最適輸送理論, Riemann 的曲率次元条件と熱分布

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1. Introduction

Purpose/History

- Unify the study of

$$\text{“Ric} \geq \boldsymbol{K} \ \& \ \text{dim} \leq \boldsymbol{N}\text{”}$$

  on metric measure spaces (Equiv. on Riem. mfds)
- Applications

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Two major approaches

Study via optimal transport

Study via diffusion generator

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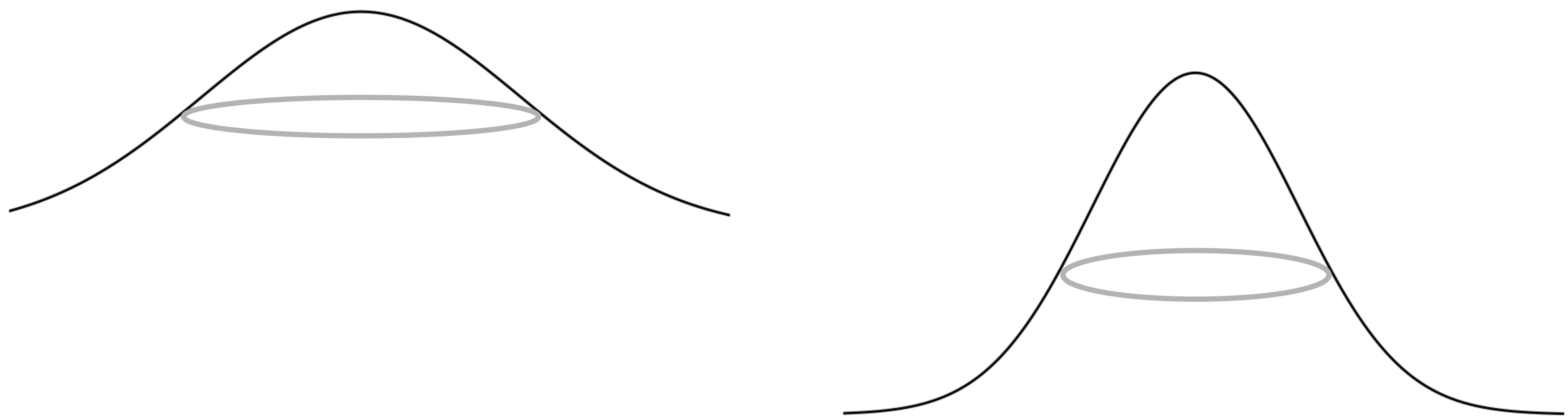
★ Heat distributions combine them!

Optimal transport approach



- OT: Energy minimization of bringing measures
- Distorsion of volume element along transport

$\leftrightsquigarrow$ Ricci curvature, dimension

[Cordero-Erausquin, McCann & Schmuckenschläger '01]
[von Renesse & Sturm '05]

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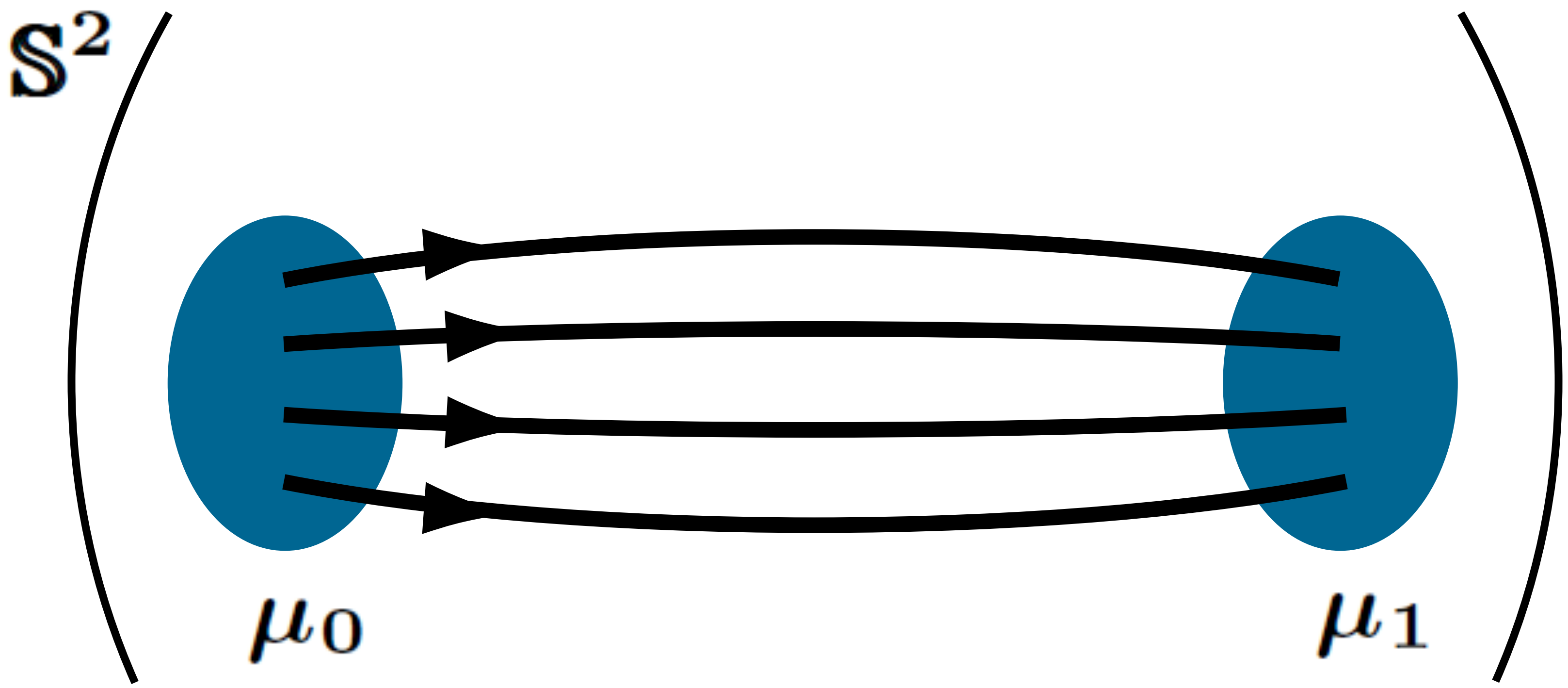

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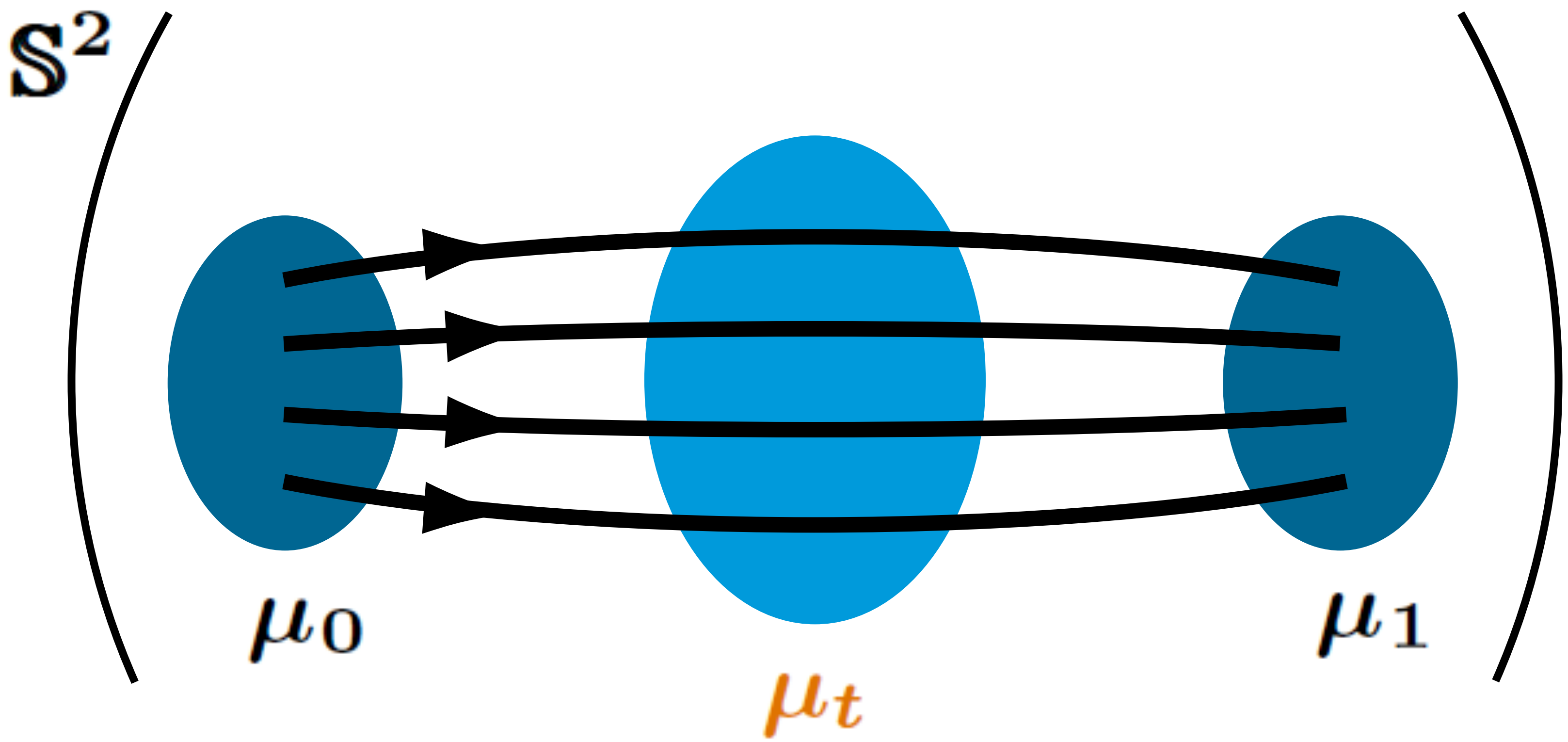

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Bakry-Émery's approach

Bochner-Weitzenböck formula

$$\Gamma_2(f,f) = \mathrm{Ric}(\nabla f, \nabla f) + \|\operatorname{Hess} f\|_{\mathsf{HS}}^2,$$
$$\Gamma_2(f,f) := \frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f\rangle$$

- "$\mathrm{Ric} \geq K$ & $\dim \leq N$"

$$\Leftrightarrow \boxed{\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2}$$

(Curv.-dim. cond. **BE**(K,N) [Bakry & Émery '85])

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(**BE**(K,N) is formulated only in terms of Δ)

Motivation

- Study of measured Gromov-Hausdorff limits of Riem. mfds with $\mathbf{Ric} \geq \boldsymbol{K}$ & $\mathbf{dim} \leq \boldsymbol{N}$
  $\leftharpoondown$ Cond. via OT has "integral form":
    - † Formulated only by distance and measure
    - † Stable under mGH conv.

  [Sturm '06 / Lott & Villani '09]
- Bakry-Émery theory produces many applications
  $\rightsquigarrow$ Applications by combination of two approaches
- Deeper understanding of Riem. geom.
- Extension to other class of spaces

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Overview of applications

- (Geometric) stability
  - $\Rightarrow$ Same stability for analytic conditions
- Regularization by heat semigroup P_t
  - $\Rightarrow$ (Almost) full strength of Bakry-Émery
- Cheeger-Gromoll splitting theorem
  - $\Rightarrow$ Rigidity of comparison estimates
  - $\Rightarrow$ Structure thm (rectifiability)
- Lévy-Gromov isoperimetric ineq.

Outline of the talk

1. **Introduction**

2. **Optimal transport: Convexity of Entropy**

3. **Bakry-Émery theory**

4. **Equivalence via evolution variational inequality**

5. **Applications**

Framework

$(X, d, \mathfrak{m})$: Polish geodesic metric measure sp.,

$\operatorname{supp}\mathfrak{m} = X$,

$$\exists c > 0 \text{ s.t. } \int \exp\left(-cd(x_0, x)^2\right)\mathfrak{m}(dx) < \infty$$

Example (weighted Riem. mfd)

(X, g): complete Riem. mfd., $\partial X = \emptyset$,

d: Riem. dist., $\mathfrak{m} = \mathrm{e}^{-V}\mathrm{vol}_g \quad (V : X \to \mathbb{R})$

Framework

L^2-Wasserstein distance W_2: For $\mu_0, \mu_1 \in \mathcal{P}(X)$,

$$W_2(\mu_0, \mu_1)^2 := \inf_{\pi} \int_{X \times X} d(x, y)^2 \, \pi(dxdy)$$

$$\pi \in \mathcal{P}(X^2), \begin{cases} \pi(A \times X) = \mu_0(A), \\ \pi(X \times A) = \mu_1(A) \end{cases}$$

$$\mathcal{P}_2(X) := \{\mu \in \mathcal{P}(X) \mid W_2(\delta_{x_0}, \mu) < \infty\}$$

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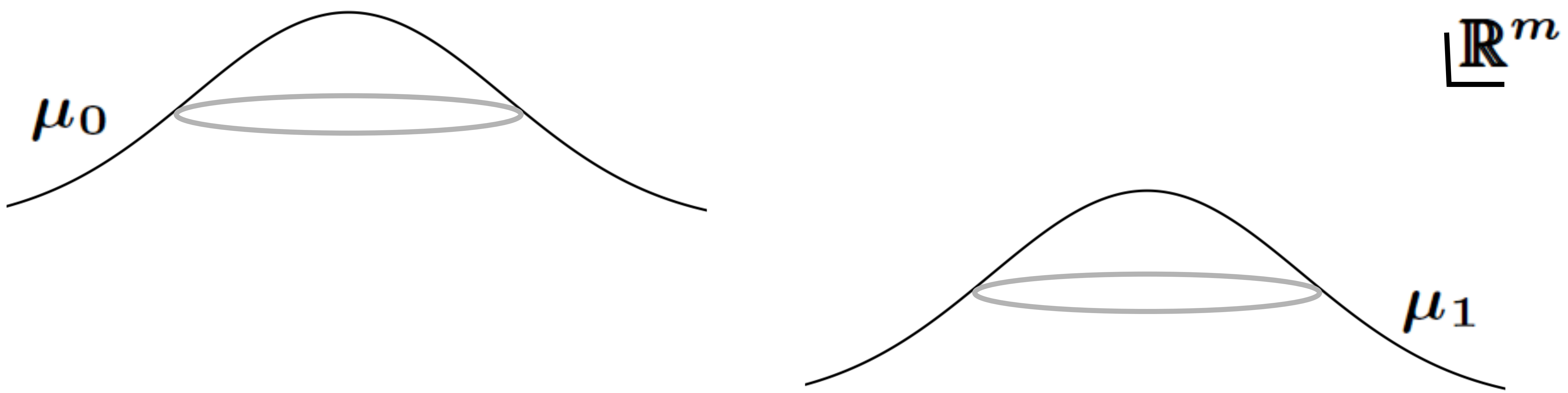

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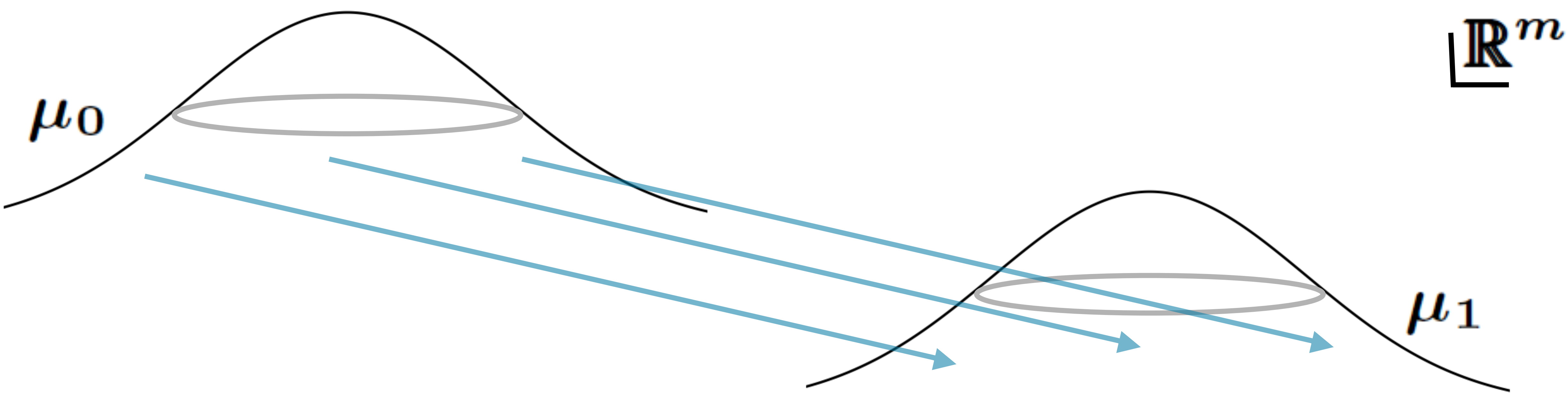

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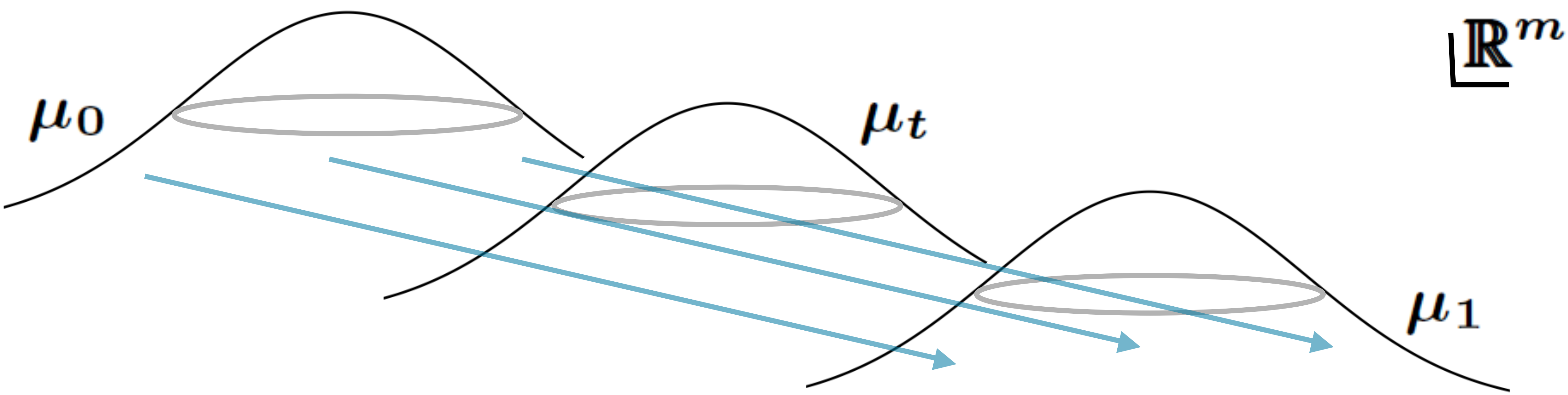

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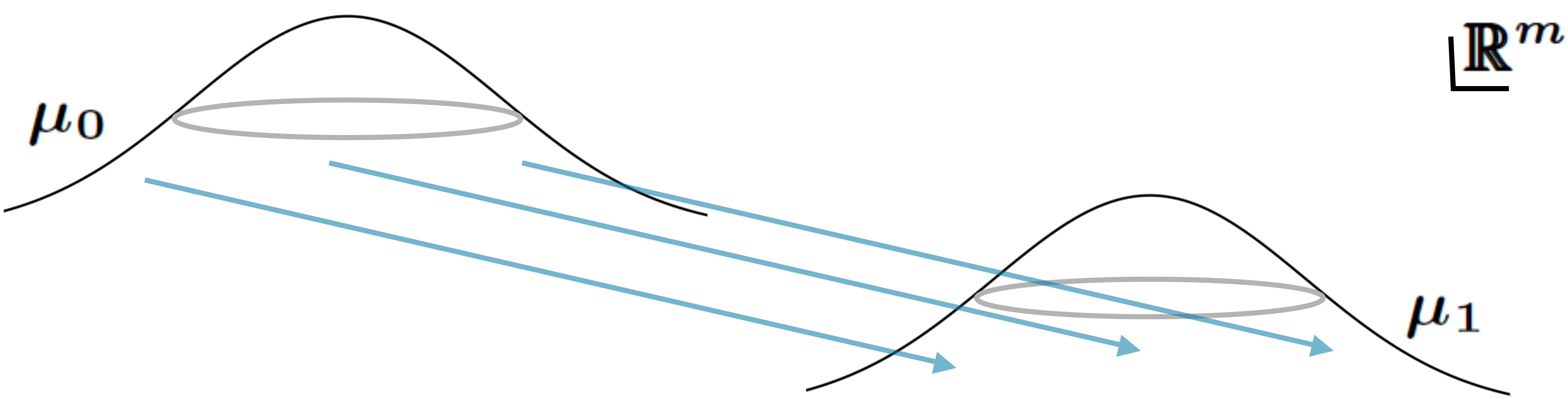

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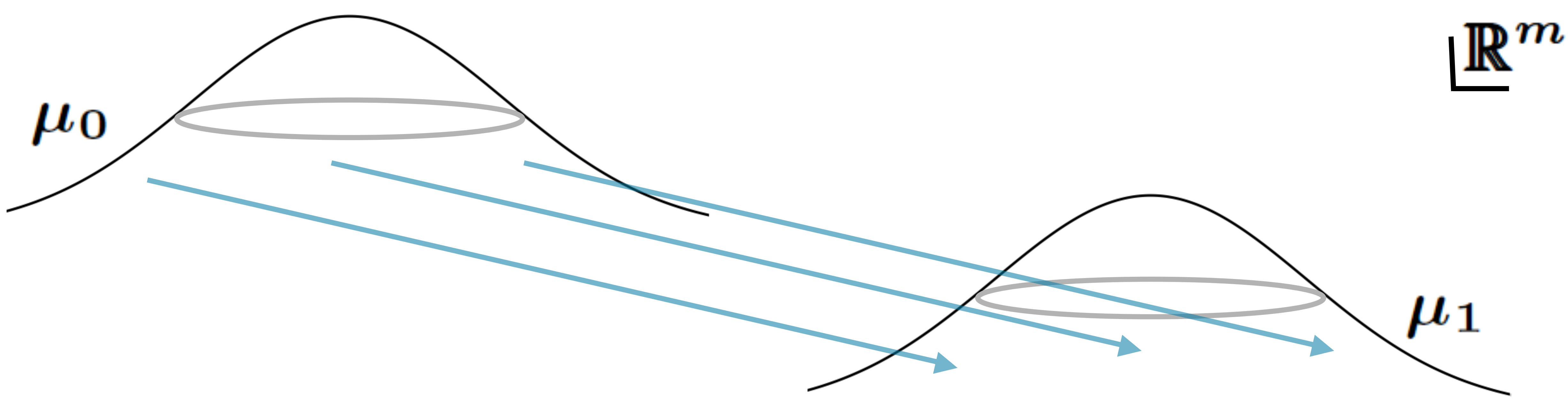

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- ${}^\exists \Gamma \in \mathcal{P}(\mathrm{Geo}(X))$ s.t.

$$\mu_t(A) = \int_{\mathrm{Geo}(X)} 1_A(\gamma(t))\Gamma(d\gamma),$$

$$W_2(\mu_s, \mu_t)^2 = \int_{\mathrm{Geo}(X)} d(\gamma(s), \gamma(t))^2 \Gamma(d\gamma)$$

($\mathrm{Geo}(X)$: sp. of const. speed geod.'s)

Entropic curvature-dimension cond.

- $\mathrm{Ent}(\rho\mathfrak{m}) := \int \rho \log \rho \, d\mathfrak{m}$ (relative entropy)

(K, N)-convexity of Ent ($K \in \mathbb{R}$, $N \in (0, \infty]$)

$$\text{“}\mathrm{Hess}\,\mathrm{Ent} - \frac{1}{N} \nabla \mathrm{Ent}^{\otimes 2} \geq K\text{”} \text{ on } (\mathcal{P}_2(X), W_2)$$

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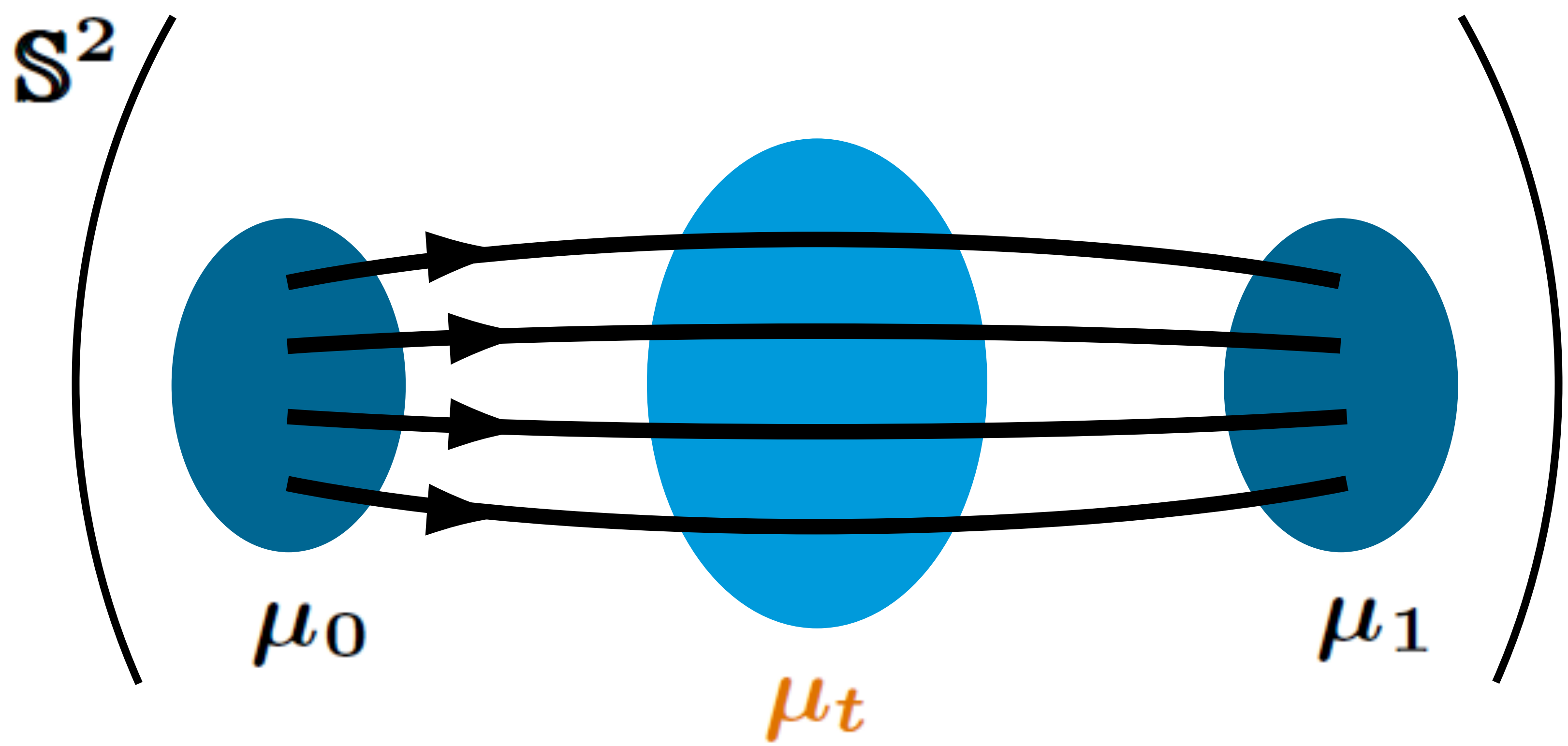

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E.g. $(0,\infty)$-convexity:
$^\forall \mu_0, \mu_1 \in \mathcal{P}_2(X)$, $^\exists(\mu_t)_{t\in[0,1]}$: W_2-min. geod. s.t.

$$\mathrm{Ent}(\mu_t) \leq (1-t)\,\mathrm{Ent}(\mu_0) + t\,\mathrm{Ent}(\mu_1)$$

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($\mathbf{CD}^e(K,N)$ cond. [Erbar, K. & Sturm '15])

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Relation with other conditions

† **CD**(K,N) [Sturm '06 / Lott & Villani '09]

† **CD**$^*(K,N)$ [Bacher & Sturm '10]

† **MCP**(K,N) [... / Sturm '06 / Ohta '07]

- **CD**$(K,\infty) =$ **CD**$^*(K,\infty) =$ **CD**$^e(K,\infty)$
- **CD**$(K,N) \Rightarrow$ **CD**$^*(K,N)$ "$\Leftrightarrow$" **CD**$^e(K,N)$
- **CD**$^*(K,N) \Rightarrow$ **MCP**(K,N)
- No Ent when $N < \infty$ in **CD**/**CD***/**MCP**

Relation with other conditions

† $\mathsf{CD}(K, N)$ [Sturm '06 / Lott & Villani '09]

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If X: m-dim. weighted Riem. mfd,

$\mathsf{CD}(K,N)/\mathsf{CD}^*(K,N)/\mathsf{CD}^e(K,N)$

$$\Leftrightarrow \mathrm{Ric} + \mathrm{Hess}\, V - \frac{\nabla V \otimes \nabla V}{N - m} \geq K$$

Framework of Bakry-Émery theory

Weighted Riem. mfd (for simplicity)

(X, g): cpl. Riem. mfd., $\partial X = \emptyset$, $\mathfrak{m} = \mathrm{e}^{-V}\,\mathrm{vol}_g$

$\mathcal{L} := \Delta - \nabla V \cdot \nabla \left(\leftrightsquigarrow \int |\nabla f|^2 \, d\mathfrak{m} \text{ on } L^2(\mathfrak{m}) \right)$

BE(K, N) cond.

$$\Gamma_2(f, f) \geq K|\nabla f|^2 + \frac{1}{N}(\mathcal{L}f)^2,$$

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Gradient estimate for heat semigroup

BE(K,N): $\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\mathcal{L}f)^2$

$P_t = \mathrm{e}^{t\mathcal{L}}$: heat semigroup

★ **BE**$(K,N) \Leftrightarrow$ **G**(K,N)
[Bakry & Émery '85 ($N=\infty$)/ Bakry & Ledoux '06]

G(K,N):

$^\exists C(t) = 1 + O(t)$ $(t \to 0)$ s.t.

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{2tC(t)}{N}(\mathcal{L}P_t f)^2$$

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($\because$ Integration/derivation in t)

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W(K, N): An estimate of $W_2(\mu P_t, \nu P_s)$

W(K, ∞):

$$W_2(\mu P_t, \nu P_t)^2 \le \mathrm{e}^{-2Kt} W_2(\mu, \nu)^2$$

W$(0, N)$:

$$W_2(\mu P_t, \nu P_s)^2 \le W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

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Why $\mathsf{G}(K,N) \Leftrightarrow \mathsf{W}(K,N)$?

$\mathsf{G}(K,\infty)$: $|\nabla P_t f|^2 \le \mathrm{e}^{-2Kt} P_t(|\nabla f|^2)$

$\mathsf{W}(K,\infty)$: $W_2(\mu P_t, \nu P_t)^2 \le \mathrm{e}^{-2Kt} W_2(\mu,\nu)^2$

- Duality in fn.'s & meas.'s
- Kantorovich duality

$$\frac{1}{2} W_2(\mu,\nu)^2 = \sup_{g,f} \left[\int g \, d\mu - \int f \, d\nu \right]$$

$$\left(\text{Constraint: } g(x) + f(y) \le \frac{d(x,y)^2}{2}\right)$$

- Dual interpolation by Hopf-Lax semigroup:

$$Q_r f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x,y)^2}{2r} \right]$$

Why $\mathsf{G}(K,N) \Leftrightarrow \mathsf{W}(K,N)$?

$\mathsf{G}(K,\infty)$: $|\nabla P_t f|^2 \le \mathrm{e}^{-2Kt} P_t(|\nabla f|^2)$

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- Duality in fn.'s & meas.'s
- Kantorovich duality

$$\frac{1}{2} W_2(\mu,\nu)^2 = \sup_{g,f} \left[\int g \, d\mu - \int f \, d\nu \right]$$

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Formally, $\mu_t = \mu P_t$ solves $\boxed{\dot\mu_t = -\nabla \operatorname{Ent}(\mu_t)}$ on $(\mathcal{P}_2(X), W_2)$ (Otto calculus)

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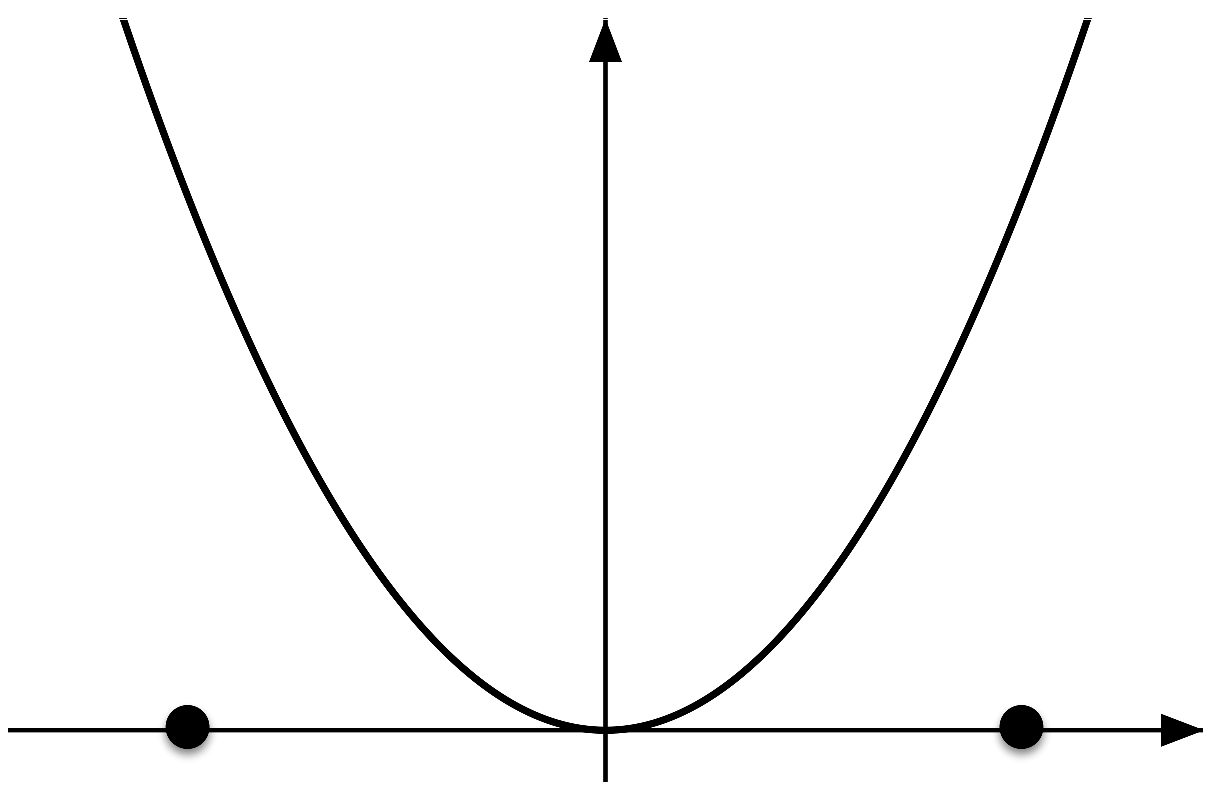

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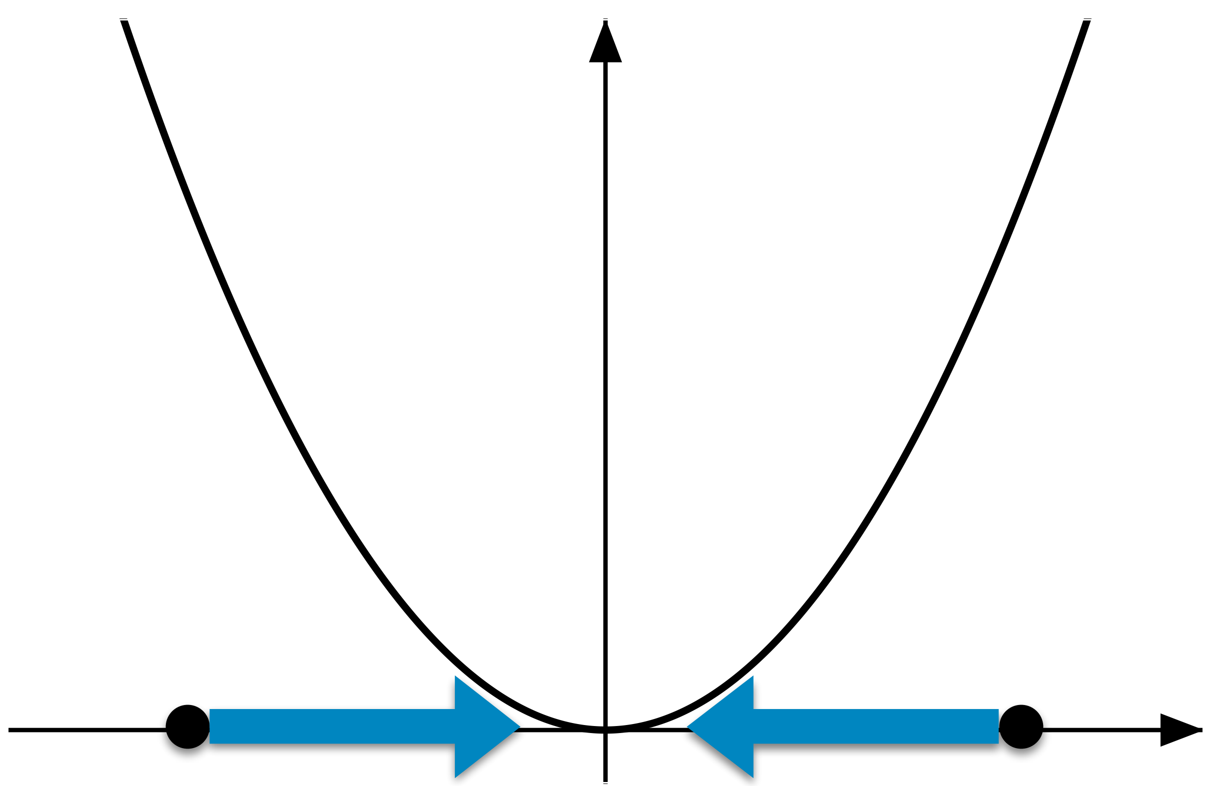

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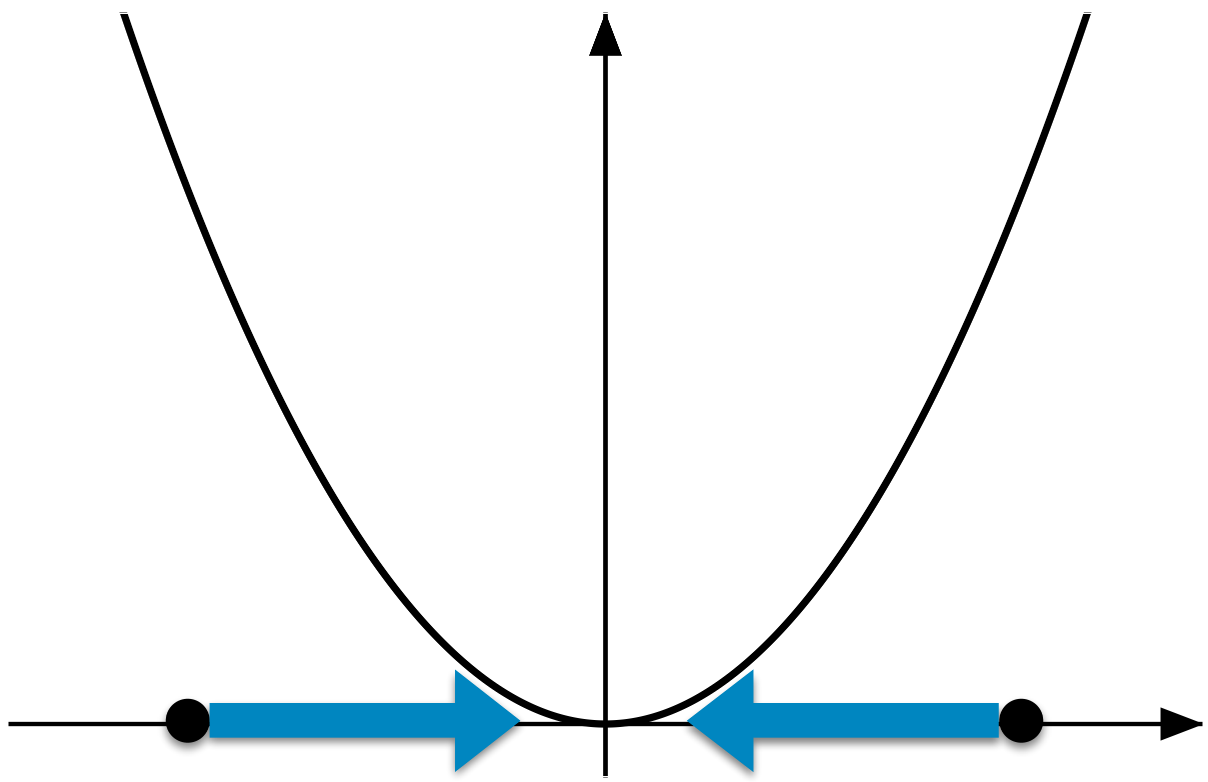

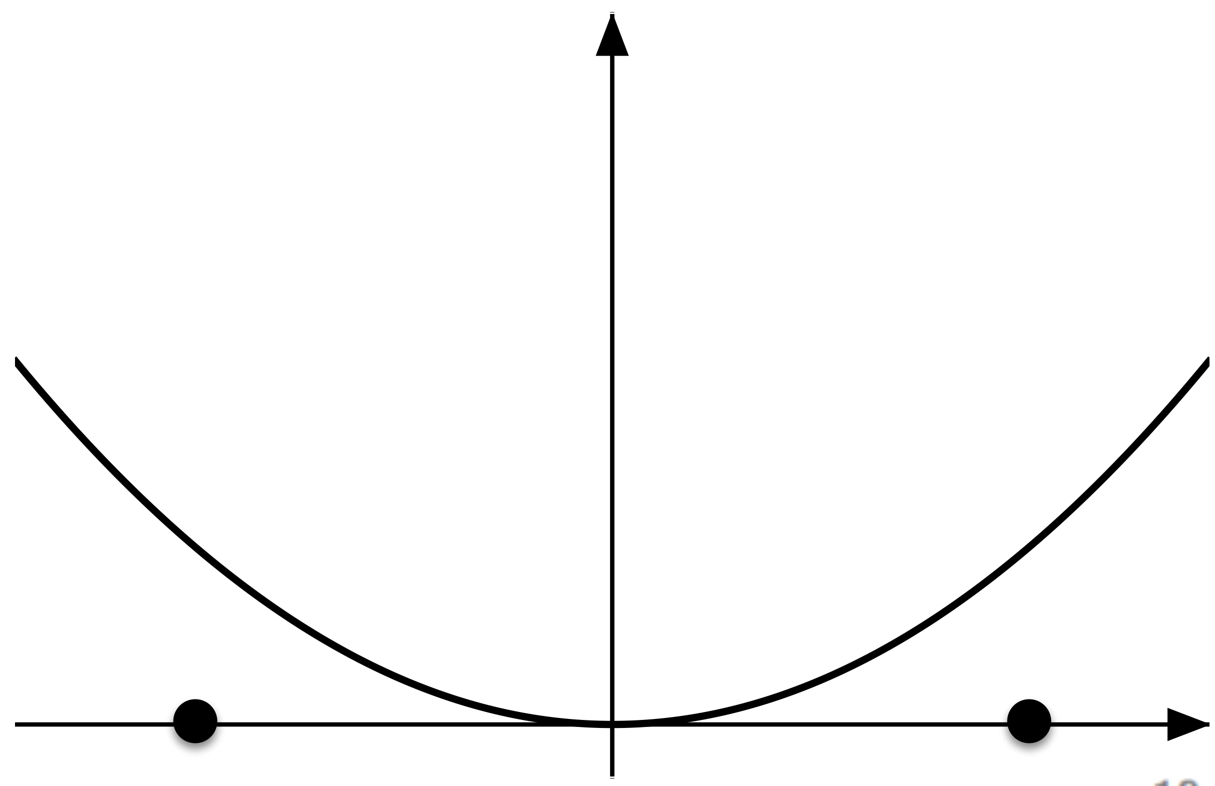

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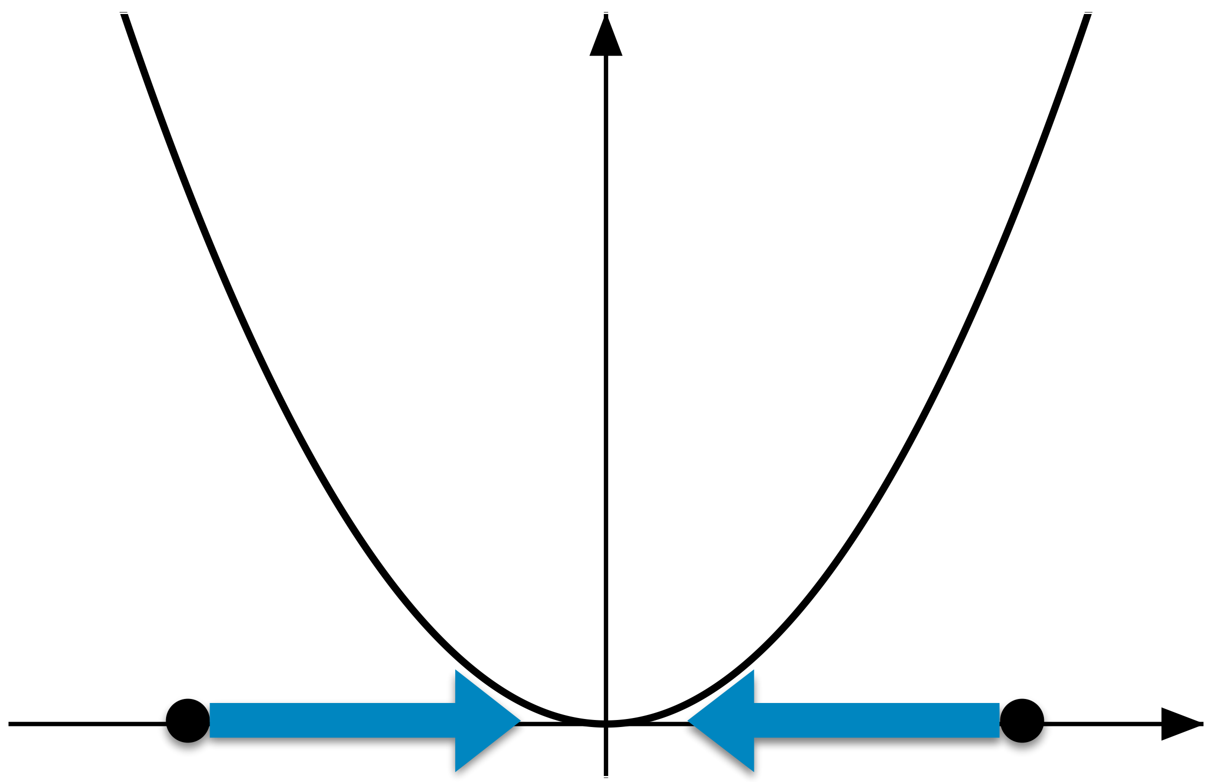

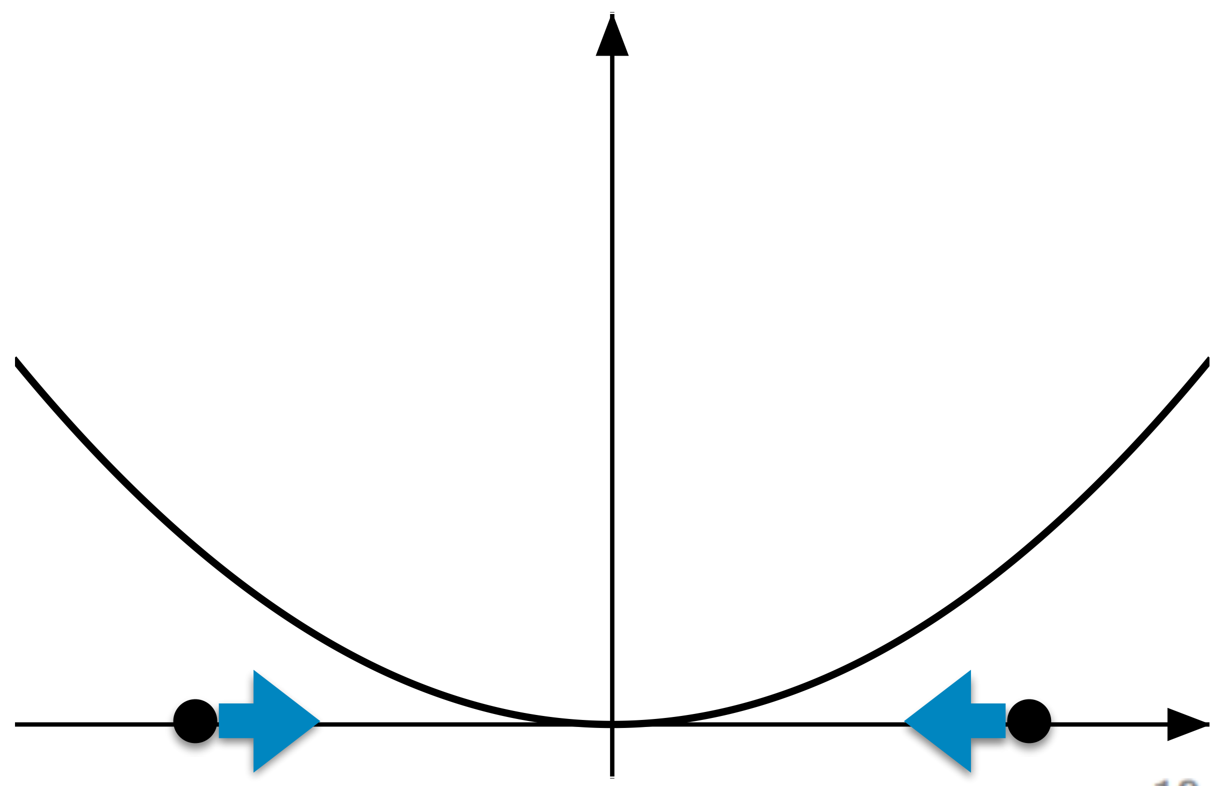

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$\Rightarrow$ Evolution variational inequality (**EVI**)

Evolution variational inequality

What's **EVI** (K, N)?

- A formulation of grad. flow on met. meas. sp. for (K, N)-convex potential
- μP_t solves **EVI**$(K, N) \Rightarrow$ **W**(K, N)

EVI $(0, \infty)$

$(\mu_t)_{t\geq 0}$: abs. conti., $^\forall \nu \in \mathcal{P}_2(X)$,

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$$\left(\begin{array}{l} \text{Formally, (LHS)} = \langle \nabla \mathrm{Ent}(\mu_t), \overrightarrow{\mu_t \nu} \rangle \\ \text{if } \dot{\mu}_t = -\nabla \mathrm{Ent}(\mu_t) \end{array} \right)$$

P_t on metric measure spaces

$(X, d, \mathfrak{m})$: met. meas. sp

$$\boxed{P_t = \mathrm{e}^{t\Delta} \leftrightarrow \text{Cheeger's } L^2\text{-energy}}$$

loc. Lip. const ↴

$$\mathsf{Ch}(f) := \inf_{\substack{f_n:\ \text{Lip.} \\ f_n \to f \text{ in } L^2}} \liminf_{n} \int_X |\nabla f_n|^2 d\mathfrak{m}$$

$$\overset{\exists}{=} \int_X |\nabla f|_w^2 \, d\mathfrak{m}$$

($|\nabla f|_w$: minimal weak upper gradient)

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★ $(X, d, \mathfrak{m})$: infinitesimally Hilbertian

$\overset{\text{def}}{\Leftrightarrow}$ Ch: quadratic form ($\Leftrightarrow$ P_t: linear)

Riemannian CD$^e(K, N)$

Theorem 4 ([Erbar, K. & Sturm '15])

For $K \in \mathbb{R}$ and $N > 0$, TFAE:

(i) CD$^*(K, N)$ *& infin. Hilb.*

(ii) CD$^e(K, N)$ *& infin. Hilb.*

(iii) $^\forall \mu_0$, $^\exists (\mu_t)_{t \ge 0}$: *sol. to* (K, N)-**EVI**

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★ **(i)**–**(iii)**: RCD$^*(K, N)$ cond.

RCD meets Bakry-Émery

Theorem 5 ([Erbar, K. & Sturm '15])

(1) *Suppose* $\mathsf{RCD}^*(K, N)$

$\mathsf{Ch}(f) < \infty$ & $|\nabla f|_w \le 1 \Rightarrow$ f*: Lip.,* $|\nabla f| \le 1$

(2) *Under* $(X, d, \mathfrak{m})$*: infin. Hilb. & concl. of* **(1)***,*

either **(iv)–(vi)** *is also equiv. to* $\mathsf{RCD}^*(K, N)$

- **(iv)** $\mathsf{W}(K, N)$
- **(v)** $\mathsf{G}(K, N)$
- **(vi)** $\mathsf{BE}(K, N)$ *(weak form)*

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★ $N = \infty$: [1,3-5]

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★ $N = \infty$: [1,3-5]

★ $N < \infty$ via Rényi entropy (porous medium eq.) [6]

Extension to other class of sp.'s

- Alexandrov sp.'s $\subset$ **RCD** sp.'s [Petrunin '11]
- Finsler mfds ([42] and references therein)
- (super) Ricci flow [McCann & Topping '10 / . . .]
- Sub-Riemannian mfds. [Juillet '09 / 9 / . . .]
- Graphs (or state sp. of Markov chains) [32 / . . .]
- Hamiltonian system [. . . / Ohta '14]
- $N < \mathbf{0}$ [. . . / Ohta '15+]
- $\boldsymbol{\infty}$-$\mathbf{dim}$ sp.'s [Fang, Shao & Sturm '10 / Erbar & Huesmann '15+ / Ambrosio, Erbar & Savaré]
- Negative result on Sierpinski gasket [Kajino '13]

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<u>Notation</u>

N: $\boldsymbol{N} < \infty$ is essential

R: "Riemannian" is essential

Analytic properties of RCD$^*(K, N)$

- L^∞/Lipschitz regularization of P_t [1,4]:

$$|\nabla P_t f| \le \sqrt{\frac{K}{e^{2Kt} - 1}} \|f\|_\infty$$

- Self-improvement of **BE**(K, ∞) [18]
  $\rightsquigarrow$ (Almost) full-strength of Bakry-Émery
- Li-Yau's ineq. [18 / Garofalo & Mondino '14]
- Gaussian heat kernel estimate,
  L^p-bddness of Riesz transf. [Jiang, Li & Zhang]
- Regularity of harmonic functions [Jiang '14 / Kell]
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- (Sharp) Bishop-Gromov volume comparison [10]
- (Sharp) Bonnet-Myers diameter bound [10]

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  $\leftrightsquigarrow$ $\mathbf{EVI}(K, N)$

- Tensorization [13]
- $\mathsf{RCD}^*_{\mathrm{loc}}(K, N) \Rightarrow \mathsf{RCD}^*(K, N)$ [13]
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Splitting theorem [14]

$\mathsf{RCD}^*(0, N)$ with $\mathbb{R} \hookrightarrow X$ isom.

$$\Rightarrow (X, d, \mathfrak{m}) \underset{\text{isom}}{\simeq} \mathbb{R} \times \mathsf{RCD}^*(0, N-1)$$

Applications

- Maximal diameter theorem [20]:

  $\mathsf{RCD}^*(N, N+1)$ with $\mathrm{diam}(X) = \pi$

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- Lichnerowicz-Obata theorem [21]:

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Further structures

- Rectifiability on $\mathbf{RCD}^*(K, N)$ [30]:
  $^\exists R_j \subset X$ $(j \in \mathbb{N})$ s.t.
  - $\mathfrak{m}(X \setminus \bigcup_{j\in\mathbb{N}} R_j) = 0$
  - $^\exists$ $R_j \to \mathbb{R}^{k_j}$ bi-Lip. embedding $(k_j \leq N)$
- Non-smooth diff. geom. on $\mathbf{RCD}(K, \infty)$ [15]:
  "Analytic" approach to 1st and 2nd order differentiable structures
  vector fields/diff. forms without (co)tangent sp.
  covariant/exterior derivative, Hessian,
  Ricci curvature

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- Non-smooth diff. geom. on **RCD**(K,∞) [15]:
  "Analytic" approach to 1st and 2nd order differentiable structures
  vector fields/diff. forms without (co)tangent sp.
  covariant/exterior derivative, Hessian,
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Further structures

- Rectifiability on $\mathbf{RCD}^*(K, N)$ [30]:
  $^\exists R_j \subset X$ $(j \in \mathbb{N})$ s.t.
  - $\mathfrak{m}(X \setminus \bigcup_{j \in \mathbb{N}} R_j) = 0$
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Lévy-Gromov

Isoperimetric ineq. [11]

On $\mathbf{RCD}^*(K, N)$ with diam $\leq D$ & $\mathfrak{m} \in \mathcal{P}(X)$,

ε-enlargement

$$\mathfrak{m}^+(A) := \varliminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(A^\varepsilon) - \mathfrak{m}(A)}{\varepsilon}$$: Minkowski content

$\mathcal{I}(r) := \inf\{\mathfrak{m}^+(A) \mid \mathfrak{m}(A) = r\}$: Isop. profile

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$\mathcal{I}_{K,N,D}$: (1-dim.) model isop. profile of

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★ Sharp and rigid

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Idea of the pf.: Localization method in [23]

- Localization: Decompose opt. trans. plan in each geod.'s (Needle decomposition)
- Sharp estimate on each "needle"
- Integration

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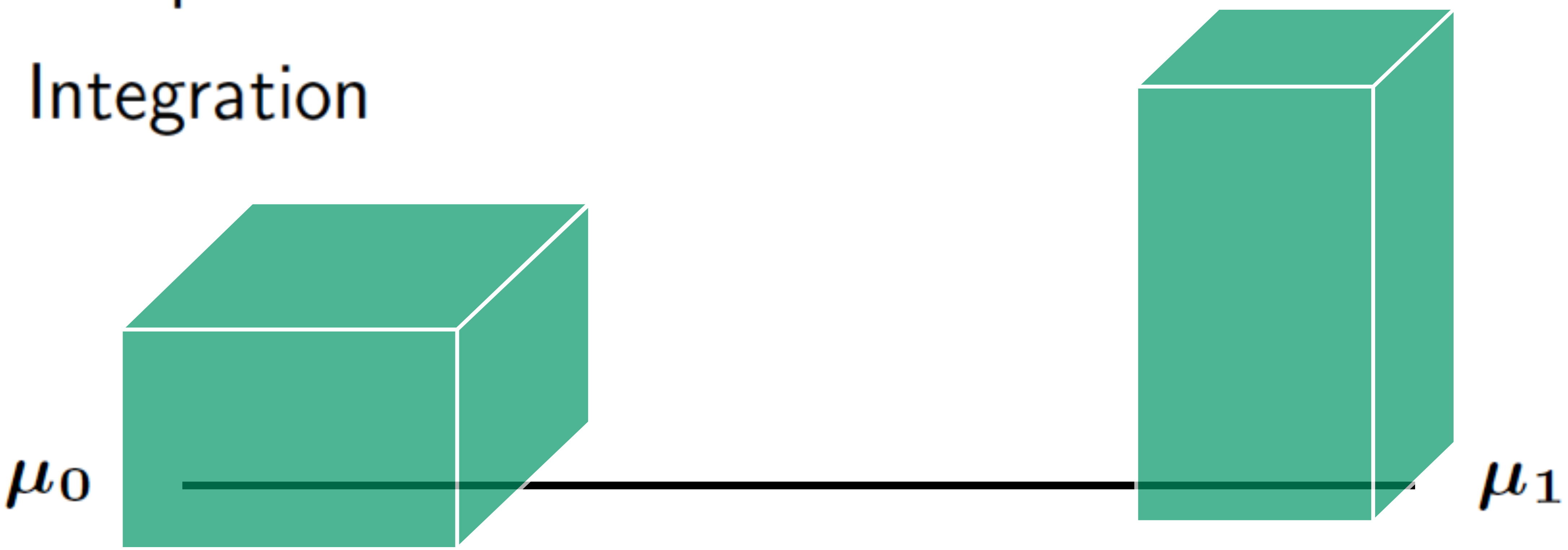

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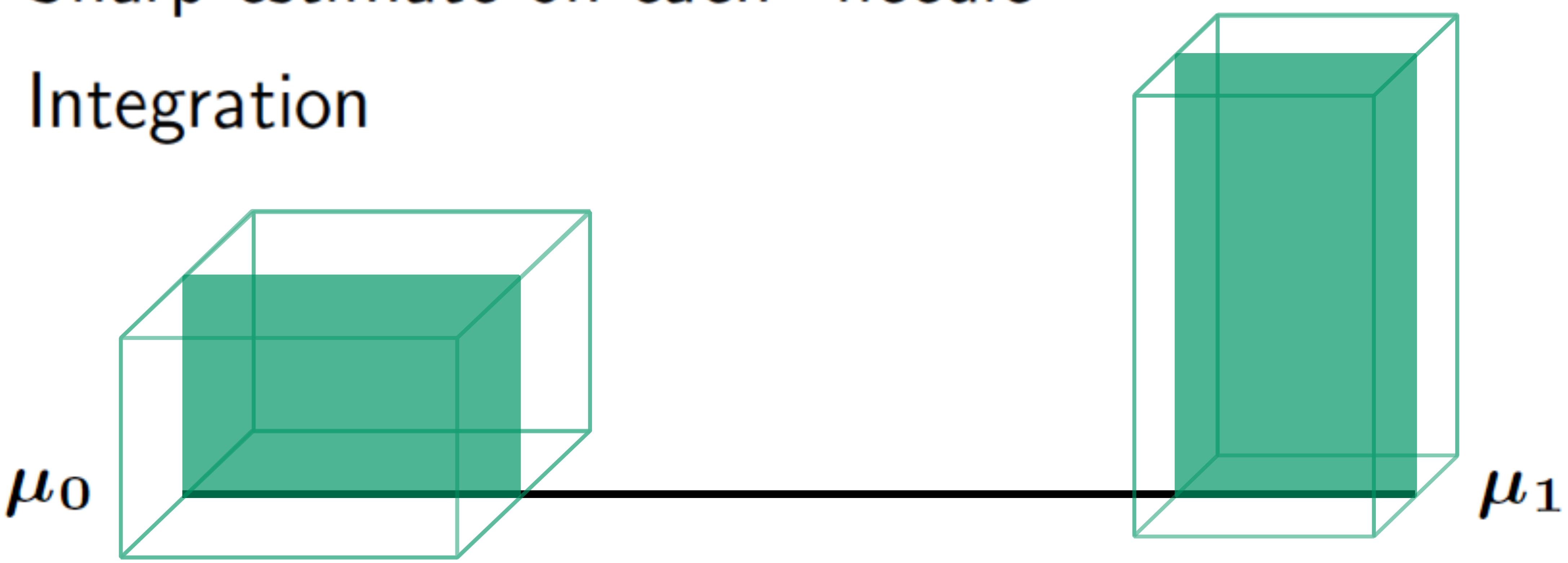

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† Apply this to $\mu_0 = \mathfrak{m}(A)^{-1}\mathbf{1}_A\mathfrak{m}$ and $\mu_1 = \mathfrak{m}$ □

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★ No use of geom. meas. theory (cf. [E. Milman '11])

Functional ineq.'s on $\mathsf{RCD}^*(K, N)$

N-extension of f'nal ineq.'s [13]:

- N-HWI ineq.

  $\Rightarrow$ N-log Sobolev/Entropy-energy ineq. $(K > 0)$

  $\Rightarrow$ N-Talagrand ineq. $(K > 0)$

Sharp and rigid f'nal ineq.'s [12]:

- N-improvement in const. of log Sobolev & Talargrand $(K > 0)$
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(P_t-)Gaussian isoperimetry

[Bakry & Ledoux '96 / K.]: On **RCD**$^*(K,\infty)$ sp.'s,

$$\sqrt{\mathcal{I}(P_t f)^2 + \alpha |\nabla P_t f|_w^2} \le P_t \sqrt{\mathcal{I}(f)^2 + c_\alpha(t) |\nabla f|_w^2}$$

$$c_\alpha(t) := \frac{1 - \mathrm{e}^{-2Kt}}{K} + \alpha \mathrm{e}^{-2Kt} \ (\alpha \ge 0)$$

$$\mathcal{I} := \Phi' \circ \Phi^{-1},\ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2} du$$

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$\mathcal{W}$-entropy

- $\mathbf{W}(0, N) \Rightarrow \mathcal{W}(t) \searrow$ [K.] (cf. [Topping '09])

$$\mathcal{W}(t) := tI(\mu P_t) - \mathrm{Ent}(\mu P_t) - \frac{N}{2}\log t + c \qquad (\mathcal{W}\text{-entropy})$$

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Q. Rigidity on $\mathbf{RCD}^*(0, N)$ sp's?
(Work in progress with X.-D. Li)

Problems

- Difference between **CD**, **CD*** & **CD**^{e}
- Examples
- Topology of **RCD**$^*(K, N)$ spaces

- Heat kernel comparison ([K. & Kuwae] in progress)
- Stochastic analysis (e.g. coupling of Brownian motions; [K.] when $N = \infty$)
- Isoperimetric ineq. via P_t when $N < \infty$