Coupling by reflection of Brownian motions on $\mathrm{RCD}(K, \infty)$ spaces

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1. Introduction

Problem

Heat eq. on a Riem. mfd X

$P_t = \mathrm{e}^{t\Delta}$: heat semigroup

$\partial_t P_t f = -\nabla \mathcal{E}(P_t f)$ on L^2, "$\mathcal{E}(f) = \dfrac{1}{2}\int |\nabla f|^2$"

$\leftrightarrow P_t \mu = -\nabla \, \mathrm{Ent}(P_t \mu)$ on $(\mathcal{P}_2(X), W_2)$

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Q. On met. meas. sp.'s with "$\mathrm{Ric} \geq K$"?

Coupling by parallel transport

★ $\mathrm{Ric} \geq K$ on Riem. mfd

$\Rightarrow$ $^{\forall}$initial data, $^{\exists}(B_t^{(0)}, B_t^{(1)})$: a coupling of BMs s.t. $\mathrm{e}^{Kt} d(B_t^{(0)}, B_t^{(1)}) \searrow$ in t.

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$\Downarrow$

- $e^{Kt} W_2(P_t \mu, P_t \nu) \searrow$ in t
- $|\nabla P_t f| \leq e^{-Kt} P_t(|\nabla f|^2)^{1/2}$

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$$\mathrm{Hess\,Ent} \ge K \text{ on } (\mathcal{P}_2(X), W_2)$$

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> s.t. $e^{Kt} d(B_t^{(0)}, B_t^{(1)}) \searrow$ in t.

$\Uparrow$ [Sturm]

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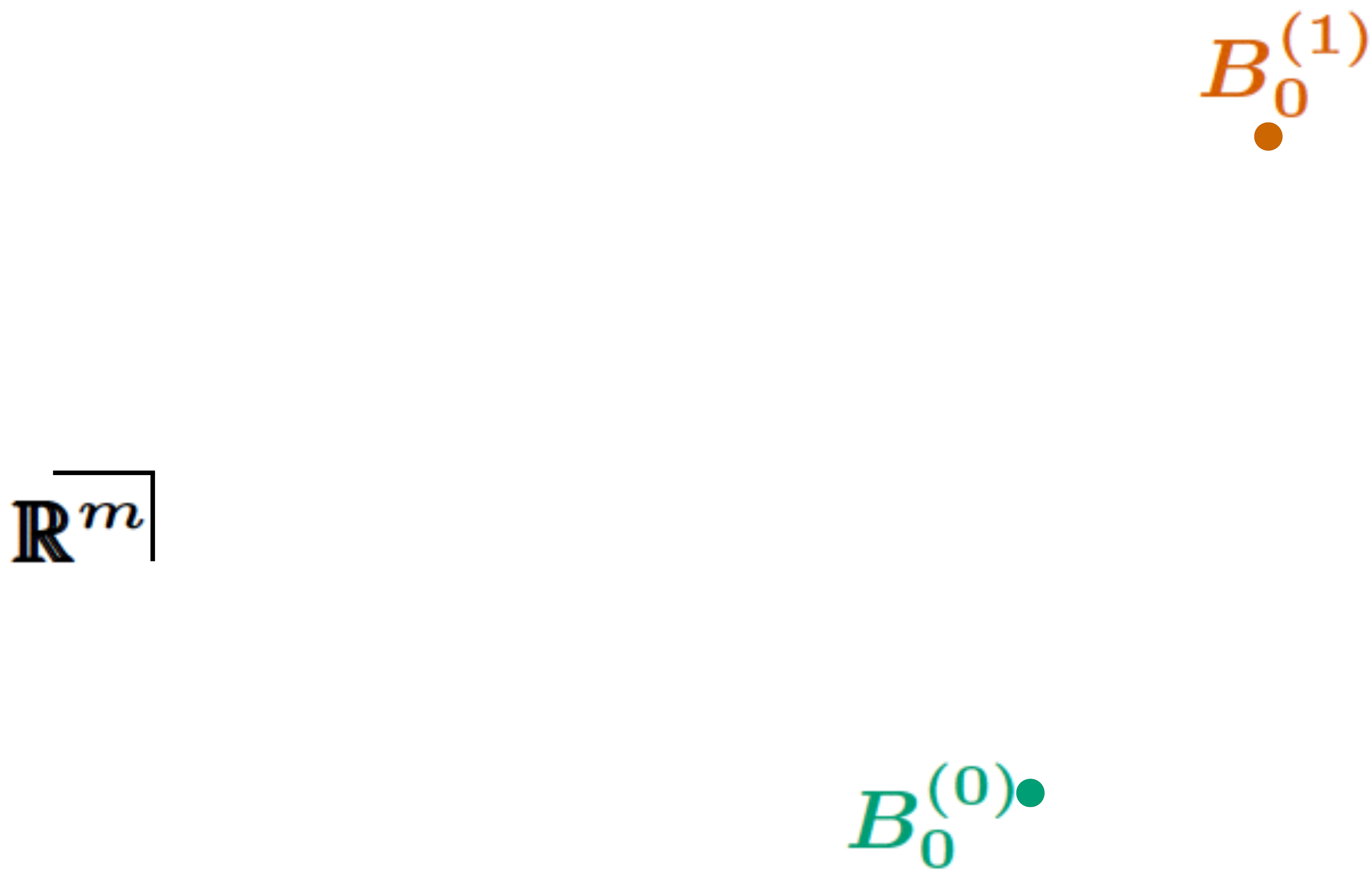

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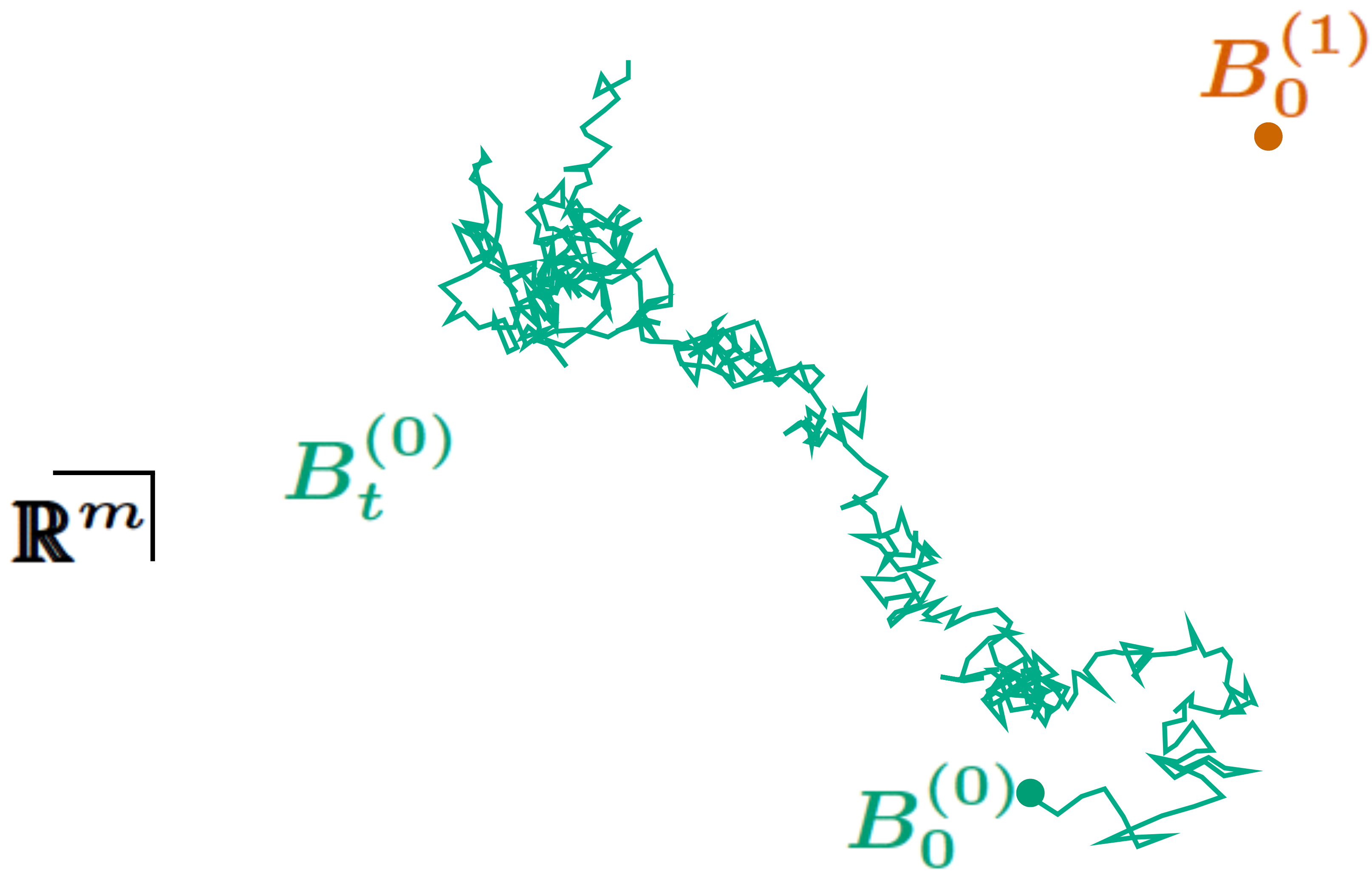

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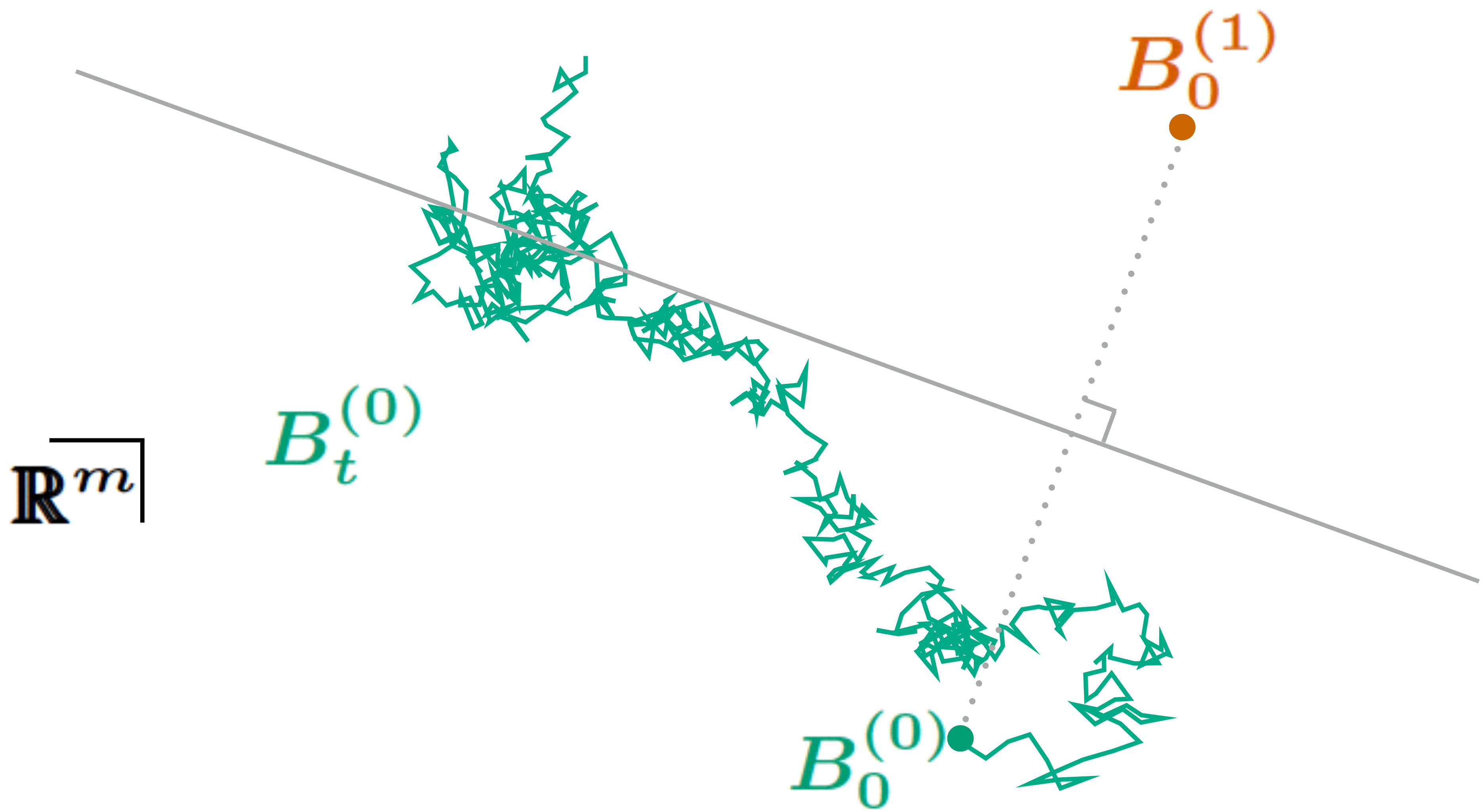

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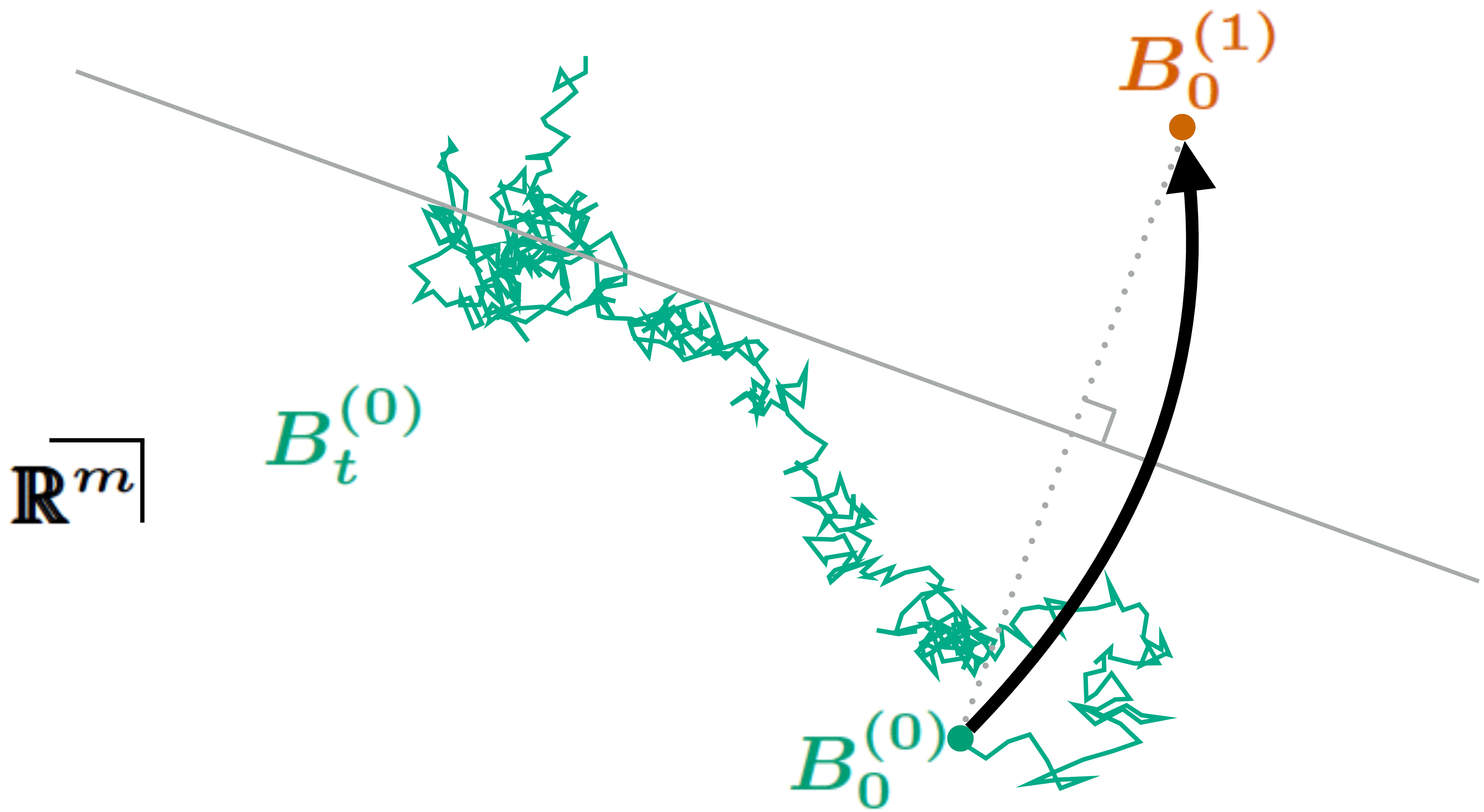

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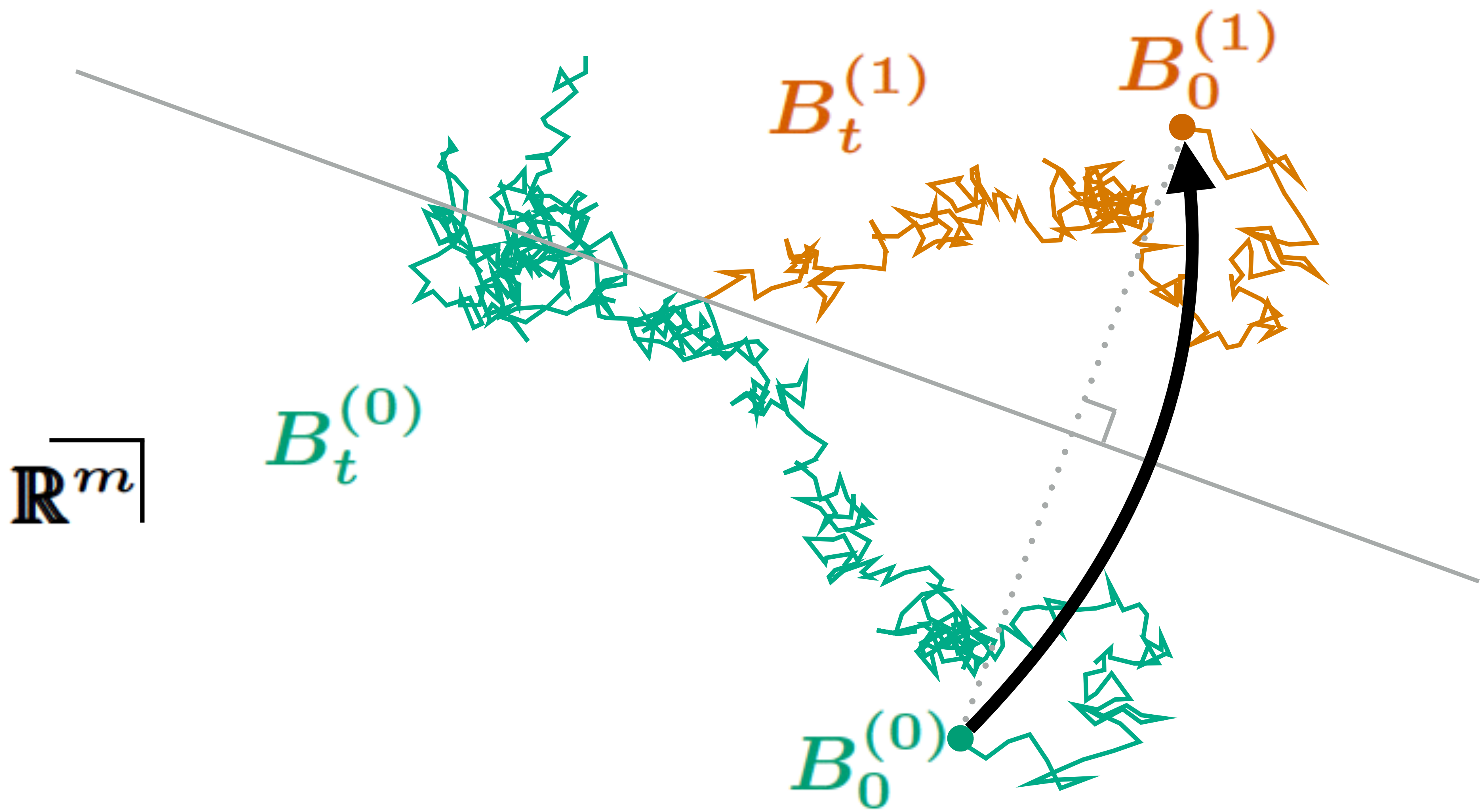

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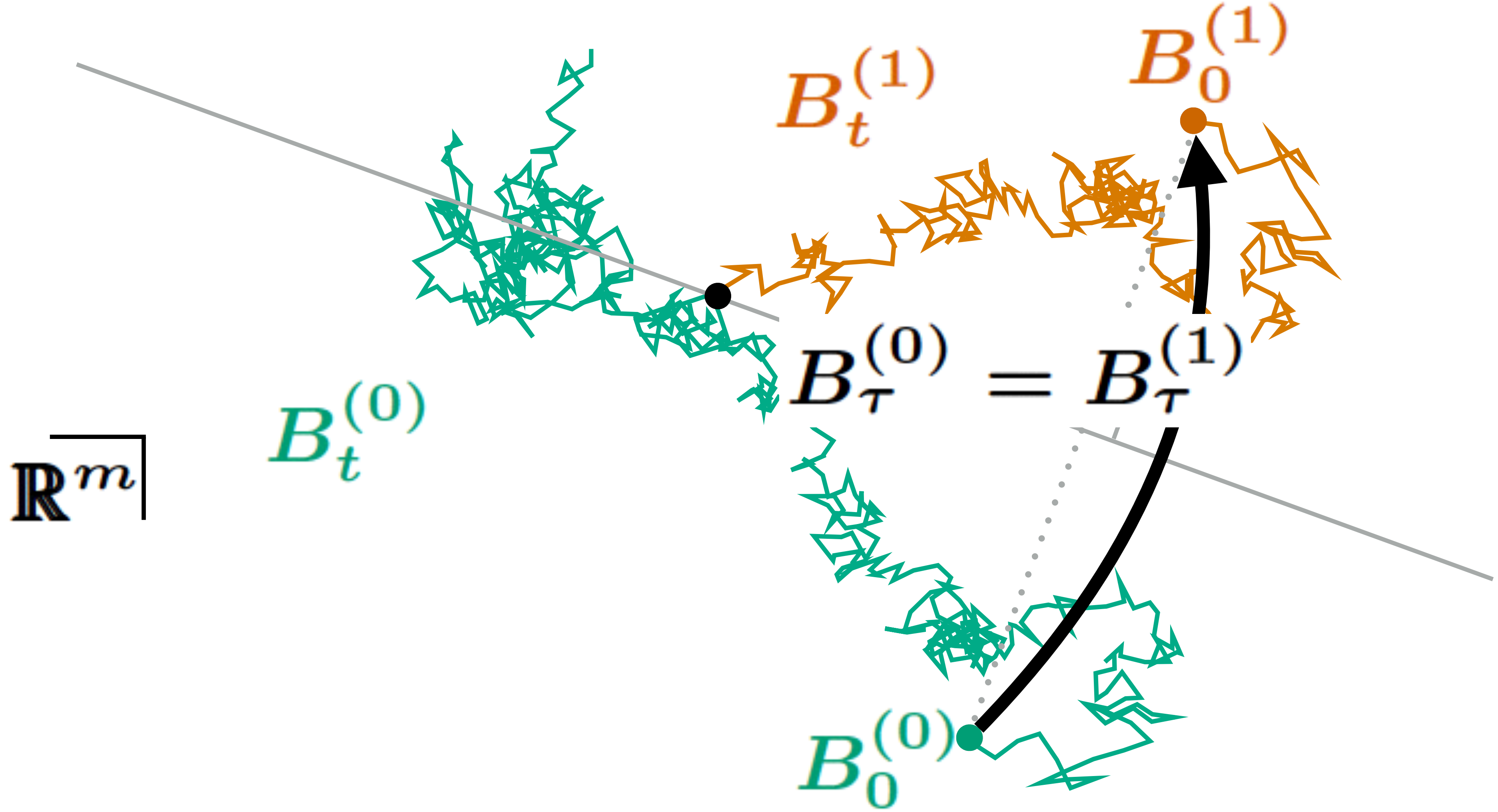

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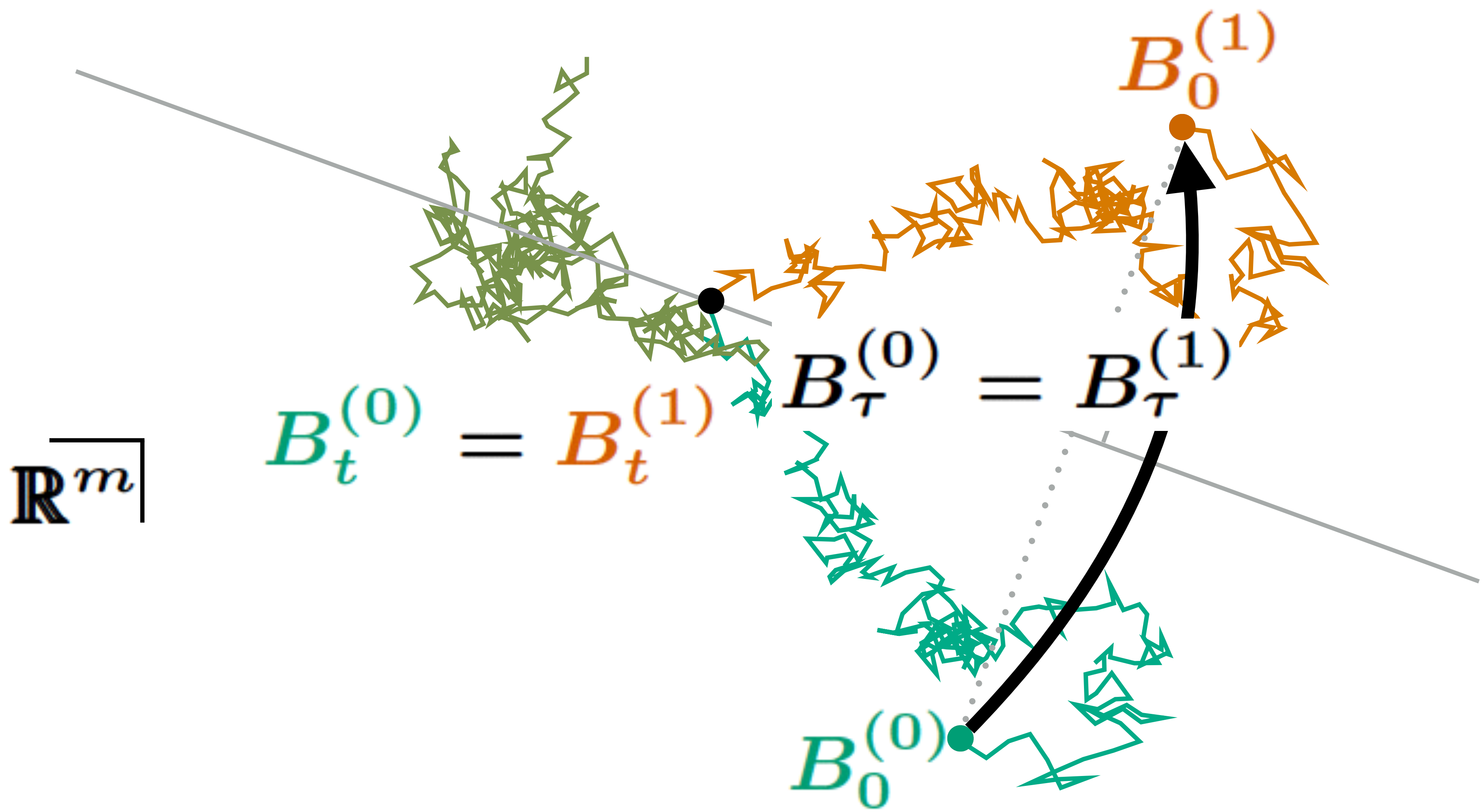

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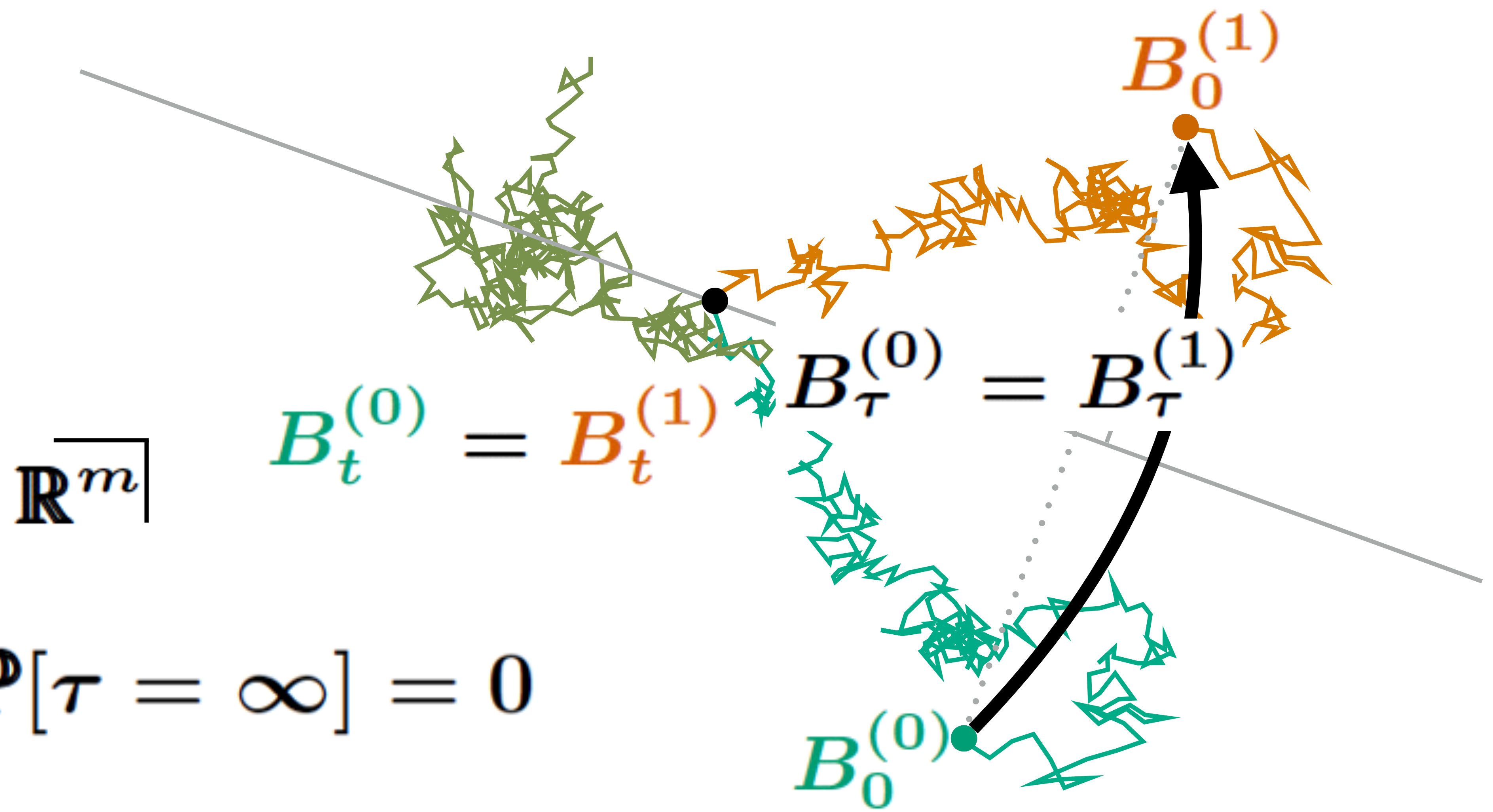

$\Rightarrow \mathbb{P}[\tau = \infty] = 0$

Outline of the talk

1. **Introduction**

2. **Framework**

3. **Characterization by optimal transport**

4. **Proof**

5. **Further questions**

$\mathrm{RCD}(K,\infty)$ spaces

$(X, d, \mathfrak{m})$: met. meas. sp., i.e.

- (X, d): Polish geodesic met. sp.,
- $\mathfrak{m}$: loc. finite, σ-finite Borel meas. on X

Def

For $K \in \mathbb{R}$, (X, d, m): $\mathrm{RCD}(K, \infty)$ sp. iff

- Ent is (weakly) K-convex on $(\mathcal{P}_2(X), W_2)$ ($\mathrm{CD}(K, \infty)$ cond.)
- $(X, d, \mathfrak{m})$: infinitesimally Hilbertian (i.e. Cheeger energy is a quadratic form)

$\mathrm{RCD}(K,\infty)$ spaces

Cheeger energy

$$\mathsf{Ch}(f) = \frac{1}{2}\inf\left\{ \varliminf_{n} \int |\nabla f_n|^2 d\mathfrak{m} \;\middle|\; \begin{array}{c} f_n : \text{ Lip.} \\ f_n \to f \text{ in } L^2 \end{array} \right\}$$
$$= \frac{1}{2}\int {}^{\exists}|\nabla f|_w^2 \, d\mathfrak{m}$$

($|\nabla f|_w$: min. weak upper grad.)

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★ **Ch**: quadratic

$\Rightarrow$

- $\langle \nabla \cdot, \nabla \cdot \rangle$: $W^{1,2}(X)^2 \to L^1(\mathfrak{m})$ s.t. $\langle \nabla f, \nabla f \rangle = |\nabla f|_w^2$
- **Ch** $\leftrightarrow \Delta \leftrightarrow P_t = e^{t\Delta}$ & Δ, P_t: linear

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$\mathrm{RCD}(K,\infty)$ spaces

Properties [Ambrosio, Gigli & Savaré '14 / Ambrosio, Gigli, Mondino & Rajala]

- $C^{\mathrm{Lip}}(X) \cap W^{1,2}(X) \subset W^{1,2}(X)$ dense
- ${}^{\forall}t$, $P_t 1 \equiv 1$ (conservativity)
- ${}^{\exists}$heat kernel of P_t
- ${}^{\exists}(B_t)_{t \geq 0}$: BM on X for each initial data
- $P_t f \in C^{\mathrm{Lip}}(X)$ for $f \in L^{\infty}(\mathfrak{m})$

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- Bakry & Émery's (K, ∞) curv.-dim. cond.:

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K|\nabla f|^2$$

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3. Characterization by optimal transport

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Coupling by refl. and OMT

M: Riem. mfd, $\mathrm{Ric} \geq K$

$\Rightarrow$ For $T > 0$ & $\mu, \nu \in \mathcal{P}(M)$,

$$\mathcal{T}_{\varphi_{T-t}(d)}(P_t\mu, P_t\nu) \searrow \text{ in } t \in [0, T]$$

[K. & Sturm '13]

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[K. & Sturm '13]

$$\varphi_t(a) := \chi\left(\frac{a}{2\sqrt{2\eta(t)}}\right),$$

$$\chi(a) := \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} \mathrm{e}^{-x^2/2} dx, \ \eta(t) := \frac{\mathrm{e}^{2Kt} - 1}{2K}$$

Coupling by refl. and OMT

$$\mathcal{T}_{\varphi_{T-t}(d)}(P_t\mu, P_t\nu) \searrow \text{ in } t \in [0,T]$$

- $0 \le \varphi_t \le 1$, $\varphi_t \nearrow$, $\varphi_t(0) = 0$ & φ_t: concave
- $\varphi_0 = 1_{(0,\infty)}$

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★ $\varphi_t(a) = \mathbb{P}[\tau_r^a > t]$

τ_r^a: coupling time of two sol.'s to the following SDE:

$$dr(t) = \sqrt{2}d\beta(t) - \frac{K}{2}r(t)dt$$

starting from $\dfrac{a}{2}$ & $-\dfrac{a}{2}$ resp. coupled by reflection

Coupling by refl. and OMT

For $t > 0$, $^\exists \pi$: coupling of $P_t\delta_x$ and $P_t\delta_y$ s.t.

$$\pi(\{\mathrm{Diag}\}^c) \le \mathbb{P}[\tau_r^{d(x,y)} > t]$$

[K. & Sturm '13]

Proof

- (LHS) $= \mathcal{T}_{\varphi_0(d)}(P_t\delta_x, P_t\delta_y)$
- (RHS) $= \varphi_t(d(x,y)) = \mathcal{T}_{\varphi_t(d)}(\delta_x, \delta_y)$ $\square$

Main Theorem

Theorem 1 ([K.])

$(X, d, \mathfrak{m})$: $\mathrm{RCD}(K, \infty)$ *mm sp. with* $K \in \mathbb{R}$

$\Rightarrow {}^\forall x_0, x_1 \in X$, ${}^\exists(B_t^{(0)}, B_t^{(1)})$: *a coupling of BMs s.t.*

- $(B_0^{(0)}, B_0^{(1)}) = (x_0, x_1)$
- ${}^\forall t > 0$, $\mathbb{P}[\tau > t] \le \mathbb{P}[\tau_r^{d(x_0, x_1)} > t]$

In particular, $\mathbb{P}[\tau = \infty] = 0$ *when* $K \ge 0$

(Recall: $\tau := \inf\{t > 0 \mid {}^\forall s > t,\ B_s^{(0)} = B_s^{(1)}\}$)

1. Reduction to the monotonicity formula

Review: Coupling by parallel transport

$^{\forall}$initial data, $^{\exists}(B_t^{(0)}, B_t^{(1)})$: a coupling of BMs

s.t. $\mathrm{e}^{Kt} d(B_t^{(0)}, B_t^{(1)}) \searrow$ in t. [Sturm]

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Proof

- Step 1: Refinement of W_2-contraction [Savaré '14]

- Step 2: Discrete-time approximation

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Proof

- Step 1: Refinement of W_2-contraction [Savaré '14]

$$W_\infty(P_t\mu, P_t\nu) \le e^{-Kt} W_\infty(\mu, \nu)$$

- Step 2: Discrete-time approximation

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<u>Proof</u>

- Step 1: Refinement of W_2-contraction [Savaré '14]
  $$W_\infty(P_{k/2^n}\delta_x, P_{k/2^n}\delta_y) \le \mathrm{e}^{-Kk/2^n} W_\infty(\delta_x, \delta_y)$$
- Step 2: Discrete-time approximation
  $\rightsquigarrow$ Trans. prob. on $X \times X$

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- Step 2: Discrete-time approximation

$\rightsquigarrow$ Trans. prob. on $X \times X$

$\rightsquigarrow$ Coupling of approximating Markov chains

Goal

$$\forall T > 0, \forall \mu, \nu \in \mathcal{P}(X), \quad \mathcal{T}_{\varphi_{T-t}(d)}(P_t\mu, P_t\nu) \searrow \text{ in } t \in [0, t]$$

Goal

$$\forall T > 0,\ \forall \mu, \nu \in \mathcal{P}(X),$$
$$\mathcal{T}_{\varphi_{T-t}(d)}(P_t\mu, P_t\nu) \searrow \text{ in } t \in [0, t]$$

$$\Uparrow$$

$$\mathcal{T}_{\varphi_s(d)}(P_t\delta_x, P_t\delta_y) \leq \varphi_{s+t}(d(x, y))$$

2. The monotonicity via Gaussian isoperimetry

Overview

Claims

Let $f : X \to [0,1]$, $f(x) - f(y) \leq \varphi_s(d(x,y))$

(1) $P_t f(x) - P_t f(y) \leq \varphi_t(d(x,y))$

(2) $P_t f(x) - P_t f(y) \leq \varphi_s(\mathrm{e}^{-Kt} d(x,y))$

$\Rightarrow P_t f(x) \leq P_t f(y) + \varphi_{s+t}(d(x,y))$

($\because \chi(\sqrt{r})$: concave)

$\Rightarrow \mathcal{T}_{\varphi_s(d)}(P_t\delta_x, P_t\delta_y) \leq \varphi_{s+t}(d(x,y))$

($\because$ Kantorovich duality) $\square$

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(1) $P_t f(x) - P_t f(y) \leq \varphi_t(d(x,y))$

(2) $\boldsymbol{P_t f(x) - P_t f(y) \leq \varphi_s(\mathrm{e}^{-Kt} d(x,y))}$

$\Rightarrow P_t f(x) \leq P_t f(y) + \varphi_{s+t}(d(x,y))$

($\because \chi(\sqrt{r})$: concave)

$\Rightarrow \mathcal{T}_{\varphi_s(d)}(P_t \delta_x, P_t \delta_y) \leq \varphi_{s+t}(d(x,y))$

($\because$ Kantorovich duality) $\square$

Overview

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Reverse Gaussian isoperimetry for P_t

For $f : X \to [0,1]$ m'ble

$$2\eta(t)|\nabla P_t f|_w^2 \leq I(P_t f)^2 - P_t(I(f))^2$$

[Bakry, Gentil & Ledoux]

$I = \Phi' \circ \Phi^{-1}$: Gaussian isoperimetric profile

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2} du$$

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Gaussian isoperimetry for P_t [Bakry & Ledoux '96]

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Let $f : X \to [0,1]$ *&* $\psi : [0,\infty) \to \mathbb{R}$

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Rem Alternatively, W_∞-contraction $\Rightarrow$ Prop 3

Questions

- (K, N)-coupling by refl. on $\mathrm{RCD}^*(K, N)$ sp.'s?
- Can we localize the construction?
- Other sample path properties?

$$\left(\begin{array}{l} \text{e.g. Comparison theorem for } d(x_0, B_t): \\ \text{OK for } K = 0 \text{ ([K. \& Kuwae]; in progress)} \end{array} \right)$$

- (K, N)-isoperimetry for P_t with $N < \infty$
- Characterize "$\mathrm{Ric} \geq K$" by the coupling by refl.?
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