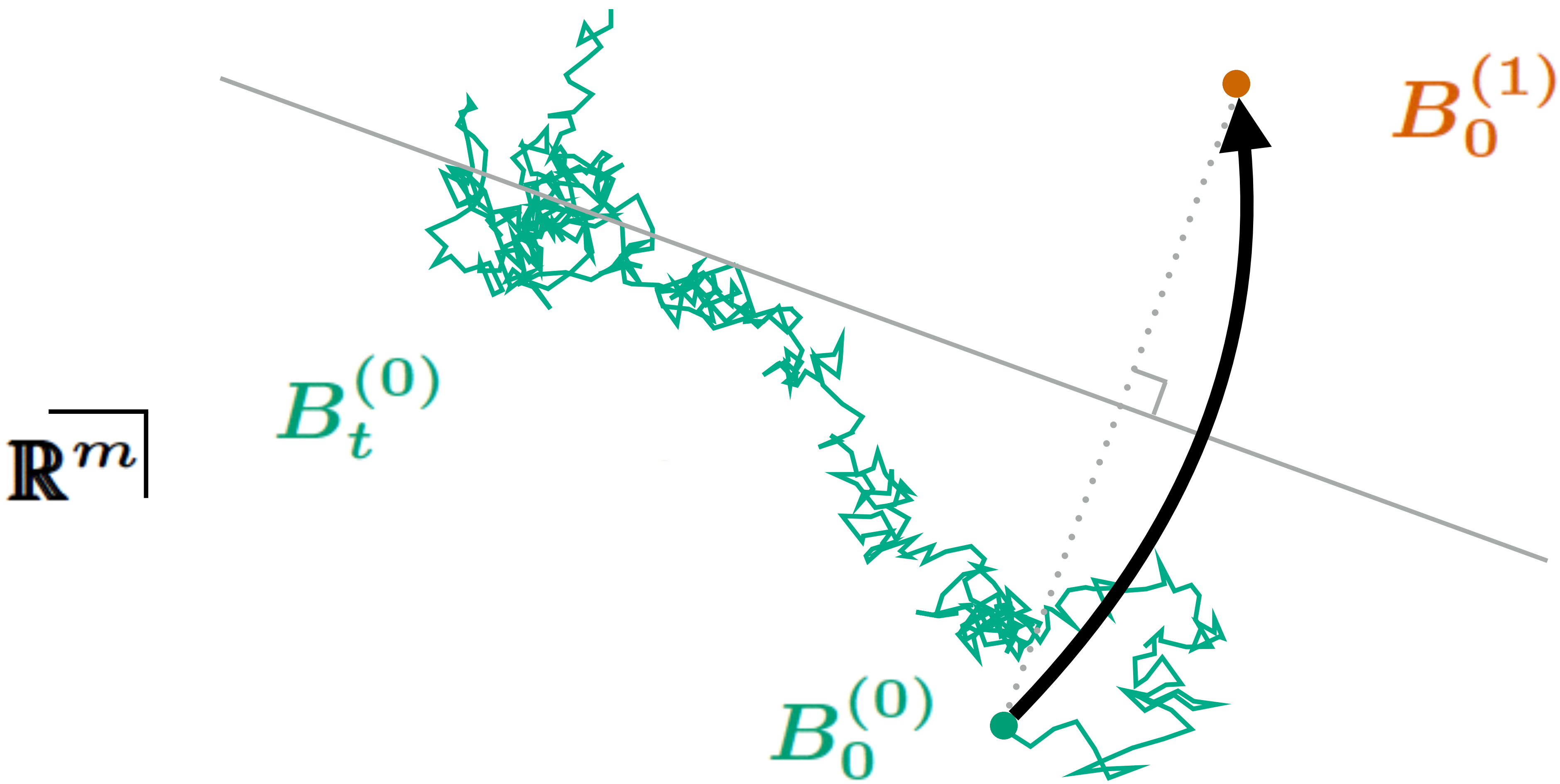

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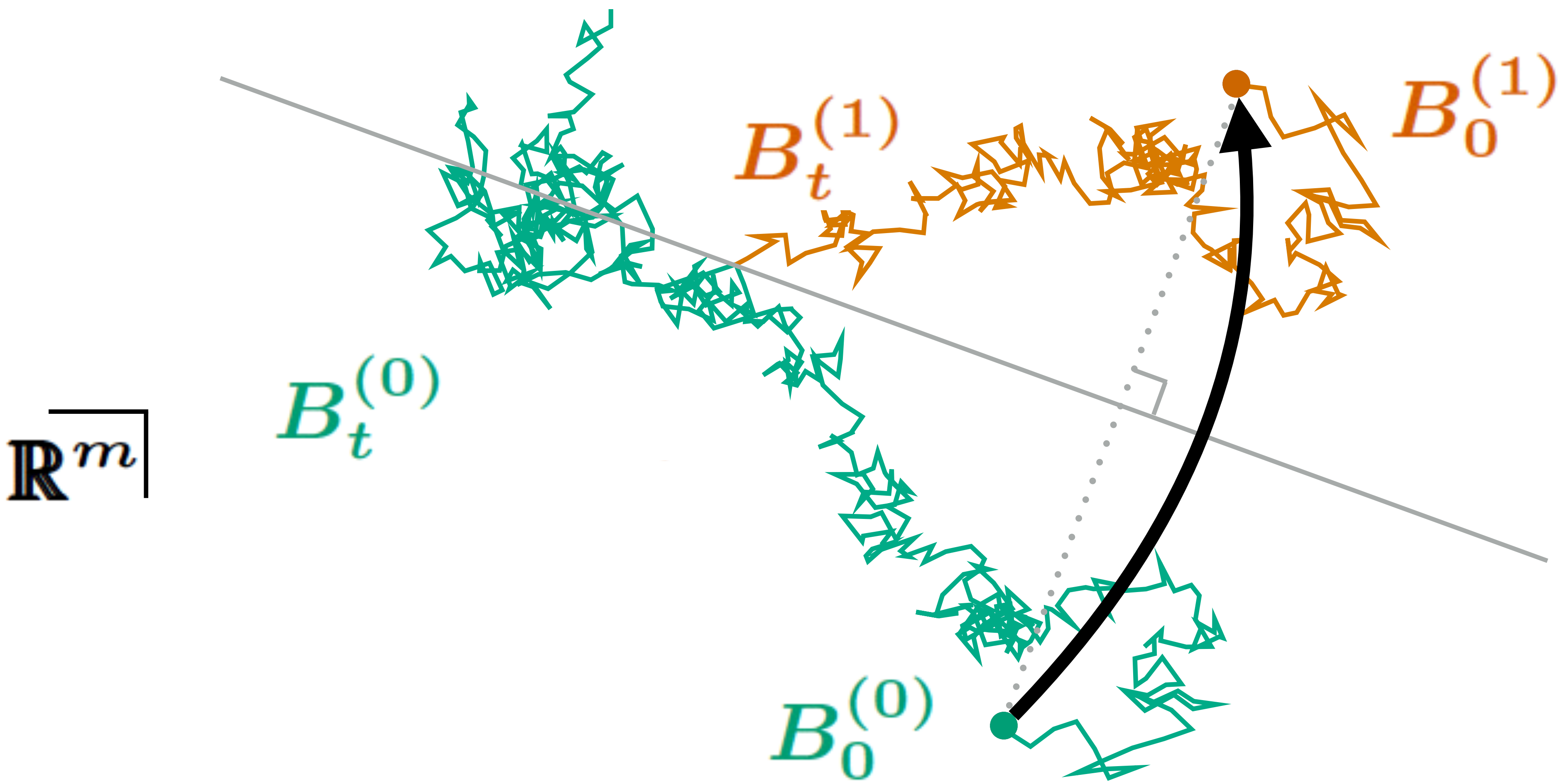

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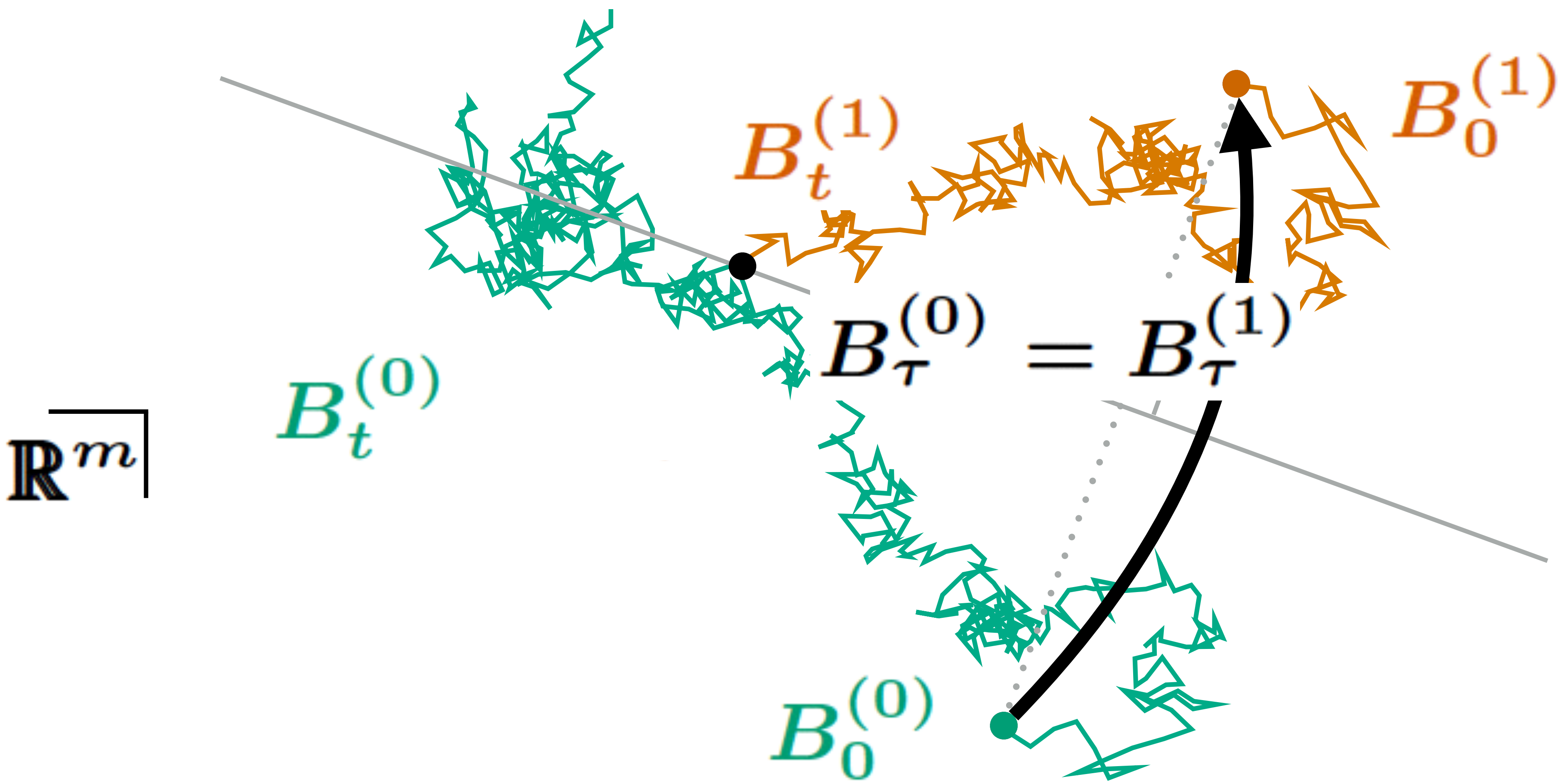

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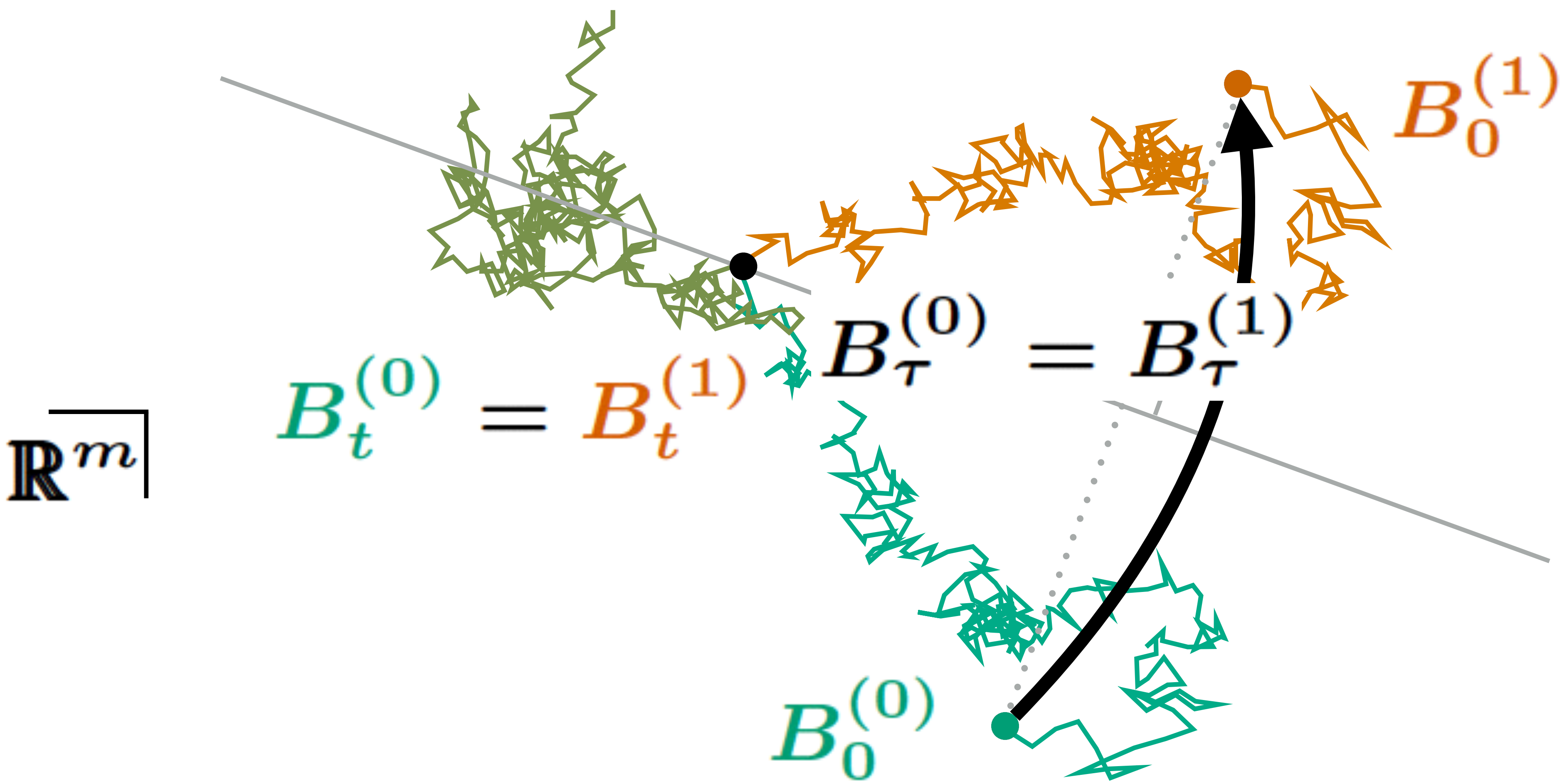

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Coupling by parallel transport/reflection $(B_t^{(0)}, B_t^{(1)})$ on Riem. mfd.:

Coupling of $dB_t^{(0)} \in T_{B_t^{(0)}}M$ & $dB_t^{(1)} \in T_{B_t^{(1)}}M$ by parallel transport/reflection $\underline{T_{B_t^{(0)}}M \to T_{B_t^{(1)}}M}$ along a minimal geodesic

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- Require the notion of parallel transport
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[Kendall '86 / Cranston '91 / F.-Y. Wang '94,'05 / E.-P. Hsu '03 / von Renesse '04 / K. '10,'12 / Arnaudon, Coulibaly & Thalmaier '09 / Neel & Popescu '15+ / $\cdots$]

How do we extend?

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$\rightsquigarrow$

Change the definition:

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$\rightsquigarrow$

Change the definition:

From the structure we used for construction
to the characteristic property they satisfies

Outline of the talk

1. **Introduction**
2. **Framework**
3. **Coupling by parallel transport**
4. **Coupling by reflection**
5. **Concluding remarks**

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Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,

$\mathfrak{m}$: loc. finite, σ-finite, $\operatorname{supp} \mathfrak{m} = X$,

$P_t = \mathrm{e}^{t\Delta} \leftrightarrow$ Cheeger's L^2-energy

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$$2\mathsf{Ch}(f) := \int_X |\nabla f_n|^2 d\mathfrak{m} \qquad \begin{matrix} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{matrix}$$

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$$= \int_X {}^{\exists}|\nabla f|_w^2 \, d\mathfrak{m}$$

($|\nabla f|_w$: minimal weak upper gradient)

$\mathsf{RCD}(K,\infty)$ space

$$W_2(\mu,\nu) := \inf\left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l}\pi\text{: coupling}\\ \text{of } \mu \ \& \ \nu\end{array} \right\}$$

$$\mathrm{Ent}(\rho\mathfrak{m}) := \int \rho \log \rho \, d\mathfrak{m}$$

Definition 1

$(X, d, \mathfrak{m})$: Riemannian **CD**(K,∞) sp. $(K \in \mathbb{R})$

$\overset{\text{def}}{\Leftrightarrow}$ "$\mathrm{Hess}\,\mathrm{Ent} \geq K$" on $(\mathcal{P}(X), W_2)$

& **Ch**: quadratic form ($\Leftrightarrow$ P_t: linear)

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Properties

- "$\partial_t(\mu P_t) = -\boldsymbol{\nabla}\,\mathrm{Ent}(\mu P_t)$" on $(\mathcal{P}(X), W_2)$
- **Ch**: str. local quasi-reg. Dirichlet form admitting carré du champ ($\rightsquigarrow$ Brownian motion $(B(t),\mathbb{P}_x)$)
- "$\dfrac{1}{2}\Delta|\nabla f|_w^2 - \langle \nabla f, \nabla\Delta f\rangle_w \ge K|\nabla f|_w^2$"

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$\mathsf{RCD}(K,\infty) \Rightarrow \mathsf{BE}(K,\infty)$

$\mathrm{Hess\,Ent} \geq K$

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$\mathsf{W}_2(K,\infty)$: $W_2(\mu P_t, \nu P_t) \leq \mathrm{e}^{-Kt} W_2(\mu,\nu)$

$\Downarrow$ [K. '10, '13 / $\cdots$]

$\mathsf{G}_2(K,\infty)$: $|\nabla P_t f|_w \leq \mathrm{e}^{-Kt} P_t(|\nabla f|_w^2)^{1/2}$

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- Reduce to construction of a coupling of trans. prob.
- Self-improvement of **BE**(K, ∞) [Savaré '14]

  $\Rightarrow W_\infty(\delta_x P_t, \delta_y P_t) \le e^{-Kt} d(x, y)$

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Observe it on Riem. mfd.

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★ "Thm. 2 ⇒ Thm. 4" is simliar to the corresponding argument in coupling by parallel transport

Idea of the proof of Thm 2

Basic idea: modify "$\mathbf{G}_2(K, \infty) \Rightarrow \mathbf{W}_2(K, \infty)$"
(cf. [Bakry, Gentil & Ledoux '15+])

- $\mathbf{W}_\infty(K, \infty)$
- Kantorovich-Rubinstein duality
- Reverse f'nal Gaussian isoperimetric ineq. for P_t
  ($\Leftarrow \mathbf{G}_1(K, \infty)$)

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$$\begin{aligned}\Rightarrow P_t f(x) - P_t f(y) &\le \frac{t}{s+t}\varphi_t(d(x,y)) + \frac{s}{s+t}\varphi_s(d(x,y)) \\ &\le \varphi_{s+t}(d(x,y)) \quad (\because \Phi(\sqrt{\cdot}): \text{ concave}) \qquad \square\end{aligned}$$

1. Introduction

2. Framework

3. Coupling by parallel transport

4. Coupling by reflection

5. Concluding remarks

Summary

- "$\mathbf{Ric} \geq \boldsymbol{K}$" on "Riemannian" met. meas. sp
  - $\rightarrow$ Brownian motion is defined
  - $\rightarrow$ Bakry-Émery theory is available
- "Coupling by parallel transport/reflection" of BMs
  - $\leftarrow$ Define them by characteristic properties
  - $\leftarrow$ Use of f'nal ineq's & Bakry-Émery theory
- New approach to construction of couplings
  - $\rightarrow$ Requires less regularity of the underlying sp.
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