A dimensional Wasserstein contraction characterizing the curvature-dimension condition

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1. Introduction

Heat flow and Ricci curvature

M: complete Riemannian manifold

$$\begin{cases} \partial_t u(t,x) = \Delta u(t,x), \\ u(0,\cdot) = f \end{cases} \quad \text{heat eq. on } M$$

$\Rightarrow u = P_t f$

- $P_t = \mathrm{e}^{t\Delta}$: heat semigroup

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★ P_t characterizes "$\mathrm{Ric} \geq K$ & $\dim \leq N$"

(curvature-dimension cond.)

Bakry-Émery's approach

Bochner-Weitzenböck formula

$$\Gamma_2(f,f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\mathrm{Hess}\, f\|_{\mathsf{HS}}^2,$$
$$\Gamma_2(f,f) := \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle$$

★ "Ric $\geq K$ & dim $\leq N$"

$\Leftrightarrow$ **BE**(K,N): $\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2$

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(**BE**(K,N) is formulated only by $\Delta \leftrightarrow P_t = e^{t\Delta}$)

Optimal transport approach

$$W_2(\mu,\nu) := \inf\left\{ \|d\|_{L^2(\pi)} \;\middle|\; \begin{array}{l}\pi\text{: coupling}\\ \text{of } \mu \,\&\, \nu\end{array} \right\}$$

$$\mathrm{Ent}_{\mathrm{vol}}(\rho\,\mathrm{vol}) := \int \rho \log \rho \, d\,\mathrm{vol}$$

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$\Leftrightarrow$ $\mathsf{CD}^e(K,N)$: "$\nabla^2\,\mathrm{Ent} - \frac{1}{N}\nabla\,\mathrm{Ent}^{\otimes 2} \geq K$"

on $(\mathcal{P}(M), W_2)$ ((K,N)-convexity of Ent)

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- $\mu P_t \in \mathcal{P}(M)$ ($\mu \in \mathcal{P}(M)$): heat distribution
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⇔ **W**(K, N): An estimate of $W_2(\mu P_t, \nu P_s)$

- **W**(K, ∞):
  $W_2(\mu P_t, \nu P_t)^2 \leq \mathrm{e}^{-2Kt} W_2(\mu, \nu)^2$
- **W**$(0, N)$:
  $W_2(\mu P_t, \nu P_s)^2 \leq W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$

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Q.
Estimate of $W_2(\mu P_t, \nu P_t)^2$
characterizing "$\mathrm{Ric} \geq K$ & $\dim \leq N$"

Outline of the talk

1. Introduction

2. Framework and main results

3. Idea of the proof

4. Further problems

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Framework

(X, g): m-dim. Riem. mfd., d: dist., $dv = \mathrm{e}^{-V} d\,\mathrm{vol}$

$\mathcal{L} := \Delta - \nabla V \cdot \nabla$, $\quad P_t := \mathrm{e}^{t\mathcal{L}}$

$\mathrm{Ent} := \mathrm{Ent}_v$

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BE(K, N) in terms of $\mathcal{L}$

$$\Gamma_2(f, f) \geq K|\nabla f|^2 + \frac{1}{N}(\mathcal{L}f)^2,$$

$$\Gamma_2(f, f) := \frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle$$

Another dimensional W_2-contraction

$\widehat{\mathbf{W}}(K,N)$

$$\left(\frac{W_2(\mu P_t, \nu P_t)}{2}\right)^2 \leq \mathrm{e}^{-2Kt} \qquad \left(\frac{W_2(\mu, \nu)}{2}\right)^2$$

Another dimensional W_2-contraction

$\underline{\widehat{\mathsf{W}}(K,N)}$

$$\mathfrak{s}_{K/N}\left(\frac{W_2(\mu P_t,\nu P_t)}{2}\right)^2 \le \mathrm{e}^{-2Kt}\mathfrak{s}_{K/N}\left(\frac{W_2(\mu,\nu)}{2}\right)^2$$

$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}\right)$$

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$$- 2N\int_0^t \mathrm{e}^{-2K(t-s)}$$
$$\times \sinh^2\left(\frac{\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s)}{2N}\right) ds,$$

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Another dimensional W_2-contraction

$\widehat{\mathsf{W}}'(K,N)$

$$W_2(\mu P_t, \nu P_t)^2 \le \mathrm{e}^{-2Kt} W_2(\mu,\nu)^2 - \frac{2}{N}\int_0^t \mathrm{e}^{-2K(t-s)}\big(\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s)\big)^2 ds$$

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$$\left(\mathfrak{s}_\kappa(r) := \frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}} \right)$$

$$\left(\mathfrak{s}_\kappa(r) \approx r \ (r \ll 1),\ \sinh(r) \ge r \quad \rightsquigarrow \widehat{\mathsf{W}}(K,N) \Rightarrow \widehat{\mathsf{W}}'(K,N) \right)$$

Another dimensional W_2-contraction

Theorem 1 ([Bolley, Gentil, Guillin & K.])

For $K \in \mathbb{R}$ *and* $N > 0$*, TFAE:*

(i) $\mathsf{BE}(K, N)$

(ii) $\widehat{\mathsf{W}}(K, N)$

(iii) $\widehat{\mathsf{W}}'(K, N)$

Remark on extensions

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- Riem. energy meas. sp. with "$\mathbf{BE}(K, N)$"
  $\Leftrightarrow$ $\mathbf{RCD}^e(K, N)$ sp.: "Riem." met. meas. sp.,
  "$\nabla^2 \,\mathrm{Ent} - \frac{1}{N} \nabla \,\mathrm{Ent}^{\otimes 2} \geq K$" on $(\mathcal{P}(M), W_2)$

  [Erbar, K. & Sturm '15]

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- $\mathbf{RCD}^e(K, N) \Rightarrow \mathbf{W}(K, N) \Rightarrow \mathbf{BE}(K, N)$
- Thm.1 implies another proof of

$$\mathbf{RCD}^e(K, N) \Rightarrow \mathbf{BE}(K, N)$$

For simplicity, we assume $\boldsymbol{K} = \mathbf{0}$ in the sequel

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(We still use "$\boldsymbol{K}$" if it holds for general $\boldsymbol{K}$)

Proof of $\mathsf{BE}(K,N) \Rightarrow \widehat{\mathsf{W}}(K,N)$

Formal derivation of $\widehat{\mathsf{W}}'(0, N)$

† $\mathsf{BE}(0, N) \Leftrightarrow \nabla^2 \operatorname{Ent} - \frac{1}{N} \nabla \operatorname{Ent}^{\otimes 2} \geq 0$

† $\mu_t^{(i)} := \mu^{(i)} P_t$ solves $\dot{\mu}_t^{(i)} = -\nabla \operatorname{Ent}(\mu_t^{(i)})$

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$(\sigma_r)_{r \in [0,1]}$: W_2-min. geod. in $\mathcal{P}(M)$ from $\mu_t^{(0)}$ to $\mu_t^{(1)}$

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$$= -\int_0^1 \nabla^2 \mathrm{Ent}(\sigma_r)(\dot{\sigma}_r, \dot{\sigma}_r) dr \quad (\because \ddot{\sigma}_r = 0)$$

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$$\leq - \frac{1}{N} \int_0^1 \langle \nabla \mathrm{Ent}(\sigma_r), \dot{\sigma}_r \rangle^2 dr$$

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&= -\int_0^1 \nabla^2 \operatorname{Ent}(\sigma_r)(\dot{\sigma}_r, \dot{\sigma}_r)dr \\
&\leq -\frac{1}{N}\int_0^1 \langle \nabla \operatorname{Ent}(\sigma_r), \dot{\sigma}_r\rangle^2 dr \\
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&= -\int_0^1 \nabla^2 \operatorname{Ent}(\sigma_r)(\dot{\sigma}_r, \dot{\sigma}_r) dr \\
&\le -\frac{1}{N}\int_0^1 \langle \nabla \operatorname{Ent}(\sigma_r), \dot{\sigma}_r \rangle^2 dr \\
&\le -\frac{1}{N}\left(\operatorname{Ent}(\sigma_1) - \operatorname{Ent}(\sigma_0)\right)^2
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&\le -\frac{1}{N}\int_0^1 \langle \nabla \operatorname{Ent}(\sigma_r), \dot\sigma_r\rangle^2 dr \\
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$$\leq -\frac{1}{N}\left(\operatorname{Ent}(\mu_t^{(1)}) - \operatorname{Ent}(\mu_t^{(0)})\right)^2$$

$$\Rightarrow \boxed{\begin{aligned} W_2(\mu_t^{(0)}, \mu_t^{(1)})^2 &\leq W_2(\mu_0^{(0)}, \mu_0^{(1)})^2 \\ &\quad -\frac{2}{N}\int_0^t (\operatorname{Ent}(\mu_s^{(1)}) - \operatorname{Ent}(\mu_s^{(0)}))^2 ds \end{aligned}}$$

Evolution variational inequality

- **BE**$(K,N) \Rightarrow$ **RCD**$^e(K,N)$
- **RCD**$^e(K,N) \Leftrightarrow$ $^{\forall}$initial data, $^{\exists}$sol. to (K,N)-**EVI** of Ent

[Erbar, K. & Sturm '15]

What's **EVI**?

- A formulation of grad. flow for (K,N)-convex pot. & "Riemannian"
- In this case, sol. to **EVI** of $\mathrm{Ent} = \mu P_t$
- (K,N)-**EVI** $\Rightarrow$ **W**(K,N) & $\widehat{W}(K,N)$

Proof of

$\widehat{\mathsf{W}}'(K, N) \Rightarrow \mathsf{BE}(K, N)$

Review: W$(K,N) \Rightarrow$ BE(K,N)

$$\mathbf{W}(K,N) \;\underset{\Leftarrow}{\Rightarrow}\; \mathbf{G}(K,N) \;\underset{\Leftarrow}{\Rightarrow}\; \mathbf{BE}(K,N)$$

[Bakry & Émery '84 ($N = \infty$)/ Bakry & Ledoux '06]
[K. '10, '13 ($N = \infty$)/K. '15/...]

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$\mathsf{G}(0,N)$ (Gradient estimate for P_t) :

$$|\nabla P_t f|^2 \le P_t(|\nabla f|^2) - \frac{2}{N}(\mathcal{L}P_t f)^2$$

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★ $\mathbf{W}(K,N)$ for Dirac meas.'s $\Rightarrow G(K,N)$

Our problem: $\widehat{\mathsf{W}}'(K,N) \Rightarrow \mathsf{BE}(K,N)$

Remarks

- No use of $\mathsf{G}(K,N)$:

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- On $\mathbb{R}^m$,

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$\because$ Recall $\widehat{\mathsf{W}}(0,N)$:

$$W_2(\mu P_t, \nu P_t)^2 \le W_2(\mu,\nu)^2 - \frac{2}{N}\int_0^t (\mathrm{Ent}(\mu P_s) - \mathrm{Ent}(\nu P_s))^2 ds$$

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"$\dot{\nu}_0 = \nabla f$ in $T_{\nu_0}\mathcal{P}(M)$" for $(\nu_r)_r \subset \mathcal{P}(M)$

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$\Rightarrow \dot{\mu}_0 = \nabla f$

Overview of the proof

$\mu = \mu_0$, $\nu = \mu_r$

& consider $\mathbf{W}(0, N)$ at $r \approx 0$, $t \approx 0$

(Use Kantorovich duality & Hopf-Lax semigroup)

$$W_2(\mu_0 P_t, \mu_r P_t)^2 \quad \leq W_2(\mu_0, \mu_r)^2$$

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$$\frac{(3)}{r^2} \geq \left(\int P_s(g \mathcal{L}^g f) \log P_s g \, dv \right)^2 + o(1)$$

Overview (cont.'d)

$$\Downarrow$$

$$\int \Big(-P_t(|\nabla f|^2) + 2\langle \nabla f, \nabla P_t f\rangle\Big) g\, dv$$

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$$\boxed{g \to \delta_x} \qquad = -\frac{2}{N}\left|\int (\mathcal{L} f) g\, dv\right|^2 \qquad \square$$

Questions

- Applications of $\widehat{\mathsf{W}}(K,N)$ to a sharper rate of conv. (when $K>0$): e.g.,

$$W_2(\mu P_t, v) \le \exp\left(-\frac{NK}{N-1}t\right) W_2(\mu, v)$$

- A self-improvement of $\widehat{\mathsf{W}}(K,N)$ or $\widehat{\mathsf{W}}'(K,N)$

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$\mathcal{W}$-entropy

- **W**$(0, \infty) \Rightarrow I(\mu P_t) \searrow$

  $(I(\rho v) := \int \frac{|\nabla \rho|^2}{\rho} dv$: Fisher information)

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Q. $\widehat{\mathbf{W}}(K,N) \Rightarrow$ Monotonicity of some f'nal?