On estimates of transportation costs for heat distributions

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1. Introduction

Framework

(X, g): m-dim. Riem. mfd., d: dist., $dv = \mathrm{e}^{-V} d\,\mathrm{vol}$

$\mathcal{L} := \Delta - \nabla V \cdot \nabla, \quad P_t := \mathrm{e}^{t\mathcal{L}}$

$\mu P_t \in \mathcal{P}(X)$: heat dist. ($\mu \in \mathcal{P}(X)$: initial data)

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<u>Ass.</u>

$^{\exists}K \in \mathbb{R}$, $^{\exists}N \in [m,\infty]$ s.t.

$$(\dagger)\ \mathrm{Ric} + \mathrm{Hess}\, V - \frac{1}{N-m} \nabla V \otimes \nabla V \geq K$$

$$(\text{“}\mathrm{Ric} \geq K \ \&\ \dim \leq N\text{”})$$

Framework

Remarks

- (†) ⇔ Bakry-Émery's curvature-dimension cond. (**BE**(K, N)):

$$\Gamma_2(f,f) \geq K|\nabla f|^2 + \frac{1}{N}(\mathcal{L}f)^2,$$

where $\Gamma_2(f,f) := \frac{1}{2}\mathcal{L}|\nabla f|^2 - \langle \nabla f, \nabla \mathcal{L} f \rangle$

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($\mathbf{BE}(K, N)$ is formulated only in terms of $\mathcal{L}$)

Purpose

$$\mathcal{T}_c(\mu,\nu) := \inf\left\{ \int_{X\times X} c\, d\pi \;\middle|\; \begin{array}{c}\pi\text{: coupling of}\\ \mu \text{ and } \nu\end{array} \right\}$$

(Optimal transportation cost for a cost function c)

$W_p := \mathcal{T}_{d^p}^{1/p}$: L^p-Wasserstein distance

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<u>Goal</u>

- Estimates of $\mathcal{T}_c(\mu P_t, \nu P_s)$ in terms of μ, ν, t, s, in conn. with $\mathbf{BE}(K,N)$ (and related conditions / their applications)
- "Robust" arguments valid on $\mathbf{RCD}^*(K,N)$ sp's (metric measure spaces with "(†)" via opt. trans.)

How and why ?

Semigroup methods
(gradient estimate)

Optimal transport
(gradient flow interpretation)

Estimates of $\mathcal{T}_c(\mu P_t, \nu P_s)$

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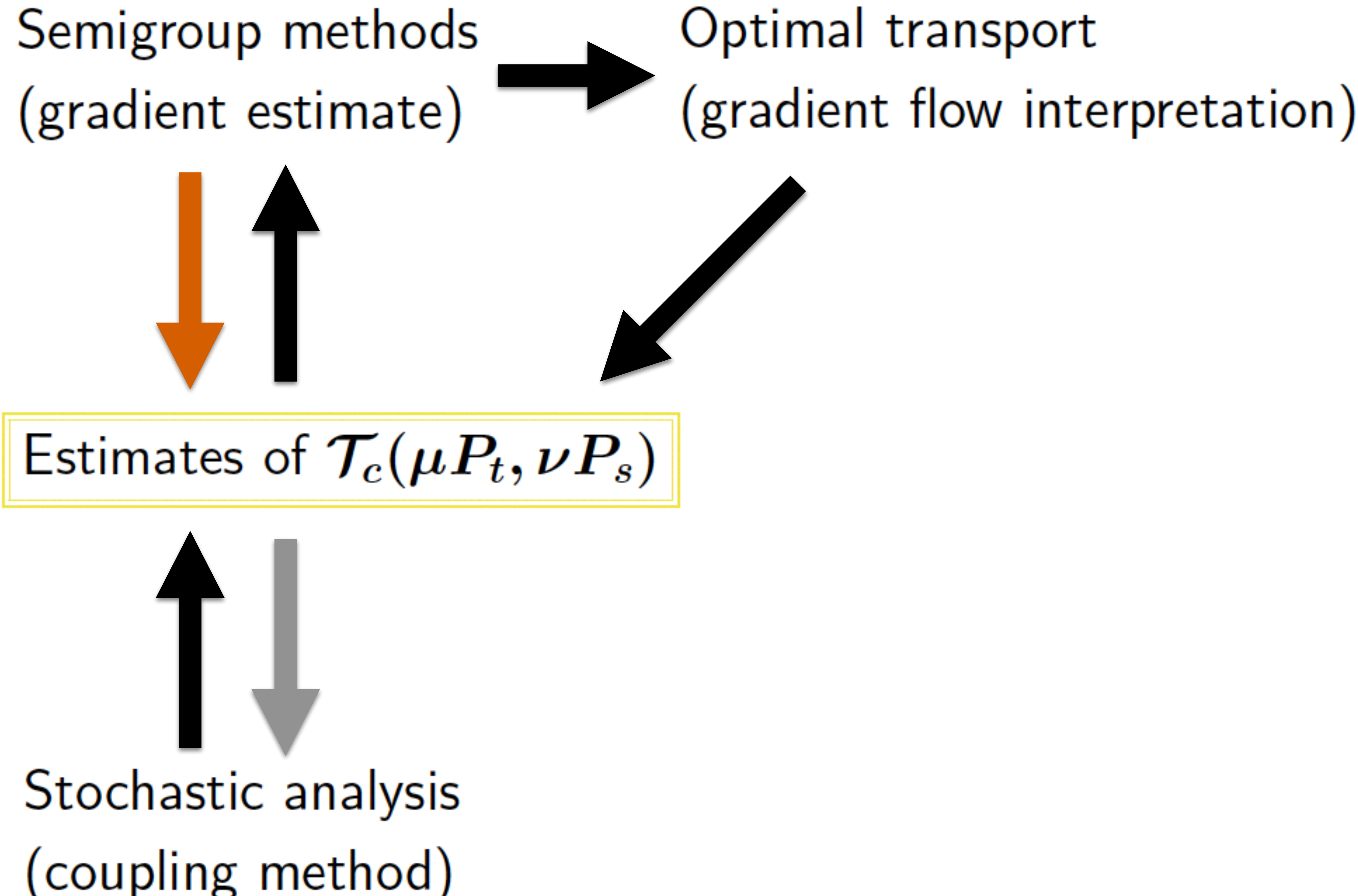
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Outline of the talk

1. **Introduction**

2. **Known results**

3. L^p/L^q**-extension**

4. **Coupling by reflection**

5. **Related results**

6. **Questions**

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2. **Known results**

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5. Related results

6. Questions

Gradient estimate of P_t

$\mathbf{G}(K,N)$:

$$|\nabla P_t f|^2 \le \mathrm{e}^{-2Kt} P_t(|\nabla f|^2) - \frac{1-\mathrm{e}^{-2Kt}}{NK} (\mathcal{L} P_t f)^2$$

★ $\mathbf{BE}(K,N) \Leftrightarrow \mathbf{G}(K,N)$
[Bakry & Émery '84 ($N=\infty$)/ Bakry & Ledoux '06]

Proof

$\Leftarrow$ "$\left.\dfrac{\partial}{\partial t}\right|_{t=0}$"

$\Rightarrow$ "$\displaystyle\int_0^t ds$ of a diff. ineq. for $P_{t-s}(|\nabla P_s f|^2)$" □

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Space-time W_2-contraction

<u>**W**(K, N)</u>:

$$\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu P_t, \nu P_s)}{2}\right) \le \mathrm{e}^{-K(s+t)}\mathfrak{s}_{K/N}^2\left(\frac{W_2(\mu,\nu)}{2}\right) + \frac{N}{2}\cdot\frac{1-\mathrm{e}^{-K(s+t)}}{K(s+t)}(\sqrt{t}-\sqrt{s})^2$$

- $\mathfrak{s}_\kappa(r) := \dfrac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}$
- $N = \infty$: $s = t$ & the last term $= 0$

★ **G**$(K, N) \Leftrightarrow$ **W**(K, N)

[K. '15+ / Erbar, K. & Sturm '15+]

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Sketch of the pf. (G $\Rightarrow$ W, $N = \infty$)

G(K,∞): $|\nabla P_t| \le \mathrm{e}^{-Kt} P_t(|\nabla f|^2)^{1/2}$

W(K,∞): $W_2(\mu P_t, \nu P_t) \le \mathrm{e}^{-Kt} W_2(\mu,\nu)$

- Kantorovich duality:

$$\frac{W_2(\nu,\mu)^2}{2} = \sup_{f \in C_b(X)} \left[\int_X Q_1 f \, d\mu - \int_X f \, d\nu \right]$$

- Hopf-Lax semigroup:

$$Q_r f(x) := \inf_{y \in X} \left[f(y) + \frac{d(x,y)^2}{2r} \right]$$

★ $\partial_r Q_r f = -\frac{1}{2} |\nabla Q_r f|^2$ (Hamilton-Jacobi eq.)

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Sketch of the pf. (G $\Rightarrow$ W, $N = \infty$)

For simplicity, $\mu = \delta_{x_0}$, $\nu = \delta_{x_1}$

$$\frac{W_2(\delta_{x_0} P_t, \delta_{x_1} P_t)^2}{2} = \sup_f \left[P_t Q_1 f(x_1) - P_t f(x_0) \right]$$

Idea: give an upper bound of $[\cdots]$ being uniform in f

Sketch of the pf. (G $\Rightarrow$ W, $N = \infty$)

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$\gamma : [0,1] \to M$: geod. joining x_0 & x_1

$$\begin{aligned}
P_t Q_1 f(x_1) - P_t f(x_0) &= P_t Q_1 f(\gamma_1) - P_t Q_0 f(\gamma_0) \\
&= \int_0^1 \partial_r P_t Q_r f(\gamma_r) dr \\
&= \int_0^1 \left(\langle \nabla P_t Q_r f, \dot{\gamma}_r \rangle - \frac{1}{2} P_t(|\nabla Q_t f|^2)(\gamma_r) \right) dr \\
&\le \cdots \le \frac{\mathrm{e}^{-2Kt}}{2} d(x_0, x_1)^2
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L^p / L^q-extension

★ $\mathsf{G}_q(K, N) \Leftrightarrow \mathsf{W}_p(K, N)$ $\left(\frac{1}{p} = 1 - \frac{1}{q} \leq \frac{1}{2}\right)$

[K. '10, K. '13, ...($N = \infty$) / K. '15+]

<u>$\mathsf{G}_q(K, N)$</u>: ($\hat{N} := N + p - 2$)

$$|\nabla P_t f|^2 \leq \mathrm{e}^{-2Kt} P_t(|\nabla f|^q)^{2/q} - \frac{1 - \mathrm{e}^{-2Kt}}{\hat{N} K} (\mathcal{L} P_t f)^2$$

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$$\mathfrak{s}^2_{K/\hat{N}} \left(\frac{W_p(\mu P_t, \nu P_s)}{2}\right) \leq \mathrm{e}^{-K(s+t)} \mathfrak{s}^2_{K/\hat{N}} \left(\frac{W_p(\mu, \nu)}{2}\right) + \frac{\hat{N}}{2} \cdot \frac{1 - \mathrm{e}^{-K(s+t)}}{K(s+t)} (\sqrt{t} - \sqrt{s})^2$$

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L^p/L^q-extension

$$\bigstar\ \mathsf{G}_q(K,N) \Leftrightarrow \mathsf{W}_p(K,N)\ \left(\frac{1}{p} = 1 - \frac{1}{q} \le \frac{1}{2}\right)$$

[K. '10, K. '13, ...($N = \infty$) / K. '15+]

$\underline{\mathsf{G}_q(K,N)}$: $(\hat{N} := N + p - 2)$

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How we obtain L^p-results?

- $\mathsf{BE}(K,\infty)$

  $\Rightarrow \Gamma_2(f,f) - K|\nabla f|^2 \geq \dfrac{|\nabla|\nabla f|^2|^2}{4|\nabla f|^2}$

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Conn. with Laplacian comparison thm

- $\mathsf{W}_2(0, N)$:

  $$W_2(\mu P_t, \nu P_s)^2 \le W_2(\mu, \nu)^2 + 2N(\sqrt{t} - \sqrt{s})^2$$

  $s = 0$, $\nu = \delta_o$, $\mu = \delta_x$:

  $$P_t d(o, \cdot)^2(x) \le d(o, x)^2 + 2Nt$$

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2. Known results

3. L^p/L^q-extension

4. Coupling by reflection

5. Related results

6. Questions

L^p/L^q-extension on non-smooth sp's

Theorem 1 ([K.])

On $\mathsf{RCD}^*(K,N)$ *sp's,*
$G_q(K,N)$ *holds for* $1 < q \leq 2$.

L^p/L^q-extension on non-smooth sp's

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BE(K,N) *yields the following:*

$$\Gamma_2(f,f) - K|\nabla f|^2 - \frac{1}{N+p-2}(\mathcal{L}f)^2 \geq \frac{2-q}{4}\frac{|\nabla|\nabla f|^2|^2}{|\nabla f|^2}$$

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- $^{\exists}$Some self-improvement of $\mathbf{BE}(K, N)$, $N < \infty$: [Bakry & Qian '00 / Sturm '14 / B.-X. Han '14]

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&(\mathrm{e}^{2Kt}-1)|\nabla P_t f|^2 \\
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I_N := \Phi_N' \circ \Phi_N^{-1}, \ \ \Phi_N(x) := \frac{1}{\mathcal{Z}}\int_0^x \mathfrak{s}_{K/(N-1)}^{N-1}(r)dr
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Coupling by refl. on Riem. mfd

[Kendall '86 / Cranston '91 / ...] $\mathrm{Ric} \geq 0$

$\Rightarrow {}^\forall x_0, x_1 \in X$, ${}^\exists(B_t^{(0)}, B_t^{(1)})$: coupling of BM's starting at (x_0, x_1) & a 1-dim (std.) BM W_t s.t.

$$d(B_t^{(0)}, B_t^{(1)}) \leq d(x_0, x_1) + 2\sqrt{2} W_t$$

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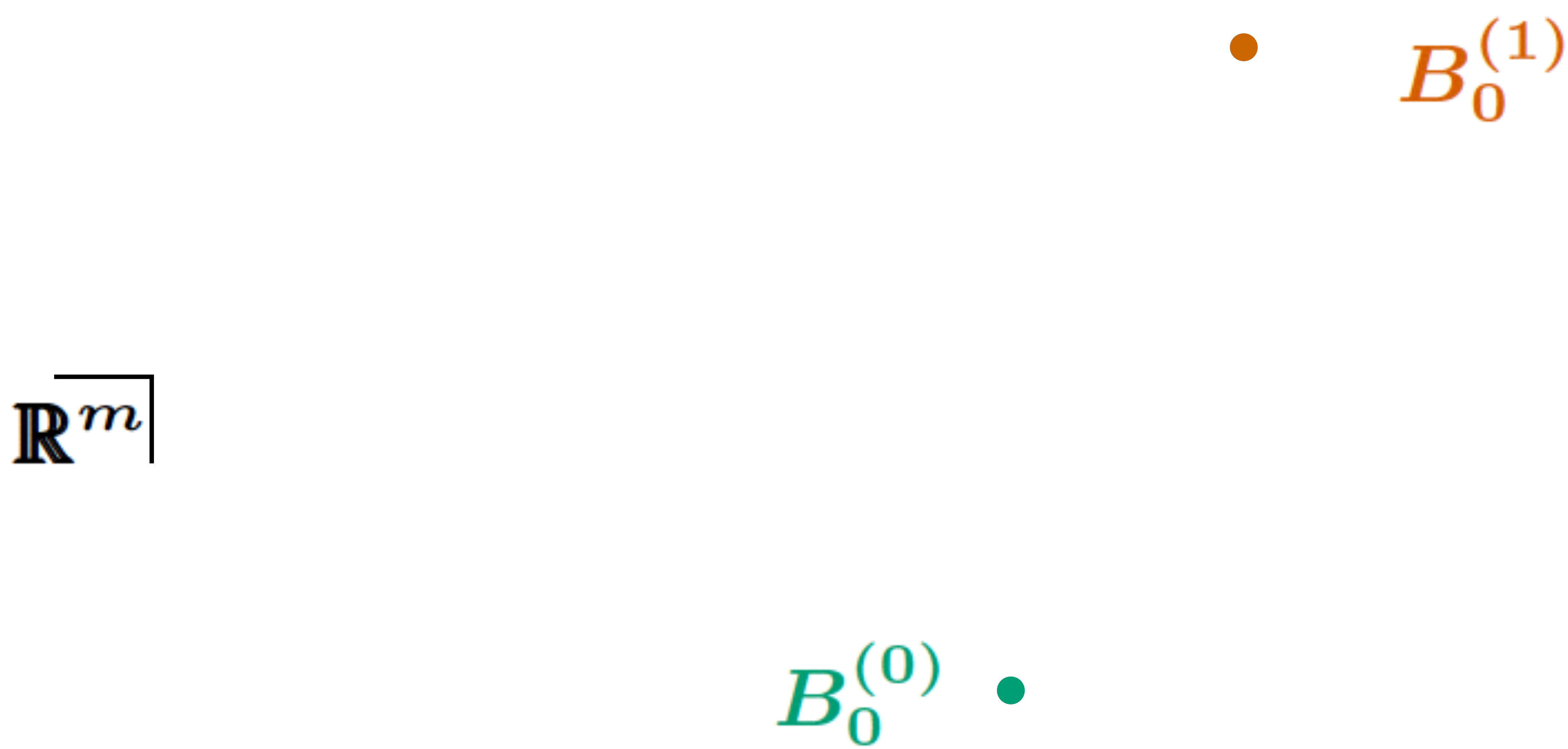

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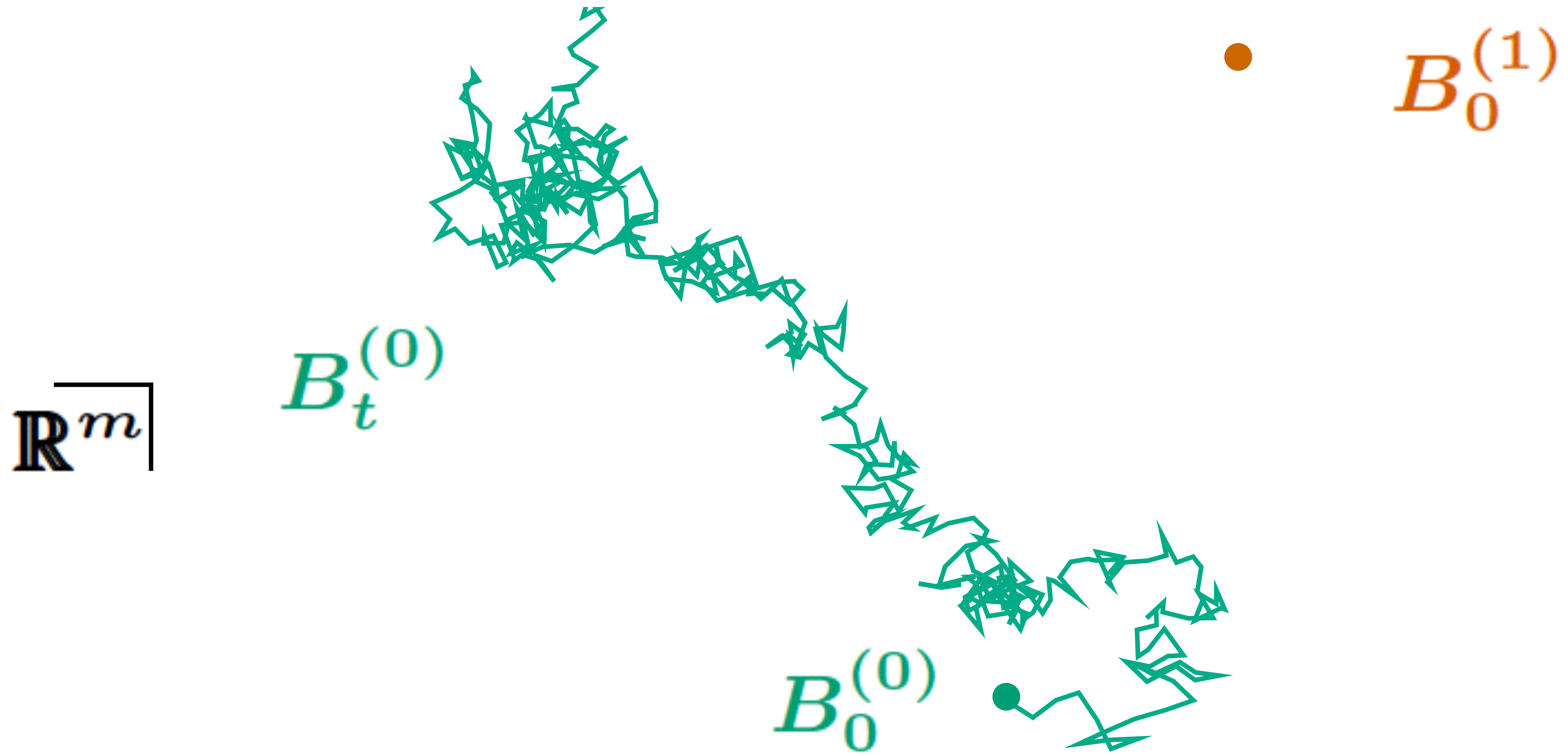

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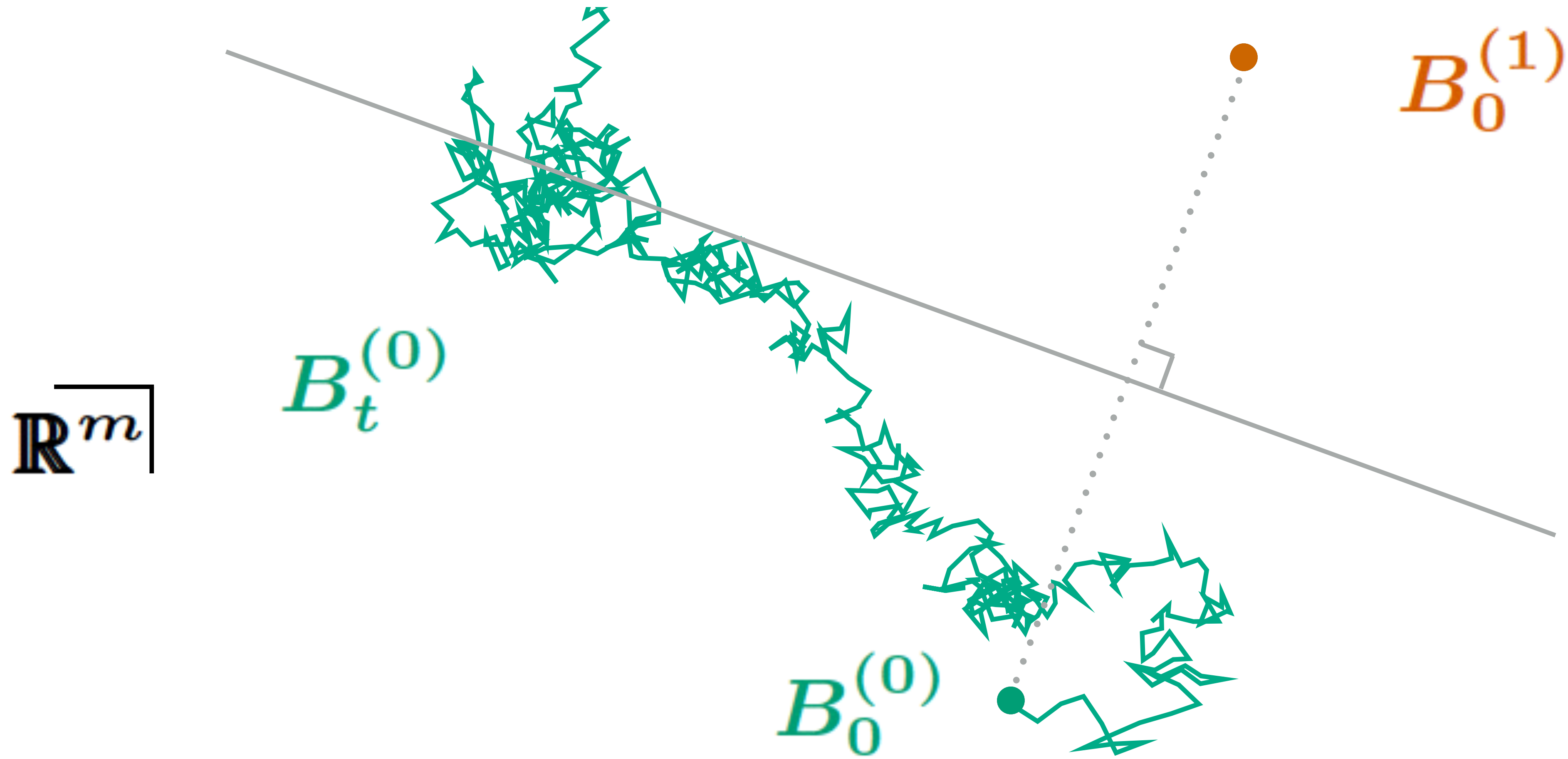

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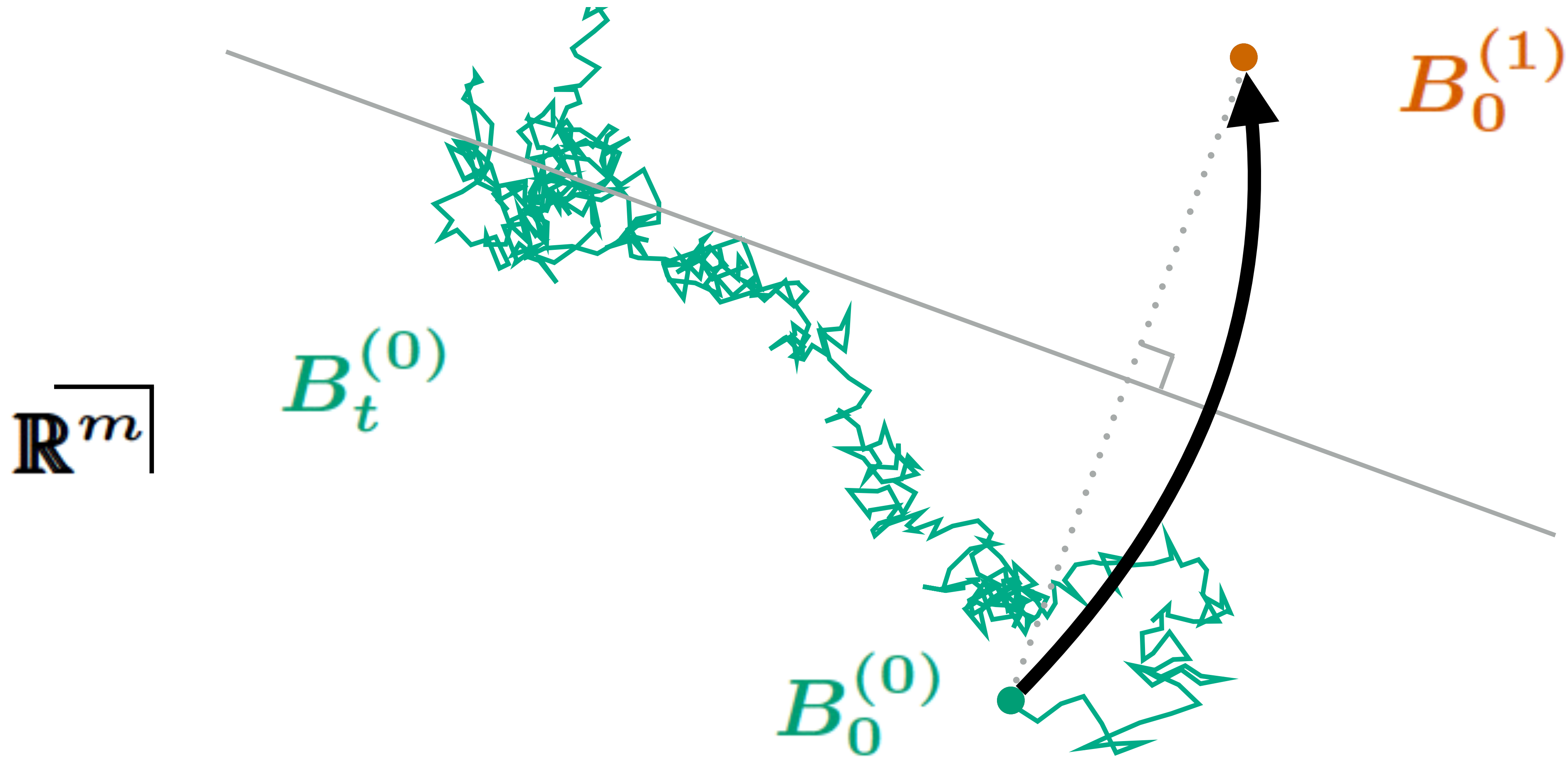

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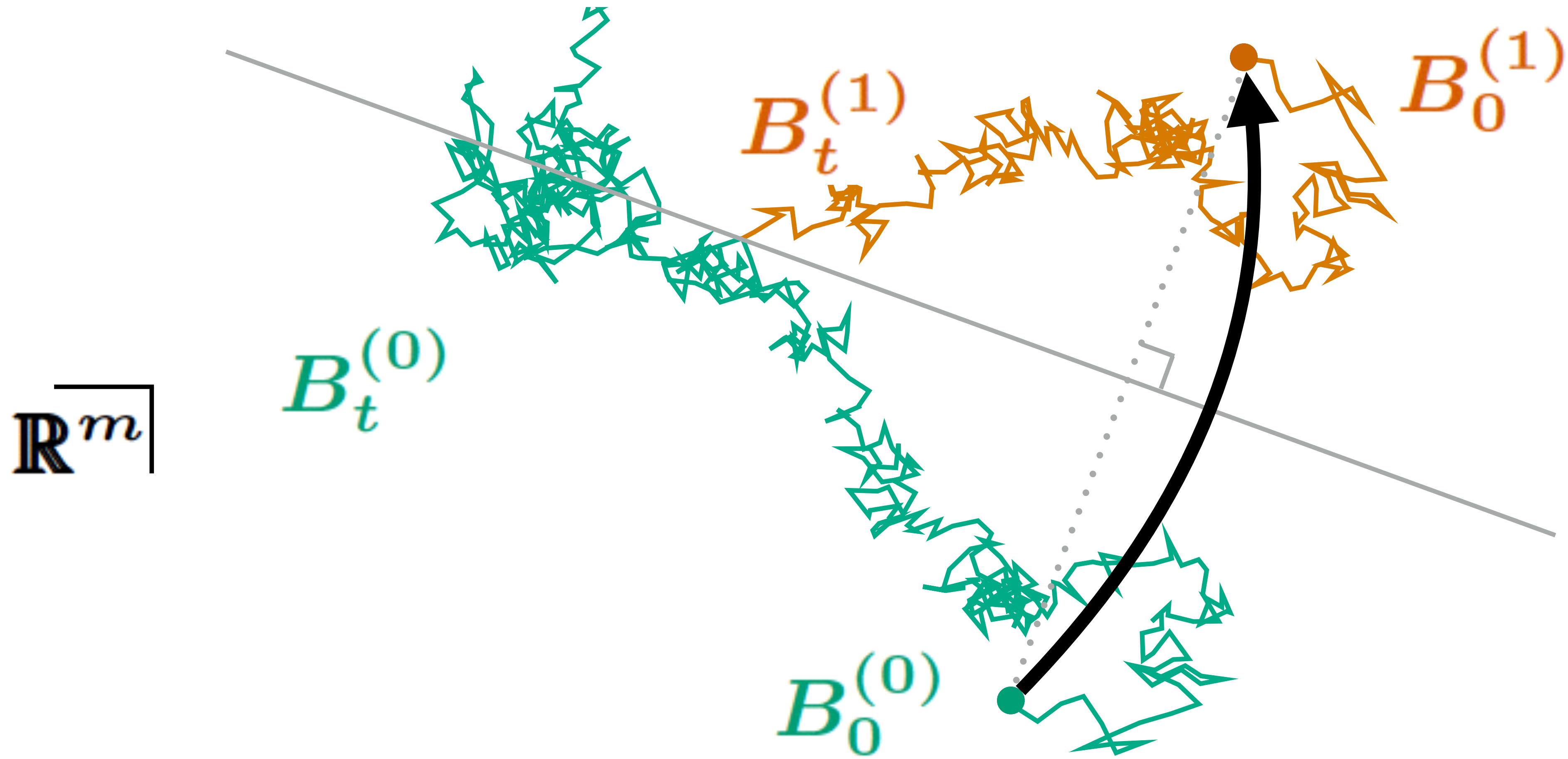

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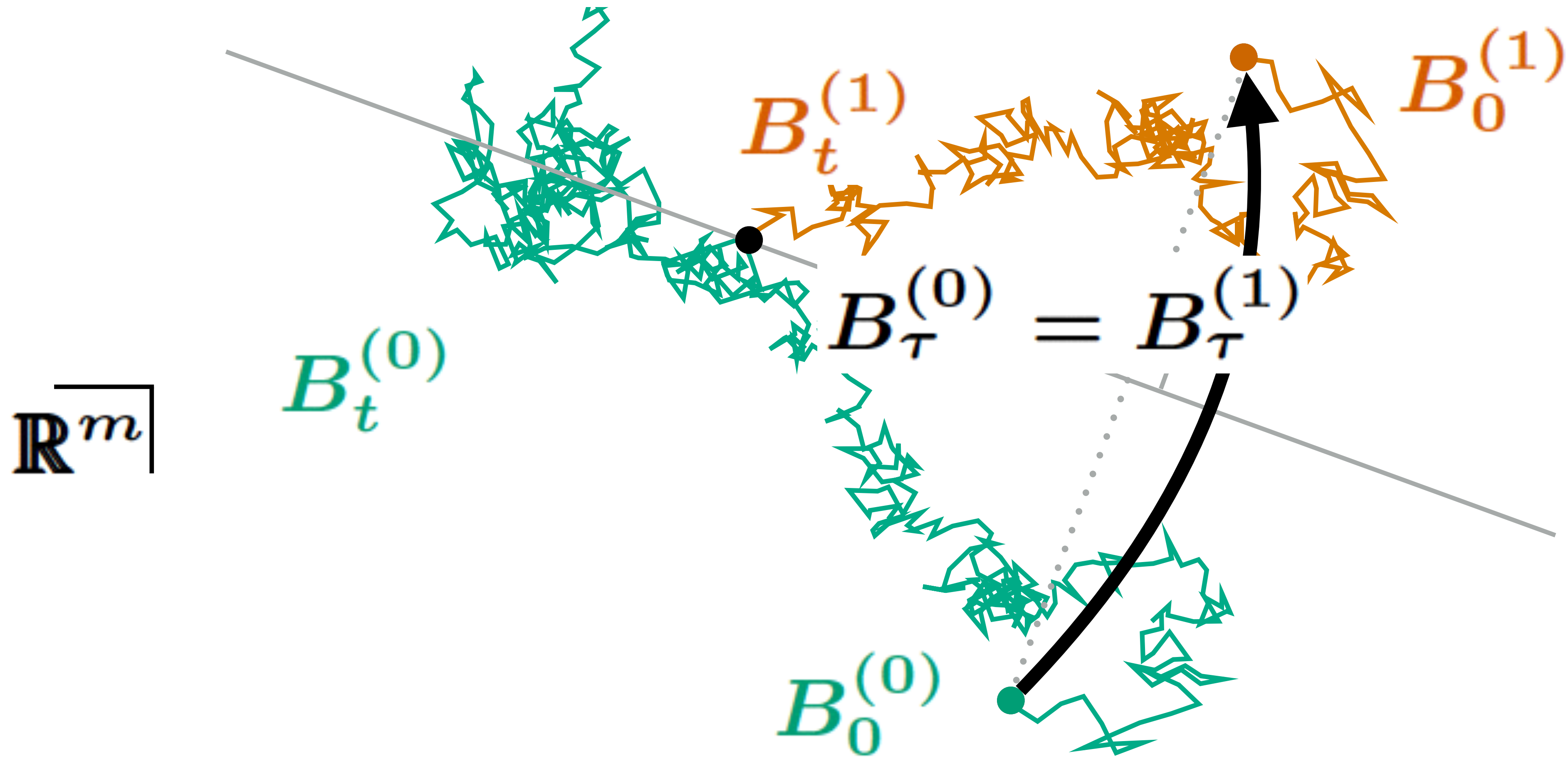

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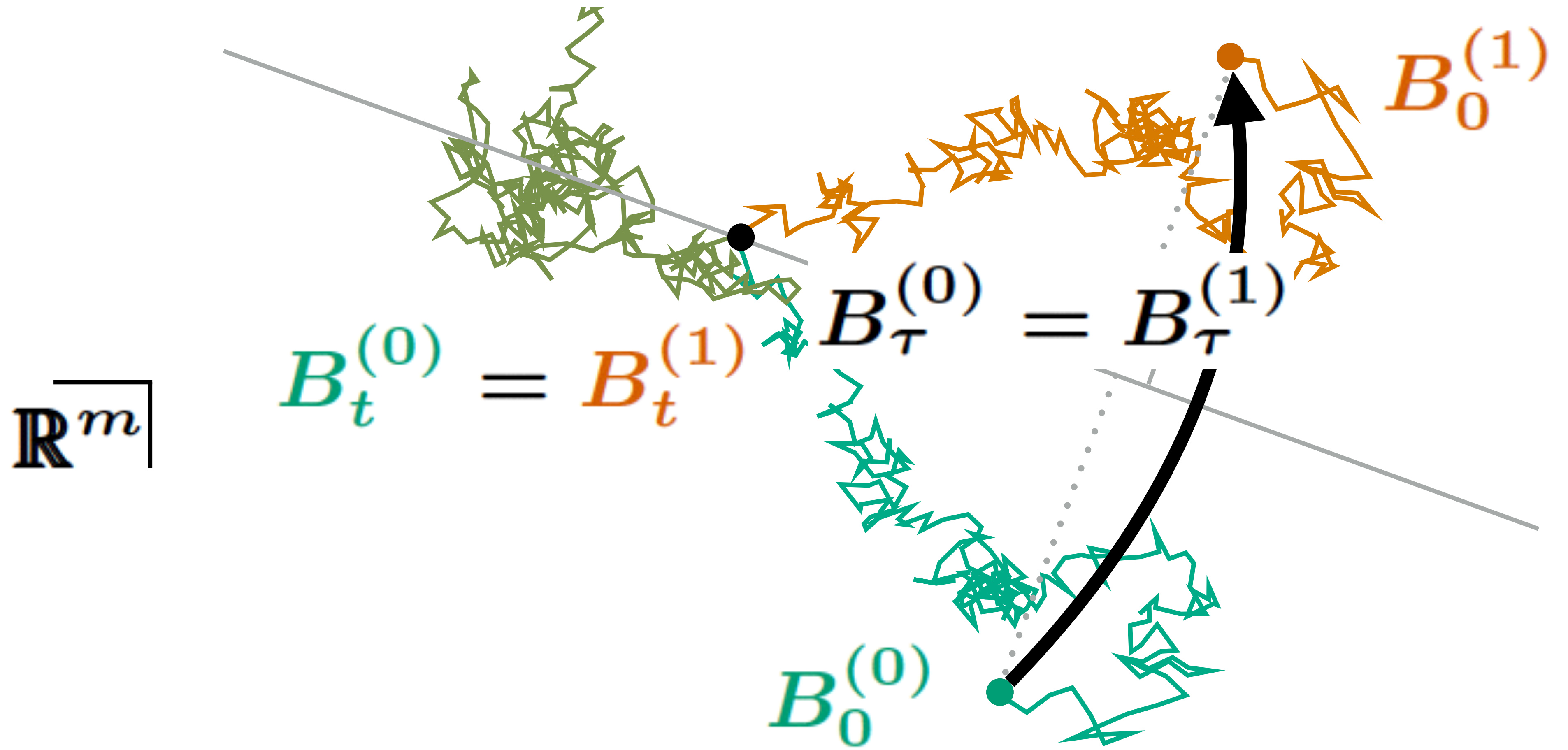

Coupling by refl. and opt. trans.

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Coupling by refl. and opt. trans.

$$\boxed{d(B_t^{(0)}, B_t^{(1)}) \le d(x_0, x_1) + 2\sqrt{2} W_t}$$

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$\Rightarrow {}^\forall s, t > 0,$

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${}^\forall T > 0$, $\mathcal{T}_{\varphi_{T-t}(d)}(\delta_{x_0}P_t, \delta_{x_1}P_t) \searrow$ [K. & Sturm '13]

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F'nal ineq. $\Rightarrow$ coupling by refl.

Theorem 2 ([K.])

On **RCD**$^*(0, \infty)$ *sp's*, $^\forall T > 0$, $\mathcal{T}_{\varphi_{T-t}(d)}(\mu P_t, \nu P_t) \searrow$

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On **RCD**$^*(0,\infty)$ *sp's,* $^\forall x_0, x_1 \in X$,

$^\exists(B_t^{(0)}, B_t^{(1)})$*: a coupling of BMs s.t.*

- $(B_0^{(0)}, B_0^{(1)}) = (x_0, x_1)$
- $^\forall t > 0$, $\mathbb{P}[\tau > t] \le \varphi_t(d(x_0, x_1))$

In particular, $\mathbb{P}[\tau = \infty] = 0$

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Theorem 3 ([K.])

On **RCD**$^*(0,\infty)$ *sp's*, $^\forall x_0, x_1 \in X$,

$^\exists(B_t^{(0)}, B_t^{(1)})$: *a coupling of BMs s.t.*

- $(B_0^{(0)}, B_0^{(1)}) = (x_0, x_1)$
- $^\forall t > 0$, $\mathbb{P}[\tau > t] \le \varphi_t(d(x_0, x_1))$

In particular, $\mathbb{P}[\tau = \infty] = 0$

★ Extension to $K \neq 0$: OK. / Ext. to $N < \infty$: Not yet

Idea of the pf. of Thm 2 & 3

Thm 2 Kantorovich duality,

Reverse f'nal Gaussian isoperimetry for P_t

$$(e^{2Kt} - 1)|\nabla P_t f|^2 \le I(P_t f)^2 - P_t(I(f))^2,$$

$$I := \Phi' \circ \Phi^{-1}, \ \Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

& $\mathbf{W}_\infty(0, \infty)$ or Gaussian f'nal isop. for P_t

(cf. [Bakry, Gentil & Ledoux '15+])

Thm 3 Making coupled trans. prob. by optimal coupling

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6. Questions

Another dimensional W_2-contraction

Theorem 4 ([Bolley, Gentil, Guillin & K.])

$\mathsf{BE}(K,N)$ *is equiv. to the following:*

$$\mathfrak{s}^2_{K/N}\left(\frac{W_2(\mu P_t,\nu P_t)}{2}\right)\le \mathrm{e}^{-2Kt}\mathfrak{s}^2_{K/N}\left(\frac{W_2(\mu,\nu)}{2}\right) - 2N\int_0^t \mathrm{e}^{-2K(t-s)} \times \sinh^2\left(\frac{\mathrm{Ent}_v(\mu P_s)-\mathrm{Ent}_v(\nu P_s)}{2N}\right) ds$$

- New even on (non-cpt.) Riem. mfds
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$\mathcal{W}$-entropy

- $\mathbf{W}_2(0, N) \Rightarrow \mathcal{W}(t) \searrow$ [K.]

$$\mathcal{W}(t) := tI_v(\mu P_t) - \mathrm{Ent}_v(\mu P_t) - \frac{N}{2}\log t + c$$

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Q. On **RCD***$(0, N)$ sp's?
(Work in progress with X.-D. Li)

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- Applications of $\mathbf{G}_q(K, N)$ / $\mathbf{W}_p(K, N)$
- Estimate involving a sharp Lap. comparison,
- Sharp f'nal isop. for P_t with $N < \infty$
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